# Hydrodynamic Collectivity in Proton + Proton Collisions and $v_2(p_T)$ and spectra at inter-medium $p_T$ in p+Pb collisions

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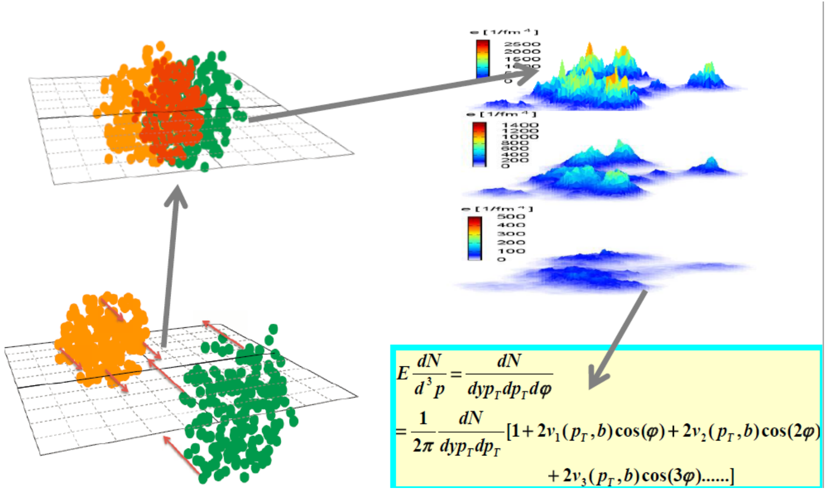
May $11^{th}$ 2019

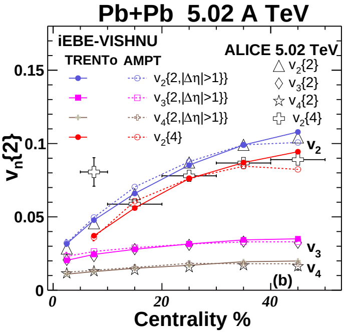
# Fluctuations and Correlations In Pb + Pb



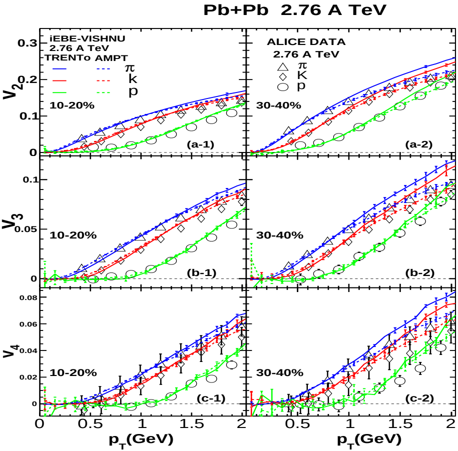

$$E\frac{dN}{d^3p}=\frac{dN}{dyp_T dp_T d\varphi}$$
$$=\frac{1}{2\pi}\frac{dN}{dyp_T dp_T}[1+2v_1(p_T,b)\cos(\varphi)+2v_2(p_T,b)\cos(2\varphi)$$
$$+2v_3(p_T,b)\cos(3\varphi)......]$$

- In heavy-ion collisions, hydrodynamics transform the initial state fluctuations to final state correlations.

- integrated $v_2\{2\}$ and $v_2\{4\}$

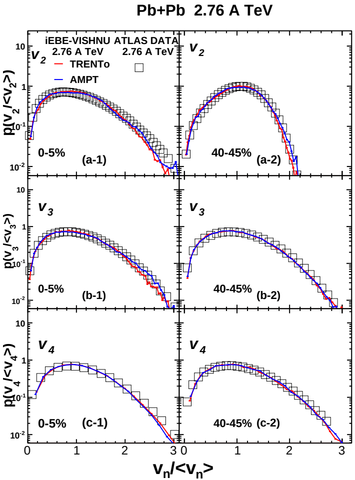



- $v_n\{p_T\}$ for $\pi$, $K$ and $p$

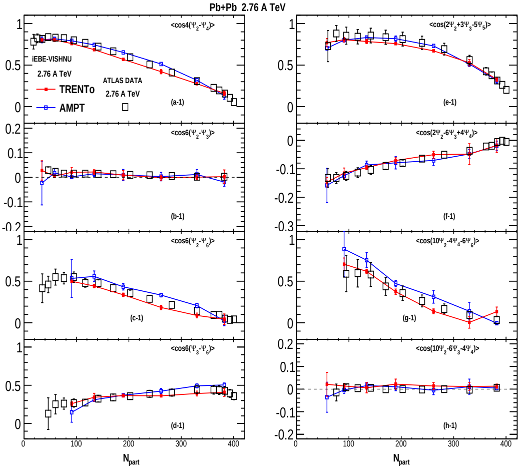



W. Zhao, H. j. Xu and H. Song, Eur. Phys. J. C **77**, no. 9, 645 (2017).

- Event-by-event $v_n$ distributions.



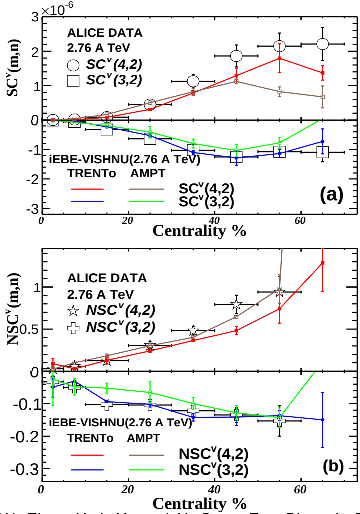

- Event-plane correlations




W. Zhao, H. j. Xu and H. Song, Eur. Phys. J. C **77**, no. 9, 645 (2017).

- Symmetry Cumulant

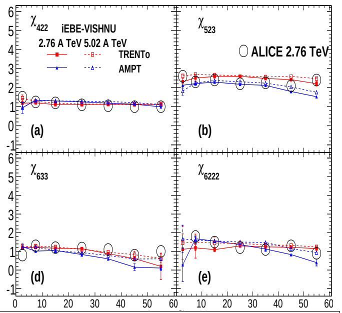
**Pb+Pb 2.76 A TeV**

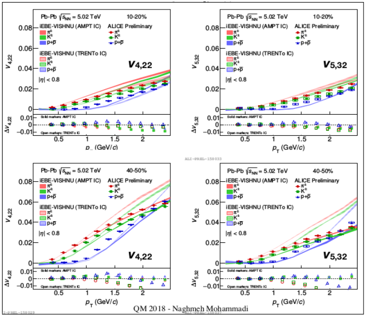



W. Zhao, H. j. Xu and H. Song, Eur. Phys. J. C **77**, no. 9, 645 (2017). N. Mohammadi [ALICE Collaboration], Nucl. Phys. A **982**, 383 (2019).

- Non-linear response cofficients

**Pb+Pb 2.76 A TeV v.s. 5.02 A TeV**

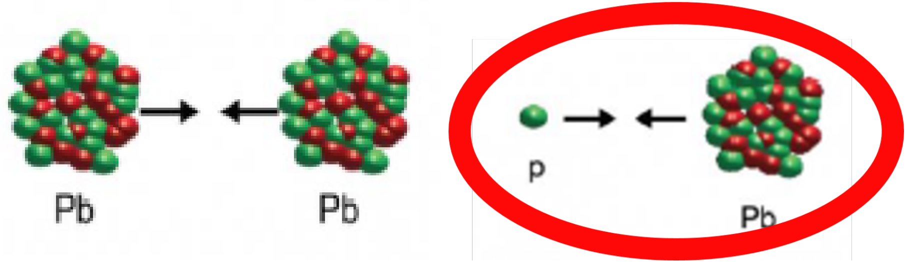


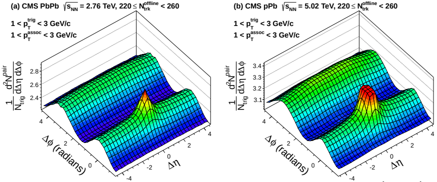
Hydrodynamic model does a great job in describing the hydrodynamic behaviors of heavy-ion collisions, including :

- integrated 2- and 4- particle cumulants, differential $v_n$, mass ordering, the event-by-event $v_n$ distributions, the event-plane correlations and Symmetric Cumulant, the nonlinear response coefficients.

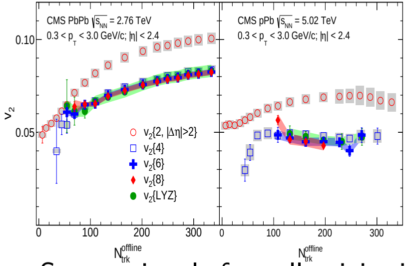
# p+Pb system

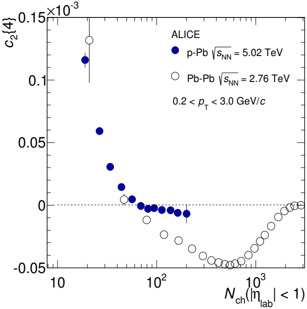


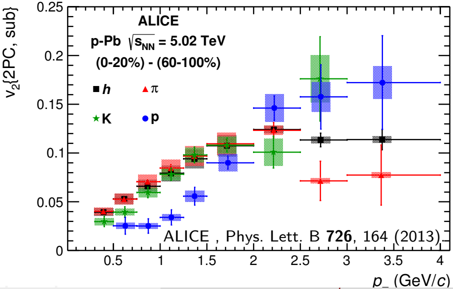
# Collective flow? p-Pb experimental Observables




CMS, Phys. Lett. B **724**, 213 (2013).



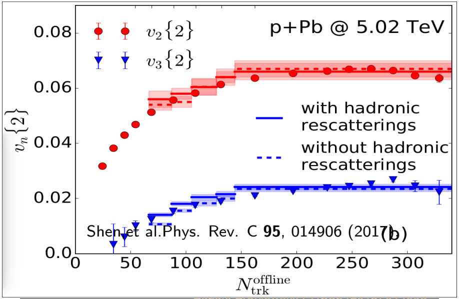



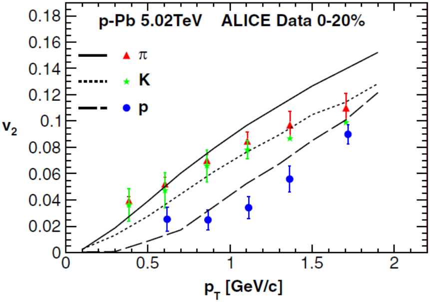




ALICE , Phys. Lett. B **726**, 164 (2013).

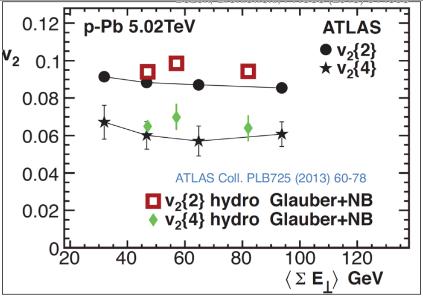
- Strong signals for collectivity in p-Pb.

# Collective flow? Hydrodynamcis simulations in p-Pb



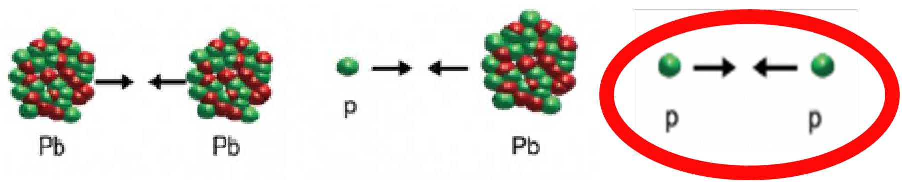




Bozek, et al.Phys. Rev. Lett. **111**, 172303 (2013).

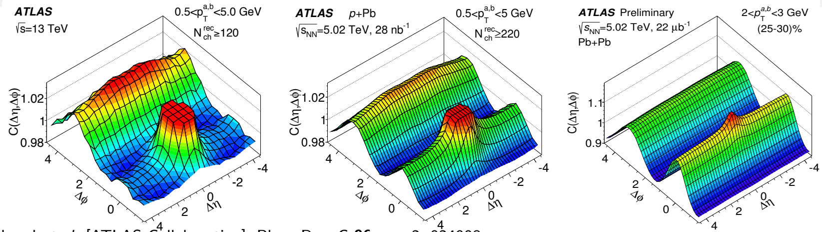



- Hydrodynamics can well reproduce the 2- and 4-pariticle correlations and mass ordering in p-Pb system.

# p+p system

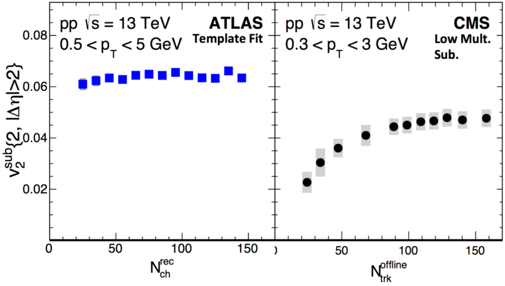


# Two-particle correlations in p-p




M. Aaboud *et al.* [ATLAS Collaboration], Phys. Rev. C **96**, no. 2, 024908

- Two-particle correlations in p-p:

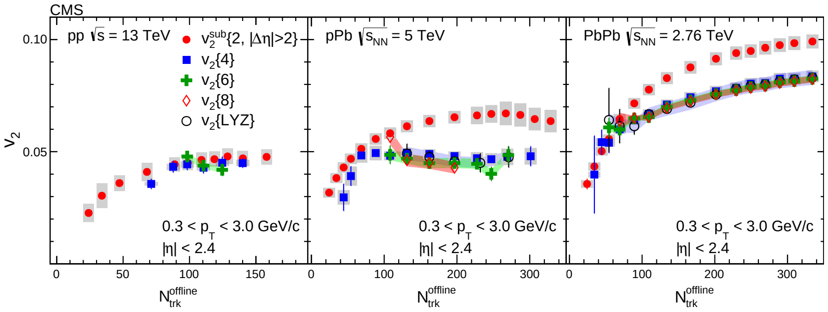



- Similar double ridge structure, but with smaller magnitudes in p-p collisions.
- Peripheral subtraction (CMS): $v_{n,n}^{peri} \approx 0$
- Template fit (ATLAS): $v_{n,n}^{cent} \approx v_{n,n}^{peri}$

# Multi-particle correlation from CMS by standard method

$$\langle 2\rangle \equiv \left\langle e^{in(\phi_1-\phi_2)}\right\rangle, \langle 4\rangle \equiv \left\langle e^{in(\phi_1+\phi_2-\phi_3-\phi_4)}\right\rangle$$

$$c_n\{4\} = \langle 4\rangle - 2\cdot\langle 2\rangle^2, v_n\{4\} = \sqrt[4]{-c_n\{4\}}$$




V. Khachatryan *et al.* [CMS Collaboration], Phys. Lett. B **765**, 193 (2017)

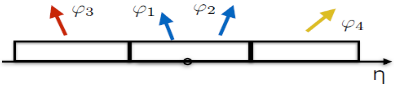
# 3-subevent

$$\langle\langle 4\rangle\rangle_{3sub} = \langle\langle \cos n(\varphi_1 + \varphi_2 - \varphi_3 - \varphi_4)\rangle\rangle$$

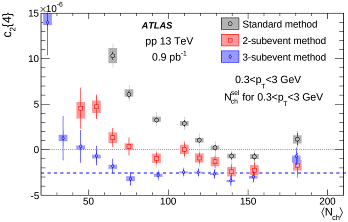
$$\langle\langle 2\rangle\rangle^2_{3sub} = \langle\langle \cos n(\varphi_1 - \varphi_3)\rangle\rangle\langle\langle \cos n(\varphi_2 - \varphi_4)\rangle\rangle$$

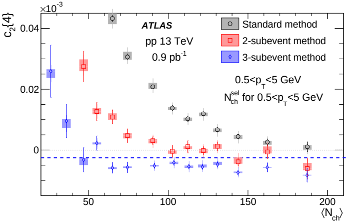
$$\langle\langle 2\rangle\rangle^2_{3sub} = \langle\langle \cos n(\varphi_1 - \varphi_4)\rangle\rangle\langle\langle \cos n(\varphi_2 - \varphi_3)\rangle\rangle$$

$$c_n\{4\}_{3sub} = \langle\langle 4\rangle\rangle_{3sub} - 2\cdot\langle\langle 2\rangle\rangle^2_{3sub}$$



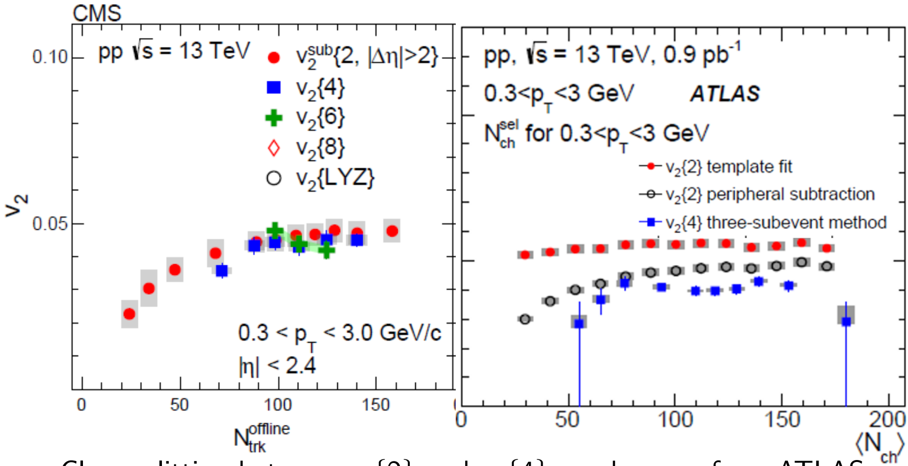

- 3 subevent cumulant can further suppress the non-flow effects.




M. Aaboud *et al.* [ATLAS Collaboration], Phys. Rev. C **97**, no. 2, 024904 (2018).

# Comparision between ATLAS and CMS results




- Clear splitting between $v_2\{2\}$ and $v_2\{4\}$ can be seen from ATLAS results.

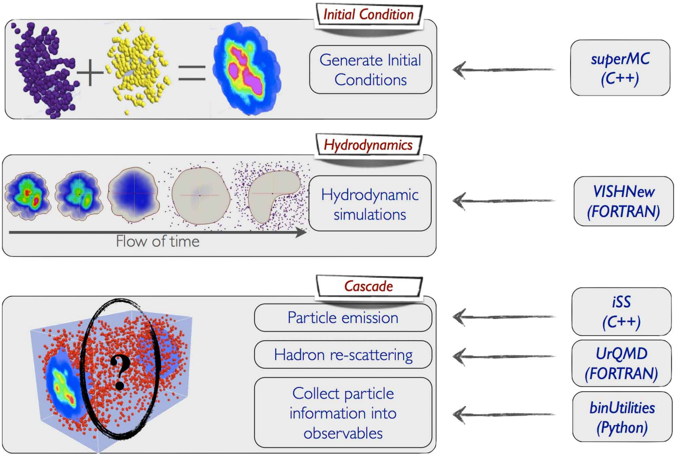
The ATLAS collaboration [ATLAS Collaboration], ATLAS-CONF-2017-002.
V. Khachatryan *et al.* [CMS Collaboration], Phys. Lett. B **765**, 193 (2017).

# Hydrodynamic simulations in p+p Collisions at 13 TeV

Collaborators: **Huichao Song**, Haojie Xu, You Zhou and Weitian Deng
Based on : Phys. Lett. B **780**, 495 (2018).

# iEBE-VISHNU hybird model

- Hydrodynamics simualtions:




C. Shen, Z. Qiu, H. Song, J. Bernhard, S. Bass and U. Heinz. Comput. Phys. Commun. **199**, 61 (2016)

# VISHNU hybrid model

- In hydrodynamics part, VISHNU solves $T^{\mu\nu}$, $\pi^{\mu\nu}$ and $\Pi$:

$$\partial_\mu T^{\mu\nu}(x) = 0, \qquad T^{\mu\nu} = eu^\mu u^\nu - (p+\Pi)\Delta^{\mu\nu} + \pi^{\mu\nu},$$
$$\dot{\Pi} = -\frac{1}{\tau_\Pi}\left[\Pi + \zeta\theta + \Pi\zeta T\partial_\mu\left(\frac{\tau_\Pi u^\mu}{2\zeta T}\right)\right], \tag{1}$$
$$\Delta^{\mu\alpha}\Delta^{\nu\beta}\dot{\pi}_{\alpha\beta} = -\frac{1}{\tau_\pi}\left[\pi^{\mu\nu} - 2\eta\nabla^{\langle\mu}u^{\nu\rangle} + \pi^{\mu\nu}\eta T\partial_\alpha\left(\frac{\tau_\pi u^\alpha}{2\eta T}\right)\right],$$

- Switch from hydrodynamics to hadron cascade (Cooper-Frye formula):

$$E\frac{d^3N_i}{d^3p}(x) = \frac{g_i}{(2\pi)^3} p\cdot d^3\sigma(x)\, f_i(x,p) \tag{2}$$

- Hadron cascade simulated by UrQMD by:

$$\frac{df_i(x,p)}{dt} = C_i(x,p) \tag{3}$$

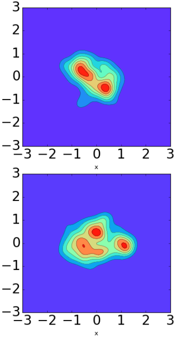

H. Song, S. A. Bass and U. Heinz, PRC **83**, 024912 (2011).

C. Shen, Z. Qiu, H. Song, J. Bernhard, S. Bass and U. Heinz, Comput. Phys. Commun. **199**, 61 (2016)

# HIJING initial condition

- In HIJING initial model, the produced jets pairs and excited nucleus are treated as independent strings, and these strings break into partons and quickly form hot spots for succeeding hydrodynamics.
- The center positions of strings $(x_c, y_c)$ are sampled by Saxon-Woods distribution, and positions of partons within the strings are sampled by, $\exp\left(-\frac{(x-x_c)^2+(y-y_c)^2}{2\sigma_R^2}\right)$
- HIJING constructs energy density by energy decompositions of individual partons via a Gaussian smearing:

$$\epsilon = K \sum_i \frac{E_i^*}{2\pi\sigma^2\tau_0\Delta\eta_s} \exp\left(-\frac{(x-x_i)^2+(y-y_i)^2}{2\sigma^2}\right),$$



# Set-up

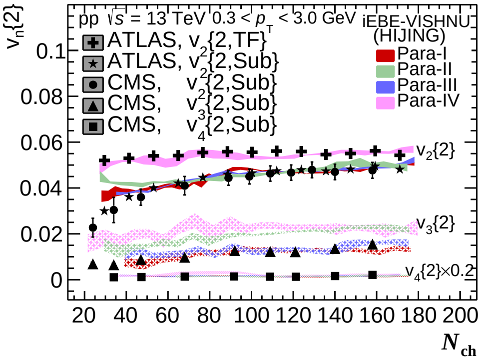
- iEBE-VISHNU + HIJING
- No initial flow, No bulk viscosity.

Table: Four sets parameters in iEBE-VISHNU + HIJING simulation of the *pp* 13 TeV.

| | $\sigma_R$ | $\sigma_0$ | $\tau_0$ | $\eta/s$ | $T_{\mathrm{sw}}$(MeV) |
|---|---|---|---|---|---|
| Para-I | 0.2 | 0.7 | 0.6 | 0.01 | 147 |
| Para-II | 0.8 | 0.4 | 0.4 | 0.08 | 148 |
| Para-III | 0.4 | 0.2 | 0.2 | 0.24 | 148 |
| Para-IV | 0.6 | 0.4 | 0.4 | 0.05 | 148 |

W. Zhao, Y. Zhou, H. Xu, W. Deng and H. Song, Phys. Lett. B **780**, 495 (2018)

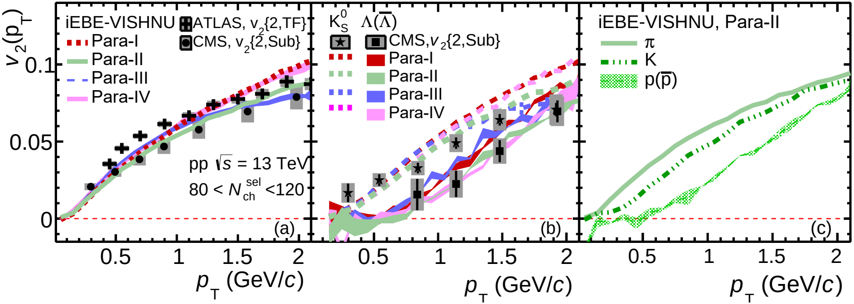
# 2-particle correlation




- In general, iEBE-VISHNU + HIJING can describe the $v_2\{2\}$, $v_3\{2\}$ and $v_4\{2\}$,from ATLAS and CMS.
- iEBE-VISHNU + HIJING fail to fit the $v_2\{2\}$ data with "peripheral subtraction" in low multiplicity.

W. Zhao, Y. Zhou, H. Xu, W. Deng and H. Song, Phys. Lett. B **780**, 495 (2018)

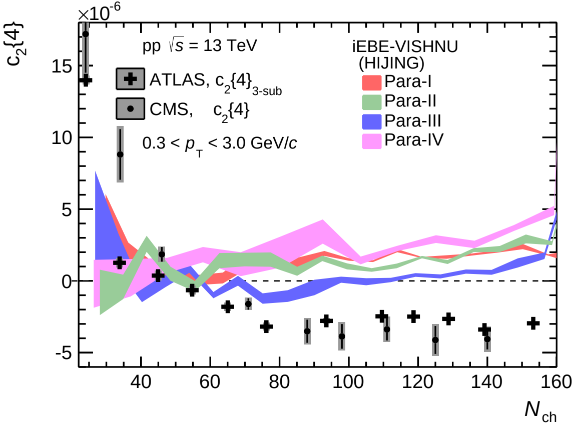
# Differential elliptic flow




- iEBE-VISHNU + HIJING can describe the $v_2(p_T)$ from ATLAS and CMS well.
- Hydrodynamics can reproduce mass ordering of experimental data.

W. Zhao, Y. Zhou, H. Xu, W. Deng and H. Song, Phys. Lett. B **780**, 495 (2018)

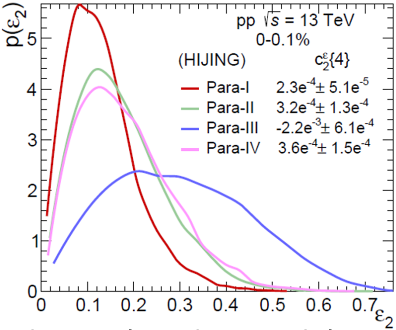
# 4-particle correlation by hydrodynamic simulations in p-p

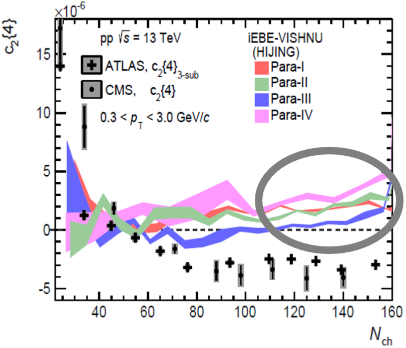



- iEBE-VISHNU + HIJING cann't get the negative $c_2\{4\}$.

W. Zhao, Y. Zhou, H. Xu, W. Deng and H. Song, Phys. Lett. B **780**, 495 (2018)

# $c_2^v\{4\}$ and $\varepsilon_2$ distributions

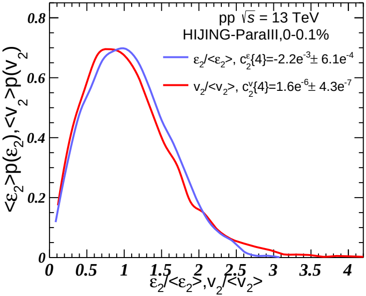


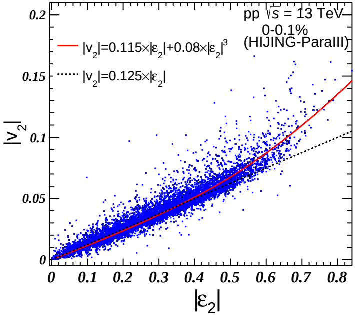




- The initial condition with large $\langle\varepsilon_2\rangle$ combined with the large fluctuation, $\sigma_\varepsilon$.
- For positive initial $c_2^\varepsilon\{4\}$ always get positive final $c_2^v\{4\}$.
- For Para-III with small negative initial $c_2^\varepsilon\{4\}$, non-linear response leading to the positive final $c_2^v\{4\}$.

W. Zhao, Y. Zhou, H. Xu, W. Deng and H. Song, Phys. Lett. B **780**, 495 (2018)

# from $c_2^{\varepsilon}\{4\}$ to $c_2^{v}\{4\}$ in p + p system

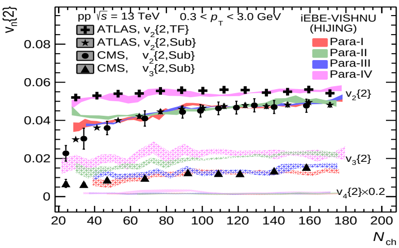


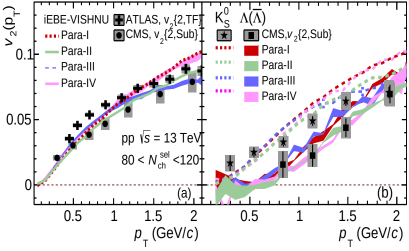



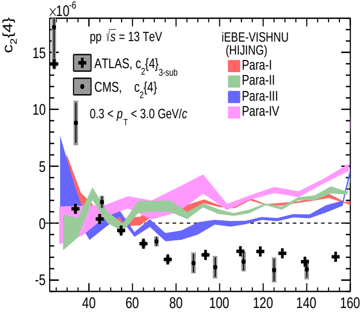

Cubic response: $|v_2| = 0.115|\varepsilon_2| + 0.080|\varepsilon_2|^3$

- Cubic response is important in large $\varepsilon_2$.
- Cubic response increases the elliptic flow fluctuations in proton + proton systems, leading some deviations between $p(v_2/\langle v_2\rangle)$ and $p(\varepsilon_2/\langle \varepsilon_2\rangle)$ that reverse the sign of $c_2\{4\}$.

# iEBE-VISHNU +HIJING simulation of p-p



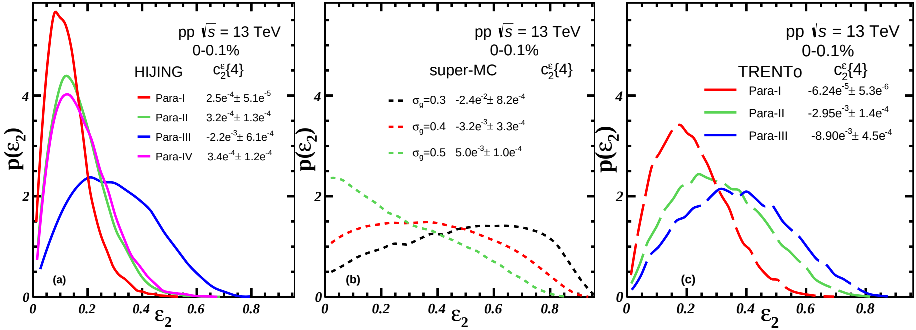




Zhao, Zhou, Xu, Deng, Song, Phys. Lett. B 780,

- Using iEBE-VISHNU + HIJING with four forms of QGP transport coefficients, we could well describe the 2-particle correlaions, $v_2(p_T)$ and mass ordering in p-p system.
- However we can't describe the 4-particle cumulants within the framework of HIJING initial condition.

# Calculate $c_2\{4\}$ in other initial models

# p($\varepsilon_2$) of HIJING , super-MC and TRENTo

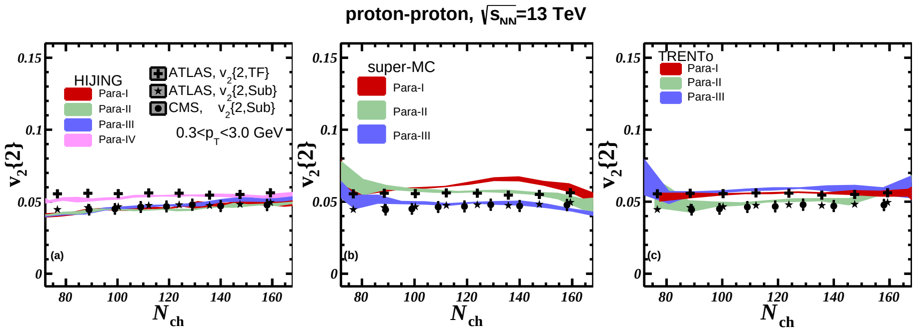



- The three typical p($\varepsilon_2$) of HIJING , super-MC and TRENTo initial models .

HIJING: Zhao, Zhou, Xu, Deng, Song, Phys. Lett. B **780**, 495 (2018)
super-MC: Welsh, Singer, Heinz, Phys. Rev. C **94**, no. 2, 024919 (2016)

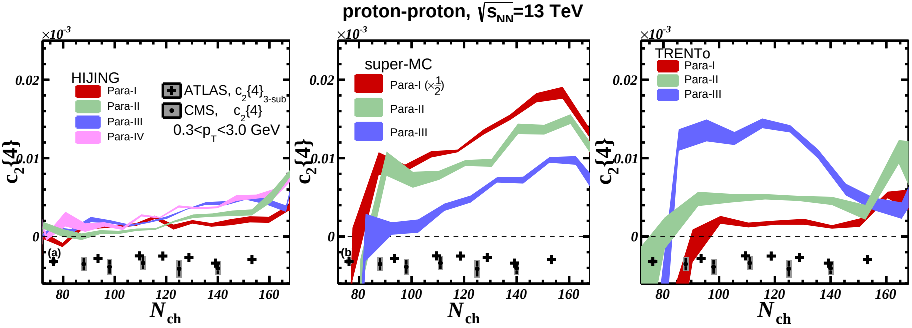
TRENTo: J. S. Moreland, J. E. Bernhard and S. A. Bass, arXiv:1808.02106 [nucl-th].

# $v_2\{2\}$ calculated by HIJING , super-MC and TRENTo




- With properly turned parameters in initial condition as well as in the VISHNU,iEBE-VISHNU + HIJING , super-MC and TRENTo can well describe the $v_2\{2\}$ at high multiplicity in p-p system.

# $c_2\{4\}$ calculated by HIJING , super-MC and TRENTo




- None parameter sets of HIJING , super-MC and TRENTo initial conditions can get the negative $c_2\{4\}$.

# Short summary

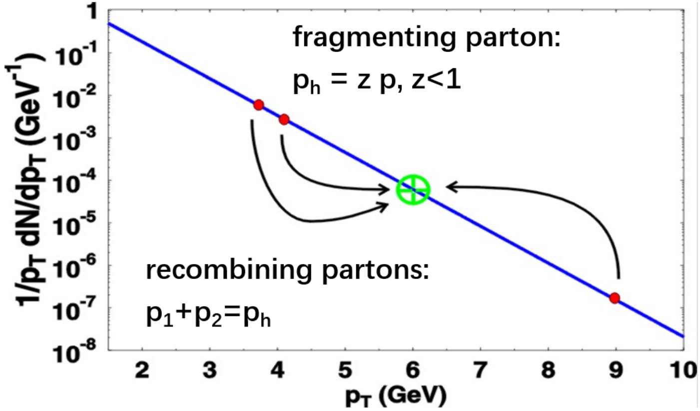
- For Pb + Pb system, hydrodynamics does a great job in describing hydrodynamic behaviors of Exp. data.
- For p + Pb system, hydrodynamics semi-quantitatively reproduce these Exp. data of 2- and 4- particle correlations, $v_2$ mass ordering .
- `iEBE-VISHNU` + `HIJING` can well describe the $v_2\{2\}$, $v_2(p_T)$ for all charge hadron and mass ordering. However fail to reproduce the negative $c_2\{4\}$ in p+p collisions. The description of negative $c_2\{4\}$ requires the further investigations on initial model as well as QGP evolutions in p-p system.

# $v_2(p_T)$ and spectra at intermedium $p_T$ in p+Pb 5.02 TeV

Collaborators: Huichao Song, Guangyou Qin and Che-Ming Ko

# Coalescence and fragmentation at intermedium $p_T$

- Fragmentation: Leading parton with $p_T$ leads to hadrons of $p_h = zp_T$ with $z < 1$.
- Coalescence: partons are already there, $p_h = np_T$, n=2,3. Quark need close in phase space. Partonic hydro behavior shifted to higher $pT$.



# Coalescence model

Mesons and baryons' momentum distributions by recombining of quarks:

$$
\begin{aligned}
\frac{dN_M}{d^3\mathbf{P}_M} &= g_M \int d^3\mathbf{x}_1 d^3\mathbf{p}_1 d^3\mathbf{x}_2 d^3\mathbf{p}_2 f_q(\mathbf{x}_1, \mathbf{p}_1) f_{\bar{q}}(\mathbf{x}_2, \mathbf{p}_2) \\
&\times W_M(\mathbf{y}, \mathbf{k}) \delta^{(3)}(\mathbf{P}_M - \mathbf{p}_1 - \mathbf{p}_2)\,, \qquad (4)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{dN_B}{d^3\mathbf{P}_B} &= g_B \int d^3\mathbf{x}_1 d^3\mathbf{p}_1 d^3\mathbf{x}_2 d^3\mathbf{p}_2 d^3\mathbf{x}_3 d^3\mathbf{p}_3 f_{q_1}(\mathbf{x}_1, \mathbf{p}_1) \\
&\times f_{q_2}(\mathbf{x}_2, \mathbf{p}_2) f_{q_3}(\mathbf{x}_3, \mathbf{p}_3) W_B(\mathbf{y}_1, \mathbf{k}_1; \mathbf{y}_2, \mathbf{k}_2) \\
&\times \delta^{(3)}(\mathbf{P}_B - \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3)\,, \qquad (5)
\end{aligned}
$$

$f_{q,\bar{q}}(\mathbf{x}_1, \mathbf{p}_1)$ is the phase-space distribution of (anti)quarks, normalized as $\int d^3\mathbf{x} d^3\mathbf{p} f_{q,\bar{q}}(\mathbf{x}, \mathbf{p}) = N_{q,\bar{q}}$,

K. C. Han, R. J. Fries and C. M. Ko, Phys. Rev. C **93**, no. 4, 045207 (2016).

# NCQ scaling

- quark's elliptic flow: $f_a(\mathbf{p}_T) = \bar{f}_a(p_T)\left(1 + 2v_{2,q}(p_T)\cos 2\phi\right)$
- the meson's elliptic flow: $v_2^M(p_T) = \frac{2v_{2,q}(p_T/2)}{1+2v_{2,q}^2(p_T/2)} \sim 2v_{2,q}(p_T/2)$
- the baryon's elliptic flow: $v_2^B(p_T) = \frac{3v_{2,q}(p_T/3)}{1+6v_{2,q}^2(p_T/3)} \sim 3v_{2,q}(p_T/3)$

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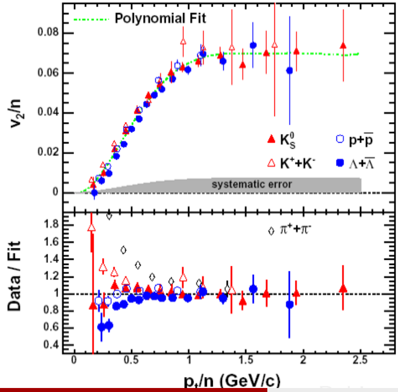
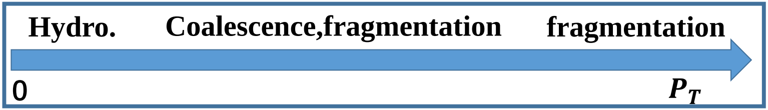
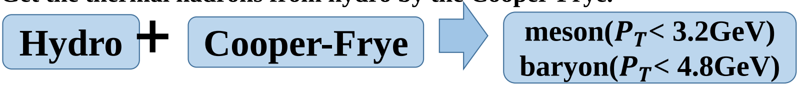



- NCQ scaling is a very clean signal of deconfinement of quark and gluons in system.

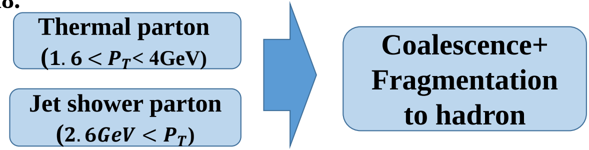
# framework of hydro+ jet




1. Get the thermal hadrons from hydro by the Cooper-Frye.

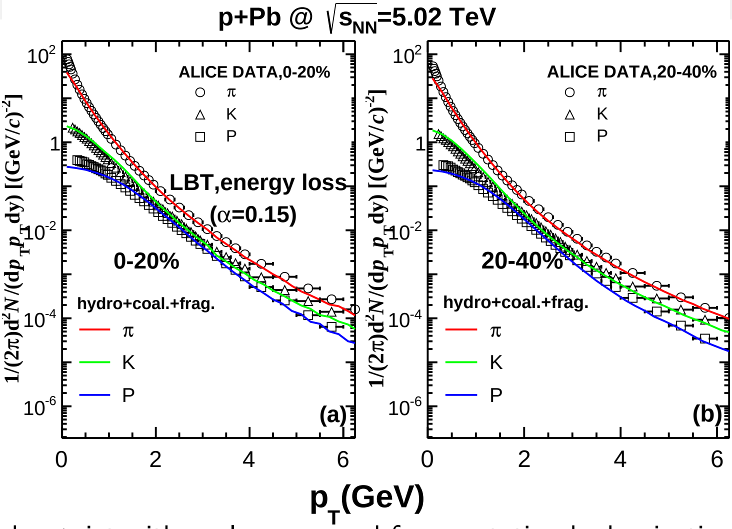



2. Get the thermal parton with $1.6 < P_T$ <4 GeV from hydro and the hard parton from Pythia8, then suffered with energy loss by LBT with $\alpha$=0.15. Coalescence the quarks, the remnant hard quarks subjected to fragmentation in Pythia8.



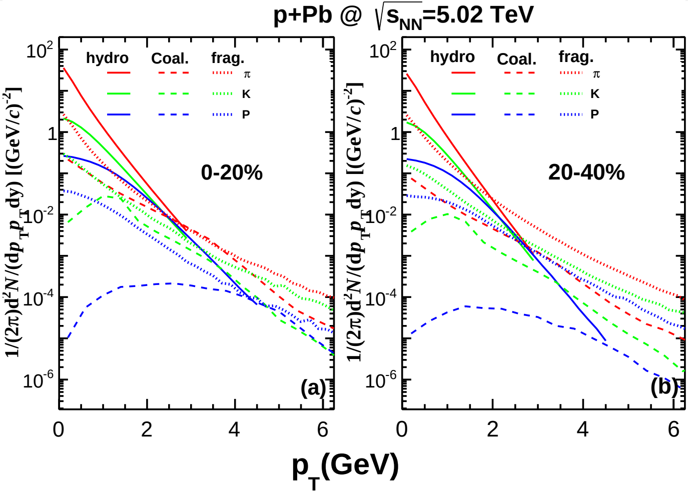

3. All hadrons feed to the UrQMD model.

# spectra



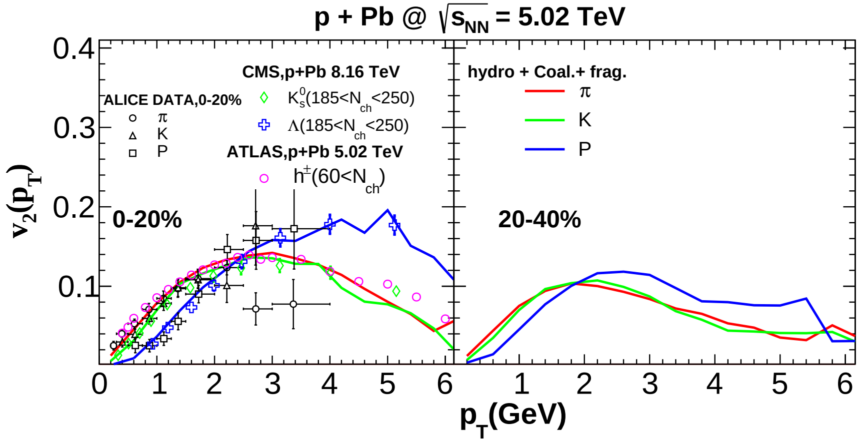

- Hydro + jet with coalescence and fragmentation hadronization mechanism, our model can well describe spectra at low and inter-medium $p_T$.

# hydro, coal. and frag.'s contributions to spectra



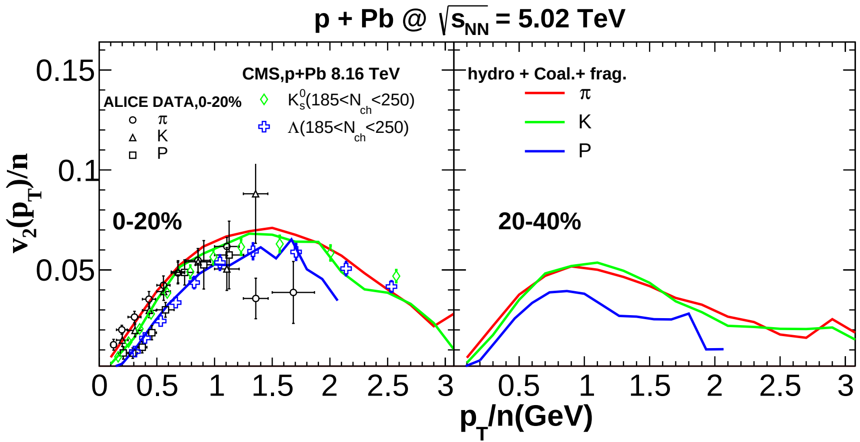

- Coalescence is important at high multiplicity at inter-medium $p_T$. It has negligible contribution at low multiplicity.

# $v_2(p_T)$




- Hydro + jet with coalescence and fragmentation hadronization mechanism, our model can well describe $v_2(p_T)$ at low and inter-medium $p_T$.

# NCQ scaling of $v_2(p_T)$




- We can get the approximately NCQ scaling of $v_2(p_T)$ at high multiplicity of p+Pb system.

# Summary

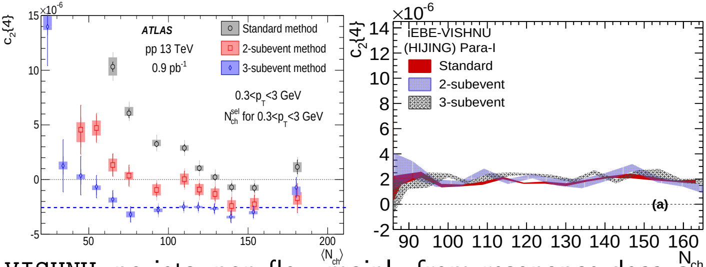
- For p-p system, hydrodynamics can well describe the $v_2\{2\}$, $v_2(p_T)$ for all charge hadron and mass ordering. However fail to reproduce the negative $c_2\{4\}$.
- NCQ scaling is a very clean signal of deconfinement of quark and gluons in heavy-ion collisions.
- Within the framework of hydro+ jet with coalescence and fragmentation hadronization mechanism, we can reproduce the spectra as well as the $v_2(p_T)$, and get the approximately NCQ scaling at high multiplicity of p+Pb system.

# Thanks

# Back up

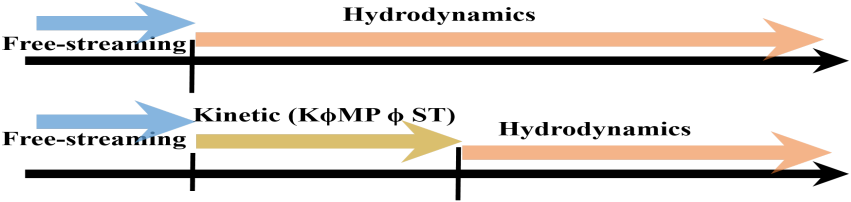
# More details on $c_2\{4\}$ calculations

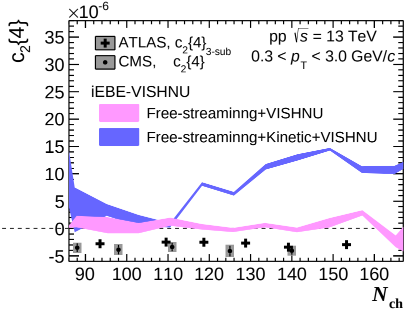
- minimize multiplicity fluctuation:
  - Cut the multiplicity class with the number of all charged hadrons $N_{ch}^{Sel}$ within $0.3 < p_T < 3.0$ GeV, $|\eta| < 2.4$
  - Calculate $c_2\{4\}$ with the same $N_{ch}^{Sel}$ to minimize multiplicity fluctuation.
  - Combined $c_2\{4\}$ to several $N_{ch}^{Sel}$.
  - Map $N_{ch}^{Sel}$ to the event activity measure $N_{ch}$ with $p_T > 0.4$ GeV, $|\eta| < 2.4$.
- Check standard method, 2-, 3-subevent in simulations




In iEBE-VISHNU, no jets, non-flow mainly from resonance decays, standard method gives same results as 2- and 3- subevent methods.

# Including pre-equilibrium effects

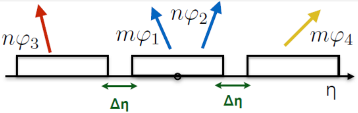



- Including pre-equilibrium effects, free-streaming + hydrodynamics or free-streaming + kinetic theory + hydrodynamics, one still can not get negative $c_2\{4\}$.

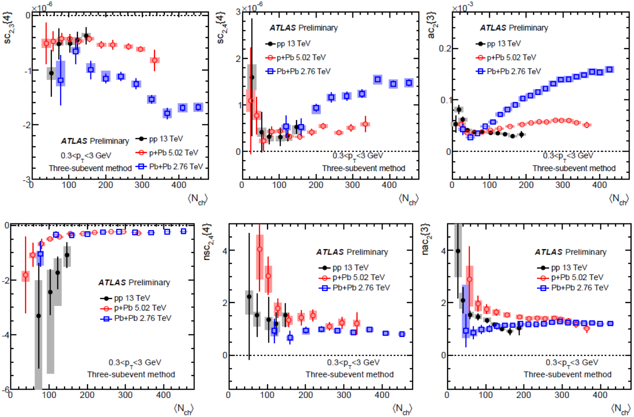
Free-streaming: J. S. Moreland, J. E. Bernhard and S. A. Bass, arXiv:1808.02106 [nucl-th].
Kinetic theory: A. Kurkela, A. Mazeliauskas, J. F. Paquet, S. Schlichting and D. Teaney, arXiv:1805.00961 [hep-ph].

# Symmetric cumulant by 3-subevent (ATLAS)




$$\langle\langle 4\rangle\rangle_{m,n,-n,-m} = \langle\langle \cos\left(m\varphi_1 + n\varphi_2 - n\varphi_3 - m\varphi_4\right)\rangle\rangle$$

$$\langle\langle 2\rangle\rangle_{n,-n}\langle\langle 2\rangle\rangle_{m,-m} = \langle\langle \cos n(\varphi_2 - \varphi_3)\rangle\rangle\langle\langle \cos m(\varphi_1 - \varphi_4)\rangle\rangle$$




- Symmetric cumulants are consistent among all three systems.

The ATLAS collaboration [ATLAS Collaboration], ATLAS-CONF-2018-012.

# Other initial models

- In `super-MC` the entropy density is:

$$s(\mathbf{r}) = \frac{\kappa_s}{\tau_0} \sum_{k=1}^{3} \gamma_k^{(i)} \frac{e^{-(\mathbf{r}-\mathbf{r}_k^{(i)})^2/(2\sigma_g^2)}}{2\pi\sigma_g^2}, \quad (6)$$

where $\gamma_k$ is sampled from $\Gamma$ distribution, $\mathbf{r}_k^{(i)}$ is quark's positions, $\sigma_g$ is width of gluons.

- In `TRENTo` the initial entropy density is:

$$s = s_0 \left( \frac{\tilde{T}_A^p + \tilde{T}_B^p}{2} \right)^{1/p}, \quad (7)$$

where $\tilde{T}(x,y) \equiv \int dz \, \frac{1}{n_c} \sum_{i=1}^{n_c} \gamma_i \, \rho_c \left(\mathbf{x} - \mathbf{x_i} \pm \mathbf{b}/2\right)$, $n_c$ is the number of the independent constituents and $\rho_c(\mathbf{x}) = \frac{1}{(2\pi v^2)^{3/2}} \exp\left(-\frac{\mathbf{x}^2}{2v^2}\right)$,

HIJING: Zhao, Zhou, Xu, Deng, Song, Phys. Lett. B **780**, 495 (2018)
super-MC: Welsh, Singer, Heinz, Phys. Rev. C **94**, no. 2, 024919 (2016)

TRENTo: J. S. Moreland, J. E. Bernhard and S. A. Bass, arXiv:1808.02106 [nucl-th].

# Wigner function

To guarantee positive value of Wigner function for stable Monto Carlo sampling, the Wigner function replaced by the overlap of hadron Wigner function $W_M$ with parton's Wigner function, $W_{q,\bar{q}}$:

$$\begin{aligned} \overline{W}_M(\mathbf{y},\mathbf{k}) &= \int d^3\mathbf{x}_1' d^3\mathbf{k}_1' d^3\mathbf{x}_2' d^3\mathbf{k}_2' \\ &\times W_q(\mathbf{x}_1',\mathbf{k}_1') W_{\bar{q}}(\mathbf{x}_2',\mathbf{k}_2') W_M(\mathbf{y}',\mathbf{k}'). \end{aligned} \tag{8}$$

Using harmonic oscillator for wave functions of excited stated of hadrons,

$$\phi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\xi^2/2}, \tag{9}$$

$\xi = \sqrt{\frac{m\omega}{\hbar}}x$, $H_n(\xi)$ are Hermite polynomials, $\omega$ is the oscillator frequency.
K. C. Han, R. J. Fries and C. M. Ko, Phys. Rev. C **93**, no. 4, 045207 (2016).

The quark wave function to be Gaussian wave packet, the wigner function of a meson in $n$-th excited state is

$$\overline{W}_{M,n}(\mathbf{y},\mathbf{k}) = \frac{v^n}{n!}e^{-v}. \tag{10}$$

with

$$v = \frac{1}{2}\left(\frac{\mathbf{y}^2}{\sigma_M^2} + \mathbf{k}^2\sigma_M^2\right). \tag{11}$$

Similarly, the Gaussian smeared Wigner function for baryon is:

$$\overline{W}_{B,n_1,n_2}(\mathbf{y}_1,\mathbf{k}_1;\mathbf{y}_2,\mathbf{k}_2) = \frac{v_1^{n_1}}{n_1!}e^{-v_1} \cdot \frac{v_2^{n_2}}{n_2!}e^{-v_2}, \tag{12}$$

with

$$v_i = \frac{1}{2}\left(\frac{\mathbf{y}_i^2}{\sigma_{Bi}^2} + \mathbf{k}_i^2\sigma_{B_i^2}\right), \quad i = 1, 2. \tag{13}$$

K. C. Han, R. J. Fries and C. M. Ko, Phys. Rev. C **93**, no. 4, 045207 (2016).