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Compositeness relations, effective range expansion and extoic hadrons

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Outline:

1. Background & Introduction
2. Compositeness relation for resonance
3. Effective range expansion
4. Exotic hadrons near thresholds
5. Summary

Background & Introduction

Prominent features of many exotic resonances:

(regardless of heavy/light flavors: X(3872), Zc/Zb, Λ(1405), Pc, ...)

★ **Beyond Standard quark Model: BSqM**

They lie quite close to the underlying thresholds.

❖ **Important question that follows:**

<u>Kinematical effects ?</u> Or <u>Hadron molecular ?</u> Or <u>Elementary/Compact state ?</u>

❖ **Theoretcial methods to probe the composition of hadrons:**

✓ **QCD sum rules**

✓ **Pole counting rule**

✓ **Nc trajecotries of resonance poles**

✓ **Weinberg's compositeness relation**

❑ Pole counting rule [Morgan, NPA'92]

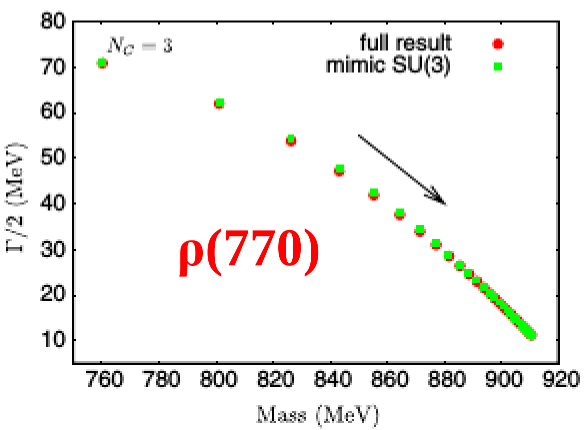
Criteria: Number of nearby poles in S-wave scattering amplitudes

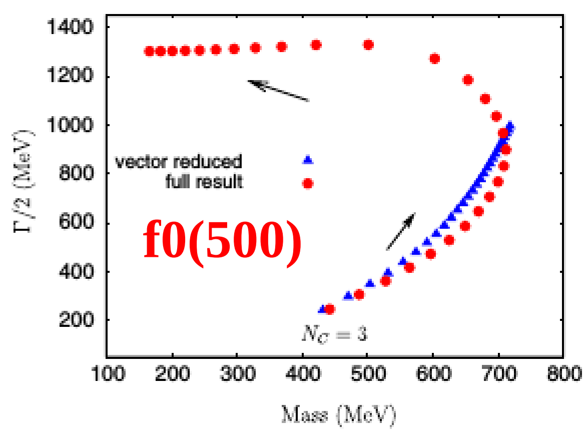
- Elementary particle: A pair of poles close to threshold
- Molecular type: Single pole close to threshold

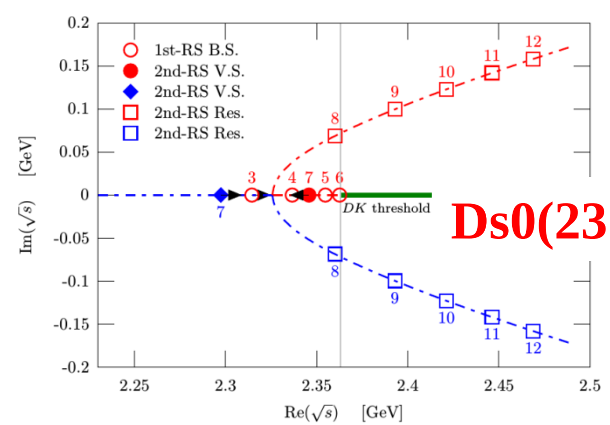
X(3872): elementary state, instead of Dbar-D* molecule

Zc(3900): molecule of Dbar-D* [Zheng et al., PRD '15 '16]

❑ Nc trajecotries of resonance poles [Guo et al., PRD '12 '15]







Ds0(2317)

❑ Pole counting rule and Nc trajectories only give qualitative conclusions.

❑ Compositeness and elementariness relation

[Weinberg, PR '63 ' 65]

$$|\psi_B\rangle = \int d\alpha \langle\varphi_\alpha|\psi_B\rangle|\varphi_\alpha\rangle + \sum_n \langle\varphi_n|\psi_B\rangle|\varphi_n\rangle$$

X + Z = 1

$$\int d\alpha |\langle\varphi_\alpha|\psi_B\rangle|^2$$

Compositeness

$$\sum_n |\langle\varphi_n|\psi_B\rangle|^2$$

Elementariness

In the non-relativistic situation, Quantum Mechanics gives

$$X = 1 - Z = \int d\alpha \frac{|\langle\varphi_\alpha|V|\psi_B\rangle|^2}{(E_\alpha - E_B)^2}$$

$$\Rightarrow \quad X_{\ell S} = \frac{1}{2\pi^2}\int_0^\infty dk k^2 \frac{g_{\ell S}^2(k^2)}{(k^2/2\mu - E_B)^2}$$

[Aceti,Oset,PRD'12]

[Oller,AnnP'18]

$$X = 1 - Z = \int d\alpha \frac{|\langle \varphi_\alpha | V | \psi_B \rangle|^2}{(E_\alpha - E_B)^2}$$

- **Bound state √: Z and X are positive real numbers, allowing probabilistic interpretations**
- **Resonance ×: Z and X are usually complex, meaningless to be interpreted as probabilities**

❑ **We propose an alternative way to generalize Weinberg compositeness relation for resonances.**

[Guo, Oller, PRD '16]

Probabilistic interpretation of compositeness relation for resonances

General form of T matrix for on-shell two-body scattering

$$T(s) = [\mathcal{K}(s)^{-1} + G(s)]^{-1}$$

$$G(s)_i = a(s_0)_i - \frac{s - s_0}{\pi} \int_{s_i}^{\infty} ds' \frac{\rho(s')_i}{(s' - s)(s' - s_0)}$$

- **Unitarity relaltion :** $\mathrm{Im}T^{-1}(s)_{ij} = -\delta_{ij}\theta(s - s_i)\rho(s)_i$

- **Relativistic phase space factor :** $\rho(s)_i = q(s)_i / 8\pi\sqrt{s}$

(with *q(s)* the CM three-momentum)

- $\mathcal{K}(s)$ **: contact interactions and crossed channel effects**

S- and *T*- matrix relation :

$$S(s) = I + i(2\rho(s))^{\frac{1}{2}} T(s) (2\rho(s))^{\frac{1}{2}}$$

n-channel thresholds: s_1 s_2 s_3 ... s_n

- **Bound-state pole:** **Physical Riemann Sheet, real axis,** $s_P < s_1$
- **Virtual-state pole:**

  Unphysical RS (adjacent to 1st RS), real axis, $s_P < s_1$
- **Resonance pole:** **Other cases**

Re-derivation of Weinberg's compositeness relation

Residues of T matrix at $s = s_P$

$$\lim_{s \to s_P} (s - s_P) T(s) = -\gamma\gamma^T, \quad \gamma^T = (\gamma_1, \ldots, \gamma_n)$$

$$T(s) = [\mathcal{K}(s)^{-1} + G(s)]^{-1}$$

Taking derivative with respect to s at $s = s_P$

$$\gamma\gamma^T = -\gamma\gamma^T \left[\frac{dG(s_P)}{ds} - \mathcal{K}^{-1} \frac{d\mathcal{K}(s_P)}{ds} \mathcal{K}^{-1} \right] \gamma\gamma^T$$

$$1 = -\gamma^T \frac{dG(s_P)}{ds} \gamma + \gamma^T G(s_P) \frac{d\mathcal{K}(s_P)}{ds} G(s_P) \gamma$$

$$= \sum_{i,j=1}^{n} \left(\underbrace{\boxed{-\delta_{ij}\gamma_i^2 \frac{dG(s_P)_i}{ds}}}_{X} + \underbrace{\boxed{\gamma_i G(s_P)_i \frac{d\mathcal{K}(s_P)_{ij}}{ds} G(s_P)_j \gamma_j}}_{Z} \right)$$

See also: [Hyodo et al., PRC'12] [Aceti et al., PRD'12]

$$\sum_{i,j=1}^{n}\left(\underbrace{\boxed{-\delta_{ij}\gamma_i^2\frac{dG(s_P)_i}{ds}}}_{X}+\underbrace{\boxed{\gamma_i G(s_P)_i\frac{d\mathcal{K}(s_P)_{ij}}{ds}G(s_P)_j\gamma_j}}_{Z}\right)$$

- **Easy to demonstrate:**
  - **X and Z are real for bound state pole s_P**
  - **X and Z become complex for other poles s_P, thus invalid for probabilisitc interpretation**

- **Way out of the problems**
  - **Taking real part of X and Z (Aceti, Oset, Hyodo, ...)**
  - **Our proposal: Rotating away the complex phases of X**

Get rid of the complex phases of the compositeness X

Laurent Expansion of the S matrix around s_P

$$S(s) = \frac{\mathcal{R}}{s - s_P} + S_0(s), \quad \mathcal{R} = -2i\rho(s_P)^{\frac{1}{2}}\gamma\gamma^T\rho(s_P)^{\frac{1}{2}}.$$

Rewrite *R* explicitly symmetric: $\mathcal{R} = i\lambda\mathcal{O}\mathcal{A}\mathcal{O}^T$

Proper choice of *O* and λ leads to projection operator *A* with rank 1

$$\mathcal{A}^\dagger = \mathcal{A} \text{ and } \mathcal{A}^2 = \mathcal{A} \quad , \mathrm{Tr}\mathcal{A} = 1$$

A deeper understanding of the compositeness reltation

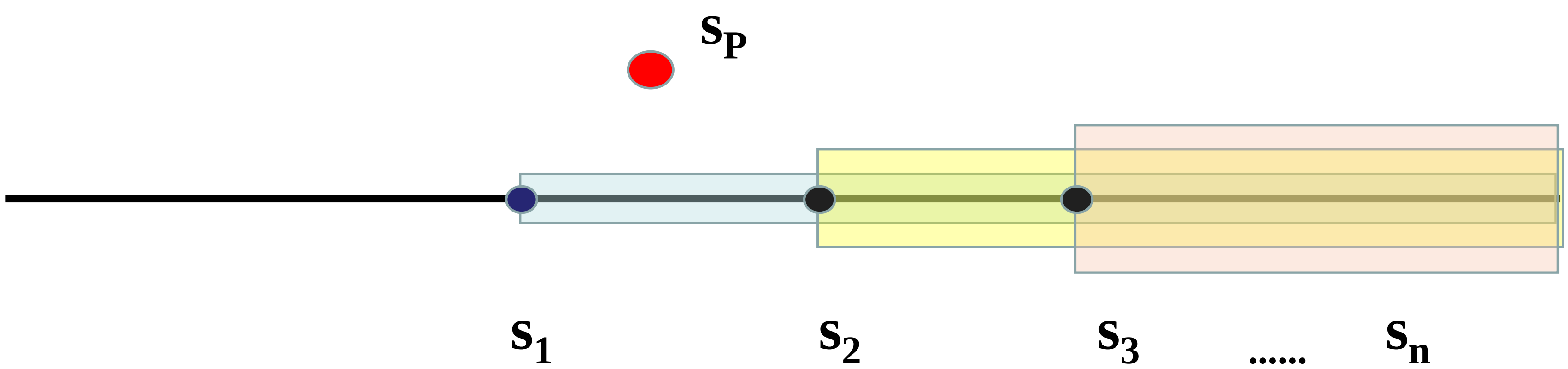
$$1 = -\gamma^T \frac{dG(s_P)}{ds}\gamma + \gamma^T G(s_P)\frac{d\mathcal{K}(s_P)}{ds}G(s_P)\gamma$$

$$\mathcal{A} = \frac{\lambda}{2}\mathcal{A}\mathcal{O}^T\rho(s_P)^{-\frac{1}{2}}\left[\frac{dG(s_P)}{ds} - G(s_P)\frac{d\mathcal{K}(s_P)}{ds}G(s_P)\right]\rho(s_P)^{-\frac{1}{2}}\mathcal{O}\mathcal{A}$$

1 stands for the proper normalization of bound/virtual/resonance states

with rank one projection operator *A*

Open and Hidden channels



Suppose there are *m* open channels

$$1 = \hat{\gamma}^T \left[-\frac{d\hat{G}_m(s_P)}{ds} + \hat{G}_m(s_P) \frac{d\hat{\mathcal{K}}_m(s_P)}{ds} \hat{G}_m(s_P) \right] \hat{\gamma}$$

After some manipulations, we can get a very important result

$$\boxed{\hat{Z} = \hat{\gamma}^T \hat{G}_m \frac{d\hat{\mathcal{K}}_m(s_P)}{ds} \hat{G}_m \hat{\gamma} = Z + \sum_{i=m+1}^{n} X_i}$$

- **Hidden-channel effects are identified as "Elementariness"！**

Transformed S matrix and sum rules

$$\hat{S}_u(s) = \hat{\mathcal{U}}_m \hat{S}_m(s) \hat{\mathcal{U}}_m^T$$

$$\hat{\gamma}_u = \hat{\rho}(s_P)^{-\frac{1}{2}} \hat{\mathcal{U}}_m \hat{\rho}(s_P)^{\frac{1}{2}} \hat{\gamma}$$

$$\hat{\mathcal{U}}_m = \mathrm{diag}(e^{i\phi_1}, \ldots, e^{i\phi_m})$$

$$1 = \hat{\gamma}^T \left[-\frac{d\hat{G}_m(s_P)}{ds} + \hat{G}_m(s_P) \frac{d\hat{\mathcal{K}}_m(s_P)}{ds} \hat{G}_m(s_P) \right] \hat{\gamma}$$

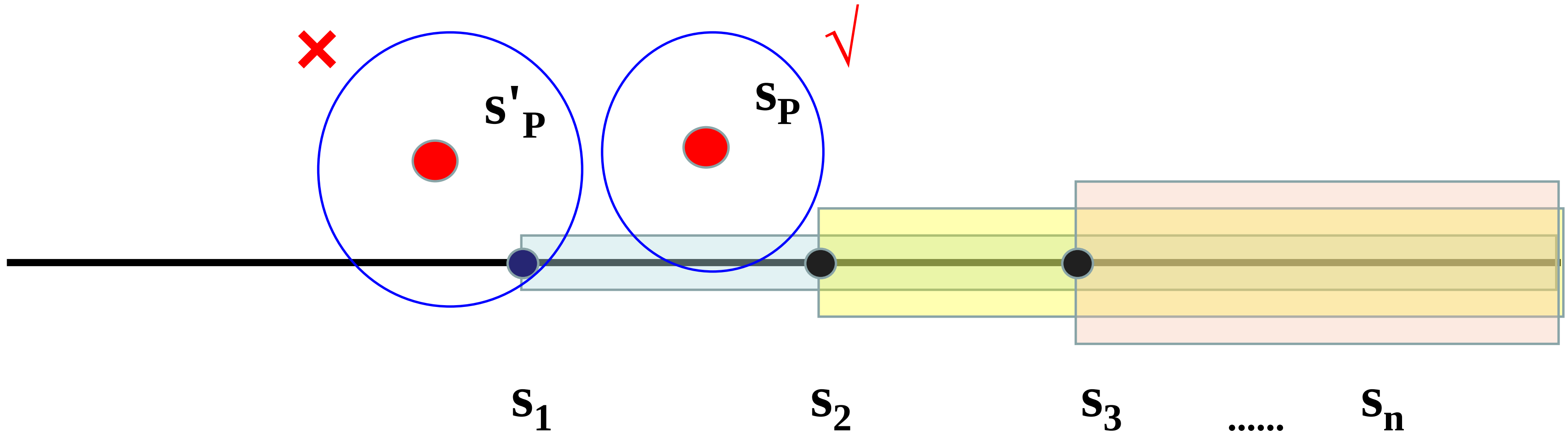
$$1 = -\hat{\gamma}_u^T \frac{d\hat{G}_m(s_P)}{ds} \hat{\gamma}_u + \hat{\gamma}_u^T \hat{G}_m \frac{d\hat{\mathcal{K}}_u(s_P)}{ds} \hat{G}_m \hat{\gamma}_u$$

This procedure gives a positive real value for X

$$X_i^R = -e^{2i\phi_i} \eta_i^2 |\gamma_i|^2 \frac{dG(s_P)_i}{ds} = |\gamma_i|^2 \left| \frac{dG(s_P)_i}{ds} \right|$$

$$X^R = \sum_{i=1}^{m} X_i^R \qquad\qquad Z^R = 1 - X^R$$

Working assumption / Condition



Summary: Once S matrix is known in a region (no matter how small) of physical real axis around M_R, we can perfrom analytic extrapolation from the real axis to the complex plane, due to the convergence of the Laurent series for s around M_R.

- **Then, compositeness is model independently determined by the properties of resonance pole (pole position and residues)**

$$X_i^R = |\gamma_i|^2 \left| \frac{dG(s_P)_i}{ds} \right|$$

Applications to some well-established hadrons

[Guo, Oller, PRD '16]

Name of the states	Pole: $\sqrt{s_P}$ [MeV]	$X^R_{\pi\pi}$	$X^R_{\bar{K}K}$	$X^R_{\eta\eta}$	$X^R_{\eta\eta'}$	X^R	Z^R
$f_0(500)$ [17]	$442^{+4}_{-4} - i246^{+7}_{-5}$	$0.40^{+0.02}_{-0.02}$	$\cdots$	$\cdots$	$\cdots$	$0.40^{+0.02}_{-0.02}$	$0.60^{+0.02}_{-0.02}$
$f_0(980)$ [17]	$978^{+17}_{-11} - i29^{+9}_{-11}$	$0.02^{+0.01}_{-0.01}$	$0.65^{+0.27}_{-0.26}$	$\cdots$	$\cdots$	$0.67^{+0.28}_{-0.27}$	$0.33^{+0.28}_{-0.27}$
$f_0(1710)$ [14]	$1690^{+20}_{-20} - i110^{+20}_{-20}$	$0.00^{+0.00}_{-0.00}$	$0.03^{+0.01}_{-0.01}$	$0.02^{+0.01}_{-0.01}$	$0.20^{+0.07}_{-0.07}$	$0.25^{+0.10}_{-0.10}$	$0.75^{+0.10}_{-0.10}$
$\rho(770)$ [17]	$760^{+7}_{-5} - i71^{+4}_{-5}$	$0.08^{+0.01}_{-0.01}$	$\cdots$	$\cdots$	$\cdots$	$0.08^{+0.01}_{-0.01}$	$0.92^{+0.01}_{-0.01}$
		$X^R_{K\pi}$				X^R	Z^R
$K_0^*(800)$ [17]	$643^{+75}_{-30} - i303^{+25}_{-75}$	$0.94^{+0.39}_{-0.52}$				$0.94^{+0.39}_{-0.52}$	$0.06^{+0.39}_{-0.52}$
$K^*(892)$ [17]	$892^{+5}_{-7} - i25^{+2}_{-2}$	$0.05^{+0.01}_{-0.01}$				$0.05^{+0.01}_{-0.01}$	$0.95^{+0.01}_{-0.01}$
		$X^R_{\pi\eta}$	$X^R_{\bar{K}K}$	$X^R_{\pi\eta'}$		X^R	Z^R
$a_0(1450)$ [17]	$1459^{+70}_{-95} - i174^{+110}_{-100}$	$0.09^{+0.03}_{-0.07}$	$0.02^{+0.12}_{-0.02}$	$0.12^{+0.22}_{-0.09}$		$0.23^{+0.37}_{-0.18}$	$0.77^{+0.37}_{-0.18}$
		$X^R_{\rho\pi}$				X^R	Z^R
$a_1(1260)$ [18]	$1260 - i250$	0.46				0.46	0.54
Hyperon with $I=0$		$X^R_{\pi\Sigma}$	$X^R_{\bar{K}N}$			X^R	Z^R
$\Lambda(1405)$ broad [19]	$1388^{+9}_{-9} - i114^{+24}_{-25}$	$0.73^{+0.15}_{-0.10}$	$\cdots$			$0.73^{+0.15}_{-0.10}$	$0.27^{+0.15}_{-0.10}$
$\Lambda(1405)$ narrow [19]	$1421^{+3}_{-2} - i19^{+8}_{-5}$	$0.18^{+0.13}_{-0.08}$	$0.82^{+0.36}_{-0.17}$			$1.00^{+0.49}_{-0.25}$	$0.00^{+0.49}_{-0.25}$
		X^R_{DK}	$X^R_{D_s\eta}$	$X^R_{D_s\eta'}$		X^R	Z^R
$D_{s0}^*(2317)$ [20]	2321^{+6}_{-3}	$0.56^{+0.05}_{-0.03}$	$0.12^{+0.01}_{-0.01}$	$0.02^{+0.01}_{-0.01}$		$0.70^{+0.07}_{-0.05}$	$0.30^{+0.07}_{-0.05}$
		$X^R_{J/\psi f_0(500)}$	$X^R_{J/\psi f_0(980)}$	$X^R_{Z_c(3900)\pi}$	$X^R_{\omega\chi_{c0}}$	X^R	Z^R
$Y(4260)$ [21,22]	$4232.8 - i36.3$	0.00	0.02	0.02	0.17	0.21	0.79

Effective range expansion

Effective-range-expansion (ERE) formalism

$$V(k) = -\frac{1}{a} + \frac{1}{2} r k^2$$

a : scattering length
r : effective range

$$k = \sqrt{2\mu(E - M_{\rm th})}$$

$$\mu = m_1 m_2/(m_1 + m_2)$$

$$M_{\rm th} = m_1 + m_2$$

$$T(k) = \frac{1}{V(k) - ik}$$

- Crossed-channel cuts neglected
- Convergent radius of ERE: dictated by the nearest singularity from the crossed channel
- Invalid if there is a near-threshold CDD pole !

$$\delta a = -\frac{M_{\rm th} - M_{\rm i,CDD}}{g_i} \qquad \delta r = -\frac{g_i}{\mu(M_{\rm th} - M_{\rm i,CDD})^2}$$

[Guo, Oller, PRD '16]

Single-channel case

$$t(E) = \frac{1}{-\frac{1}{a} + \frac{1}{2} r k^2 - ik} \qquad k = \sqrt{2\mu(E - M_{\text{th}})}$$

Determine *a* and *r* using the mass and width of resonance *R*

$$E_R = M_R - i\frac{\Gamma_R}{2}, \quad k_R = \sqrt{2\mu(E_R - M_{\text{th}})}, \quad k_R = k_r + ik_i, \qquad k_i > 0.$$

Partial wave amplitude in the 2nd Riemann Sheet ：

$$t_{II}(E) = \frac{1}{-\frac{1}{a} + \frac{1}{2} r k^2 + ik}$$

$$0 = -\frac{1}{a} + \frac{1}{2} r k_R^2 + ik_R$$

$$a = -\frac{2k_i}{|k_R|^2}, \qquad r = -\frac{1}{k_i}$$

Once *a* and *r* are determined, the partial-wave amplitude is completely fixed.

Residue in the variable of three-momentum *k*

Expand the denorminator in *k* - *k*$_R$:

$$t_{II}(E) = \frac{1}{-\frac{1}{a} + \frac{1}{2}rk^2 + ik}$$

$$\Rightarrow \quad t_{II}(k) = \frac{1}{(rk_R + i)(k - k_R)} + \ldots = \frac{-k_i/k_r}{k - k_R} + \ldots$$

$$\Rightarrow \quad \gamma_k^2 = -\frac{k_i}{k_r} > 0$$

(Residue also fixed by the pole position)

Remember: $k_R = k_r + ik_i, \qquad k_i > 0.$

Residue in the variable of CM energy *E*:

$$t_{II}(E) \xrightarrow[{[E \to E_R]}]{} -\frac{\gamma^2}{s - E_R^2}$$

$$\gamma_k^2 = -\gamma^2 \frac{dk}{ds}\bigg|_{k_R} = -\frac{\mu\gamma^2}{2E_R k_R}$$

Compositeness for a resonance within ERE

$$X = \left|\gamma^2 \frac{dG(E_R)}{ds}\right|$$

[Kang, Guo, Oller, PRD '16]

$$= \left|\gamma^2 \frac{dk}{ds}\frac{dG(E_R)}{dk}\right|^2 = |\gamma_k|^2 = -\frac{k_i}{k_r} = \left(\frac{2r}{a} - 1\right)^{-\frac{1}{2}}$$

After some manipulations, one can also write *X* explicitly in terms of M_R and Γ_R

$$X = -\frac{2(M_R - M_{\rm th})}{\Gamma_R} + \sqrt{1 + \left[\frac{2(M_R - M_{\rm th})}{\Gamma_R}\right]^2}$$

$$= 1 - \frac{2(M_R - M_{\rm th})}{\Gamma_R} + 2\left[\frac{(M_R - M_{\rm th})}{\Gamma_R}\right]^2 + \cdots$$

(The second equality is valid for $(M_R - M_{th})/\Gamma_R \ll 1$)

- **A very rough but also very simple rule for a molecule (taking with caution !!!): $(M_R - M_{th})/\Gamma_R \ll 1$**

Exotic hadrons near thresholds: ERE & Compositeness relations

$$t(E) = \frac{1}{-1/a + (1/2)\,rk^2 - ik} \qquad t_{\mathrm{II}}(E) = \frac{1}{-1/a + (1/2)\,rk^2 + ik}$$

$$E_R = M_R - i\Gamma_R/2 \qquad k_R = \sqrt{\mu(E_R - m_{\mathrm{th}})} \qquad k_r = \mathrm{Re}\,k_R \text{ and } k_i = \mathrm{Im}\,k_R.$$

$$a = -\frac{2k_i}{|k_R|^2}, \quad r = -\frac{1}{k_i}, \qquad X = |\gamma_k|^2 = -\frac{k_i}{k_r}$$

[Gao, Guo, Kang, Oller, AdvHEP '19]

Resonance	Mass (MeV)	Width (MeV)	Threshold (MeV)	a (fm)	r (fm)	X
$Z_c(3900)$	3886.6 ± 2.4	28.2 ± 2.6	$D\overline{D}^*$ (3875.8)	-0.94 ± 0.12	-2.40 ± 0.21	0.49 ± 0.06
$X(4020)$	4024.1 ± 1.9	13 ± 5	$D^*\overline{D}^*$ (4017.1)	-1.04 ± 0.26	-3.89 ± 1.42	0.39 ± 0.12
$\psi(4260)$	4230 ± 8	55 ± 19	$D_1\overline{D}$ (4289.2)	-1.04 ± 0.06	-0.54 ± 0.03	– – –
$\psi(4660)$	4643 ± 9	72 ± 11	$\Lambda_c\overline{\Lambda}_c$ (4572.9)	-0.22 ± 0.04	-1.98 ± 0.28	0.24 ± 0.04
$\chi_{c1}(4140)$ (LHCb)	4146.5 ± 6.4	83 ± 30	$D_s\overline{D}_s^*$ (4080.5)	-0.27 ± 0.06	-1.79 ± 0.61	0.29 ± 0.08
$\chi_{c1}(4140)$ (~~LHCb~~)	4147.1 ± 2.4	15.7 ± 6.3	$D_s\overline{D}_s^*$ (4080.5)	-0.06 ± 0.02	-9.10 ± 3.86	0.06 ± 0.02
$\chi_{c1}(4140)$ (PDG)	4146.8 ± 2.4	22 ± 8	$D_s\overline{D}_s^*$ (4080.5)	-0.09 ± 0.03	-6.49 ± 2.40	0.08 ± 0.03

- **Single-channel scattering is assumed for each state. X is only calculated when the working condition is satisfied.**

Scrutinizaiton of Zb(10610) and Zb(10650)

Inputs

$$M_{Z_b} = 10607.2 \pm 2.0, \qquad \Gamma_{Z_b} = 18.4 \pm 2.4,$$
$$M_{Z'_b} = 10652.2 \pm 1.5, \qquad \Gamma_{Z'_b} = 11.5 \pm 2.2.$$

Outputs from ERE and compositeness relation

	$Z_b(10610)$	$Z_b(10650)$
a (fm)	-1.03 ± 0.17	-1.18 ± 0.26
r (fm)	-1.49 ± 0.20	-2.03 ± 0.38
$X = \gamma_k^2$	0.75 ± 0.15	0.67 ± 0.16
	$B\bar{B}^*$	$B^*\bar{B}^*$

- **Implicit assumption in the single-channel ERE: the full widths of Z_b and Z_b' are exclusively contributed by BB* and B*B*, respectively .**

[Kang, Guo, Oller, PRD '16]

Resonance's width and its compositeness X

In our notation

$$\Gamma^{(1)} = \frac{k(M_R)|\gamma^2|}{M_R}$$

Define

$$g^2 = |\gamma^2 8\pi E_R| \simeq |\gamma^2| 8\pi M_R$$

(for $M_R >> \Gamma_R / 2$)

Then we have the standard formula for narrow resonance

$$\Gamma^{(1)} = \frac{k(M_R) g^2}{8\pi M_R^2}$$

	$Z_b(10610)$	$Z_b(10650)$
a (fm)	-1.03 ± 0.17	-1.18 ± 0.26
r (fm)	-1.49 ± 0.20	-2.03 ± 0.38
$X = \gamma_k^2$	0.75 ± 0.15	0.67 ± 0.16
g^2 (GeV^2)	362 ± 71	263 ± 63

Resonance's width and its compositeness X

Decay width expression beyond the narrow resonance formula

$$\Gamma^{(2)} = \frac{g^2}{16\pi^2}\int_{M_{\rm th}}^{+\infty}\frac{dW k(W)}{W^2}\frac{\Gamma_R}{(M_R-W)^2+\Gamma_R^2/4}$$

$$= \frac{X|k_R|M_R^2}{\pi\mu}\int_{M_{\rm th}}^{+\infty}\frac{dW k(W)}{W^2}\frac{\Gamma_R}{(M_R-W)^2+\Gamma_R^2/4}$$

Integration up to ∞ denoted by $\Gamma^{(2)}{}_{>}$, up to $M_{th} + n\,\Gamma_R$ by $\Gamma^{(2)}{}_{<}$

Γ (MeV)	$Z_b(10610)$	$Z_b(10650)$
$\Gamma^{(1)}$	14.9 ± 2.3	9.5 ± 2.1
$\Gamma^{(2)}_{>}$	21.9 ± 3.3	13.4 ± 2.8
$\Gamma^{(2)}_{<}$	18.5 ± 2.4	11.3 ± 2.1

[with n=8 ($M_{th} + n\,\Gamma_R$) in $\Gamma^{(2)}{}_{<}$ to exactly reproduce the inputs]

Resonance's width and its compositeness X

$$\Gamma^{(2)} = \frac{g^2}{16\pi^2}\int_{M_{\rm th}}^{+\infty}\frac{dWk(W)}{W^2}\frac{\Gamma_R}{(M_R-W)^2+\Gamma_R^2/4}$$

$$= \frac{X|k_R|M_R^2}{\pi\mu}\int_{M_{\rm th}}^{+\infty}\frac{dWk(W)}{W^2}\frac{\Gamma_R}{(M_R-W)^2+\Gamma_R^2/4}$$

$$X = \frac{\mu\pi}{|k_R|M_R^2\int_{M_{\rm th}}^{W_+}\frac{dWk(W)/W^2}{(M_R-W)^2+\Gamma_R{}^2/4}}$$

($W_+ = M_{th} + 8\,\Gamma_R$)

Taking branching ratios to redetermine the compositeness

$\mathrm{Br}(Z_b(10610)^+) \to B\bar{B}^*) = (86.0 \pm 3.6)\%$,

$\mathrm{Br}(Z_b(10650)^+) \to B^*\bar{B}^*) = (73.4 \pm 3.6)\%$.

	$Z_b(10610)$	$Z_b(10650)$
$\Gamma = \Gamma_R$	$X = 0.76 \pm 0.12$	$X = 0.69 \pm 0.14$
$\Gamma = \Gamma_R \cdot \mathrm{Br}$	$X_{\rm ex} = 0.66 \pm 0.11$	$X_{\rm ex} = 0.51 \pm 0.10$

Application to the newly observed Pentaquark candidates

Single-channel ERE

$$t(E) = \frac{1}{-1/a + (1/2)\, r k^2 - ik}$$

[Guo, Oller, PLB '19]

Resonance	Mass (MeV)	Width (MeV)	Threshold (MeV)	a (fm)	r (fm)
$P_c(4312)$	4311.9 ± 6.8	9.8 ± 5.2	$\Sigma_c^+ \bar{D}^0$ (4317.7)	-2.9 ± 0.8	-1.7 ± 0.7
			$\Sigma_c^{++} D^-$ (4323.6)	-2.4 ± 0.6	-1.2 ± 0.3
$P_c(4440)$	4440.3 ± 4.9	20.6 ± 11.2	$\Sigma_c^+ \bar{D}^{*0}$ (4459.8)	-1.7 ± 0.2	-0.9 ± 0.1
			$\Sigma_c^{++} D^{*-}$ (4464.2)	-1.6 ± 0.2	-0.8 ± 0.1
$P_c(4457)$	4457.3 ± 4.1	6.4 ± 6.0	$\Sigma_c^+ \bar{D}^{*0}$ (4459.8)	-3.8 ± 1.6	-2.3 ± 1.3
			$\Sigma_c^{++} D^{*-}$ (4464.2)	-3.0 ± 0.7	-1.6 ± 0.4

- **Natural values of a and r at the order of 1 fm indicate the moleculare natures of the three Pc states.**
- **Probabilistic interpretation of X=$|\gamma|^2$ is not valid here, since the resonance pole position does not meet the working condition !**

Coupled-channel analyses using compositeness relations

Now we include the J/ψ p, Σc D/ Σc D* to study the three Pc's.

The probabilistic interpretation of X is valid now due to the introduction of the J/ψ p channel !

$$X_j = |g_j|^2 \left| \frac{\partial G_j(s_R)}{\partial s} \right| , \qquad G(s) = i \int \frac{d^4 q}{(2\pi)^4} \frac{1}{(q^2 - m_1^2 + i\epsilon)[(P-q)^2 - m_2^2 + i\epsilon]}$$

$$|g_1|^2 \left| \frac{\partial G_1^{II}(s_R)}{\partial s} \right| + |g_2|^2 \left| \frac{\partial G_2(s_R)}{\partial s} \right| = X$$

J/ψ p **Σc D/ Σc D***

$$|g_1|^2 \frac{q_1(M_R^2)}{8\pi M_R^2} + |g_2|^2 \int_{m_{\rm th}}^{M_R+2\Gamma_R} dw \, \frac{q_2(w^2)}{16\pi^2 w^2} \frac{\Gamma_R}{(M_R - w)^2 + \Gamma_R^2/4} = \Gamma_R$$

J/ψ p **Σc D/ Σc D***

Results obtained with $X = X_1 + X_2 = 1$. The $J/\psi p$ and $\Sigma_c \bar{D}^{(*)}$ channels, which are labeled as 1 and 2 respectively, are included.

Resonance	$\lvert g_1 \rvert$ (GeV)	$\lvert g_2 \rvert$ (GeV)	Γ_1 (MeV)	Γ_2 (MeV)	X_1	X_2
$P_c(4312)$						
$m_{\Sigma_c^+} + m_{\bar{D}^0}$	$2.1^{+0.8}_{-2.1}$	$10.9^{+2.1}_{-2.9}$	$6.5^{+4.9}_{-6.5}$	$3.3^{+10.5}_{-3.3}$	$0.006^{+0.005}_{-0.006}$	$0.994^{+0.006}_{-0.005}$
$m_{\Sigma_c^{++}} + m_{D^-}$	$2.5^{+0.6}_{-0.9}$	$12.6^{+1.6}_{-2.6}$	$8.5^{+4.7}_{-4.6}$	$1.3^{+6.1}_{-1.3}$	$0.008^{+0.005}_{-0.005}$	$0.992^{+0.005}_{-0.005}$
$P_c(4440)$						
$m_{\Sigma_c^+} + m_{\bar{D}^{*0}}$	$3.2^{+0.6}_{-0.9}$	$14.9^{+1.2}_{-1.4}$	$16.3^{+6.7}_{-7.4}$	$4.3^{+9.2}_{-4.3}$	$0.010^{+0.005}_{-0.004}$	$0.990^{+0.004}_{-0.005}$
$m_{\Sigma_c^{++}} + m_{D^{*-}}$	$3.3^{+0.6}_{-0.9}$	$15.6^{+1.0}_{-1.1}$	$17.7^{+6.9}_{-8.2}$	$2.9^{+8.3}_{-2.9}$	$0.011^{+0.005}_{-0.005}$	$0.989^{+0.005}_{-0.005}$
$P_c(4457)$						
$m_{\Sigma_c^+} + m_{\bar{D}^{*0}}$	$1.5^{+0.7}_{-1.0}$	$9.5^{+2.2}_{-5.1}$	$3.5^{+4.2}_{-3.5}$	$2.9^{+9.5}_{-2.9}$	$0.002^{+0.003}_{-0.002}$	$0.998^{+0.002}_{-0.003}$
$m_{\Sigma_c^{++}} + m_{D^{*-}}$	$1.8^{+0.6}_{-0.9}$	$11.2^{+1.6}_{-2.5}$	$5.4^{+4.2}_{-4.0}$	$1.0^{+6.1}_{-1.0}$	$0.003^{+0.003}_{-0.002}$	$0.997^{+0.002}_{-0.003}$

Variation of X

Results obtained for $X = 0.8$ and $X = 0.5$ by including the $J/\psi p$ (labeled as 1) and $\Sigma_c \bar{D}^{(*)}$ (labeled as 2) channels. The values in the table are calculated by using the masses Σ_c^+ and $\bar{D}^{(*)0}$.

Resonance	$\lvert g_1 \rvert$ (GeV)	$\lvert g_2 \rvert$ (GeV)	Γ_1 (MeV)	Γ_2 (MeV)	X_1	X_2
$P_c(4312)$						
$X = 0.8$	$2.3^{+0.7}_{-1.8}$	$9.8^{+1.8}_{-2.5}$	$7.1^{+5.0}_{-6.8}$	$2.7^{+7.3}_{-2.7}$	$0.007^{+0.005}_{-0.007}$	$0.793^{+0.007}_{-0.005}$
$X = 0.5$	$2.4^{+0.7}_{-1.2}$	$7.7^{+1.5}_{-2.0}$	$8.1^{+5.1}_{-6.2}$	$1.7^{+5.1}_{-1.7}$	$0.008^{+0.005}_{-0.006}$	$0.492^{+0.006}_{-0.005}$
$P_c(4440)$						
$X = 0.8$	$3.2^{+0.7}_{-0.9}$	$13.3^{+1.0}_{-1.3}$	$17.2^{+7.6}_{-8.2}$	$3.4^{+7.4}_{-3.4}$	$0.011^{+0.005}_{-0.005}$	$0.789^{+0.005}_{-0.005}$
$X = 0.5$	$3.4^{+0.7}_{-1.0}$	$10.5^{+0.7}_{-1.0}$	$18.5^{+9.0}_{-9.3}$	$2.1^{+4.5}_{-2.1}$	$0.012^{+0.006}_{-0.006}$	$0.488^{+0.006}_{-0.006}$
$P_c(4457)$						
$X = 0.8$	$1.6^{+0.7}_{-1.5}$	$8.5^{+2.0}_{-4.5}$	$4.1^{+4.6}_{-4.1}$	$2.3^{+7.9}_{-2.3}$	$0.002^{+0.003}_{-0.002}$	$0.798^{+0.003}_{-0.003}$
$X = 0.5$	$1.7^{+0.8}_{-1.6}$	$6.7^{+1.5}_{-3.3}$	$5.0^{+5.1}_{-5.0}$	$1.4^{+5.0}_{-1.4}$	$0.003^{+0.003}_{-0.003}$	$0.497^{+0.003}_{-0.003}$

Try to include a different open channel

Results obtained when including the $\Lambda_c^+ \bar{D}^{*0}$ (labeled as 1) and $\Sigma_c^+ \bar{D}^{*0}$ (labeled as 2) channels for $X = 1.0$, 0.8 and 0.5.

Resonance	$\lvert g_1 \rvert$ (GeV)	$\lvert g_2 \rvert$ (GeV)	Γ_1 (MeV)	Γ_2 (MeV)	X_1	X_2
$P_c(4312)$						
$X = 1.0$	$4.0^{+2.0}_{-3.8}$	$10.5^{+1.3}_{-2.5}$	$6.8^{+5.4}_{-6.8}$	$3.0^{+10.6}_{-3.0}$	$0.09^{+0.16}_{-0.09}$	$0.91^{+0.09}_{-0.16}$
$X = 0.8$	$4.2^{+2.0}_{-3.4}$	$9.2^{+1.2}_{-2.0}$	$7.5^{+5.5}_{-7.2}$	$2.3^{+8.1}_{-2.3}$	$0.10^{+0.16}_{-0.10}$	$0.70^{+0.10}_{-0.16}$
$X = 0.5$	$4.5^{+2.0}_{-2.5}$	$6.8^{+0.9}_{-1.2}$	$8.5^{+5.7}_{-6.5}$	$1.3^{+4.3}_{-1.3}$	$0.11^{+0.17}_{-0.09}$	$0.39^{+0.09}_{-0.17}$
$P_c(4440)$						
$X = 1.0$	$3.8^{+0.7}_{-1.0}$	$14.8^{+1.0}_{-1.3}$	$16.4^{+6.8}_{-7.5}$	$4.2^{+9.1}_{-4.2}$	$0.03^{+0.01}_{-0.02}$	$0.97^{+0.02}_{-0.01}$
$X = 0.8$	$3.9^{+0.8}_{-1.1}$	$13.1^{+0.9}_{-1.1}$	$17.3^{+7.7}_{-8.3}$	$3.3^{+7.2}_{-3.3}$	$0.03^{+0.01}_{-0.02}$	$0.77^{+0.02}_{-0.01}$
$X = 0.5$	$4.0^{+1.0}_{-1.2}$	$10.2^{+0.6}_{-0.8}$	$18.6^{+9.2}_{-9.4}$	$2.0^{+4.3}_{-2.0}$	$0.03^{+0.02}_{-0.01}$	$0.47^{+0.01}_{-0.02}$
$P_c(4457)$						
$X = 1.0$	$1.7^{+0.9}_{-1.6}$	$9.4^{+2.3}_{-5.0}$	$3.5^{+3.7}_{-3.5}$	$2.9^{+9.5}_{-2.9}$	$0.005^{+0.007}_{-0.005}$	$0.995^{+0.005}_{-0.007}$
$X = 0.8$	$1.9^{+0.8}_{-1.9}$	$8.4^{+2.0}_{-4.4}$	$4.1^{+4.6}_{-4.1}$	$2.3^{+7.9}_{-2.3}$	$0.006^{+0.008}_{-0.006}$	$0.794^{+0.006}_{-0.008}$
$X = 0.5$	$2.0^{+0.9}_{-2.0}$	$6.6^{+1.6}_{-3.2}$	$5.0^{+5.1}_{-5.0}$	$1.4^{+4.9}_{-1.4}$	$0.008^{+0.008}_{-0.008}$	$0.492^{+0.008}_{-0.008}$

Summary

- We have extended Weinberg's compositeness relation for canonical resonance. And the compositeness coefficient is model independently given by the pole position and residues.

- Combination of effective range expansion and our compositeness relation provides useful tool to probe the inner structure of hadrons near thresholds.

- Successful descriptions achieved for: Zc(3900)、X(4020)、X(4140)、Psi(4660)、Zb(10610)、Zb(10650)、Pc(4312)、Pc(4440)、Pc(4457).

谢谢大家！