The impact of unstable K^* on the decays of $\eta(1405/1475)$ via triangle diagrams

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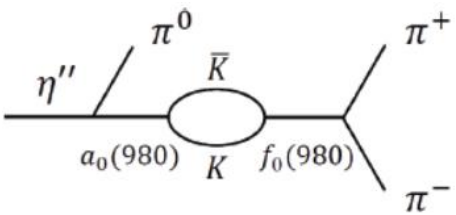
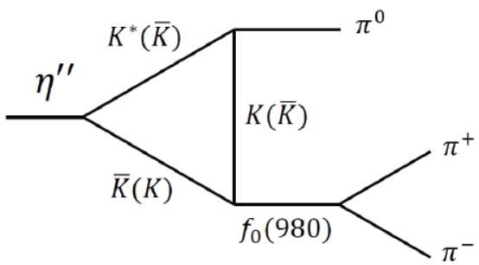
Motivation

Large isospin violation in $J/\Psi \to \gamma\eta(1405/1475) \to \gamma f_0(980)\pi \to \gamma\pi^+\pi^-\pi^0$ (we denote $\eta(1405/1475)$ *by* η'')[1][2][3]:

$$\frac{BR(J/\Psi \to \gamma\eta'' \to \gamma f_0\pi \to \gamma\pi^+\pi^-\pi^0)}{BR(J/\Psi \to \gamma\eta'' \to \gamma a_0\pi \to \gamma\eta\pi^0\pi^0)} = (17.9 \pm 4.2)\%$$



$\sim 1\%$



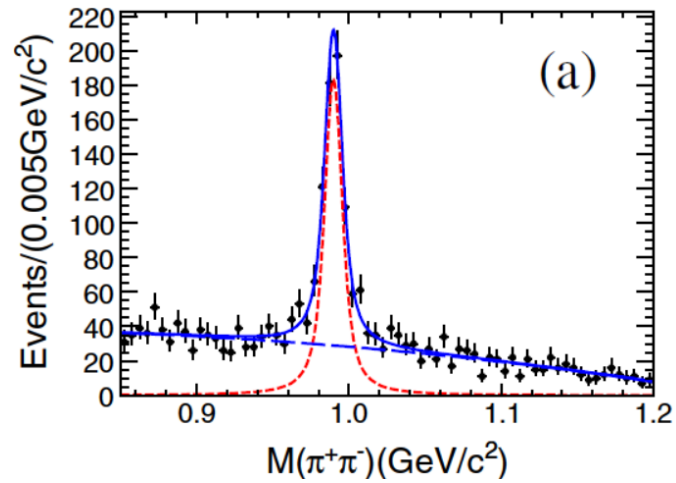
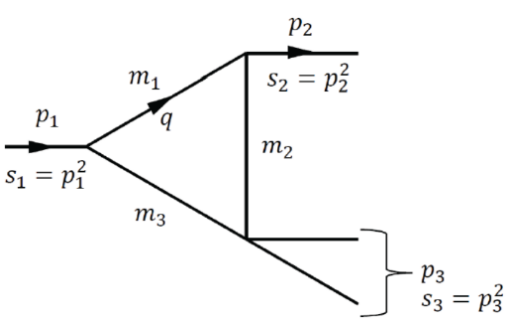


Fig.1. $\pi^+\pi^-$ spectrum from Ref[1].

Triangle singularity (TS) [4,5,6]:



Landau equation

$$I = \int \frac{d^4q}{(2\pi)^4} \frac{Numerator}{(q^2 - m_1^2)((q-p_2)^2 - m_2^2)((q-p_2-p_3)^2 - m_3^2)}$$

For $\sqrt{s_1} \in (m_1 + m_3, \sqrt{s_{1c}})$, TS occurs when $s_3 = s_3^-$, $\sqrt{s_3^-} \in (m_2 + m_3, \sqrt{s_{3c}})$.

For $\sqrt{s_3} \in (m_2 + m_3, \sqrt{s_{3c}})$, TS occurs when $s_1 = s_1^-$, $\sqrt{s_1^-} \in (m_1 + m_3, \sqrt{s_{1c}})$.

The kinematics allows the internal particles to be simultaneously on-shell.

[1]M. Ablikim et al. [BESIII Collaboration] PRL 108, 182001 (2012); [2] Jia-Jun Wu, Xiao-Hai Liu, Qiang Zhao, Bing-Song Zou, PRL 108, 081803 (2012) ;

[3] Xiao-Gang Wu, Xiao-Hai Liu, Qiang Zhao, Bing-Song Zou, PR D87, 014023 (2013); [4] L. D. Landau, NP 13, 181 (1959);

[5] G. Bonnevay, I. J. R. Aitchison and J. S. Dowker, Nuovo Cim. 21, 3569 (1961); [6] R. E. Cutkosky, J. Math. Phys. 1, 429 (1960)

The effect of non zero Γ_{K^*}

Ref[7] shows the suppression from Γ_{K^*} is unexpectedly large

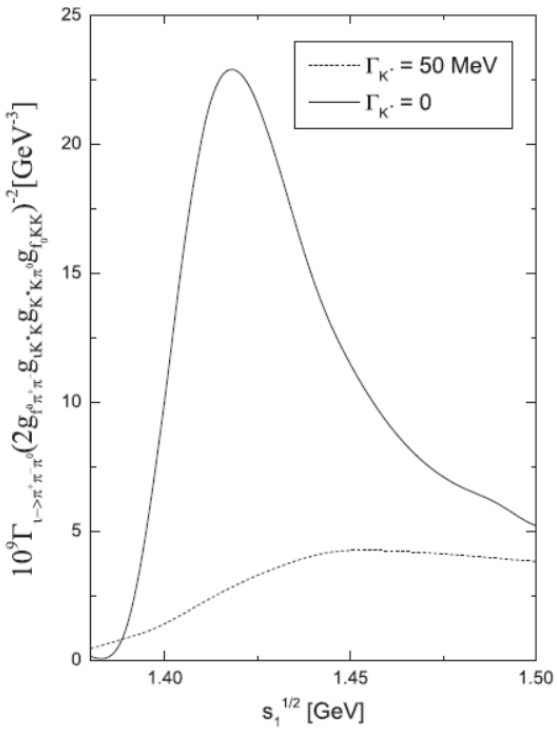
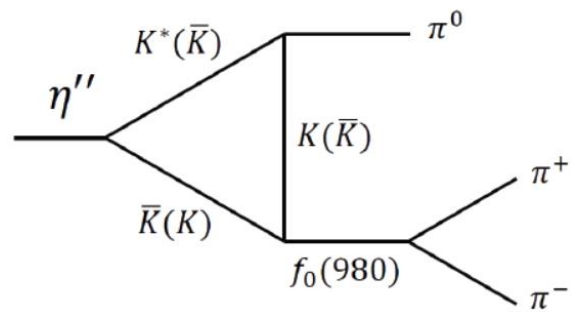


Fig.2. The decay width calculated with $\Gamma_{K^*} = 50$MeV (the lower one)and $\Gamma_{K^*} = 0$ MeV (the higher one) from Ref[7].



Naïve estimation $\frac{\Gamma_{K^*}}{m_{K^*}} \simeq \frac{1}{18}$

$$I = \int \frac{d^4q}{(2\pi)^4} \frac{(2p_1 - q)_\mu \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right)(q - 2p_2)_\nu}{(q^2 - m_1^2 + im_1\Gamma_1)((q-p_2)^2 - m_2^2)((q - p_2 - p_3)^2 - m_3^2)}$$

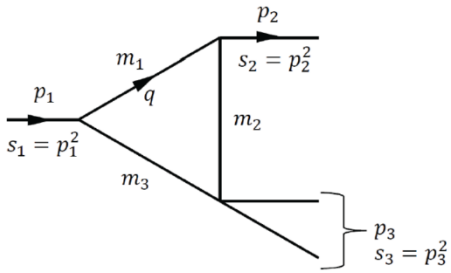
$$M_{\eta''\to f_0\pi^0\to\pi^+\pi^-\pi^0} \sim \left(I^{(n)} - I^{(c)}\right) \approx \left(\Im I^{(n)} - \Im I^{(c)}\right)$$

1. How does the non-zero Γ_{K^*} affect the absorptive part of I ?
2. Taken into account Γ_{K^*}, does the TS still suffice to explain the large isospin violation?

[7] N.N. Achasov, A.A. Kozhevnikov and G.N. Shestakov, PR D92, 036003 (2015)

Analytical properties of triangle loop integral

Decompose the amplitude



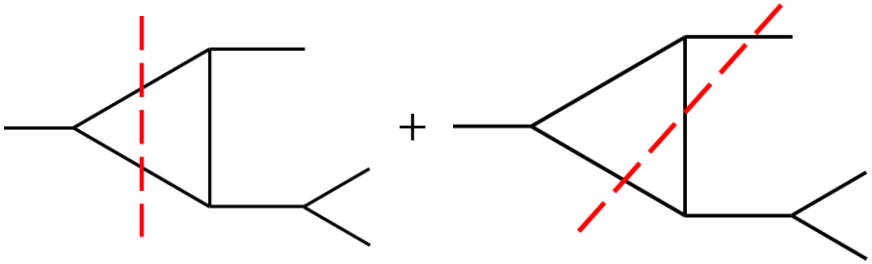
$$
\begin{aligned}
I \;=\;& -i(s_1 - m_1^2 + im_1\Gamma_1 + s_2 - 2s_3 + 2m_K^2 - \frac{(s_1 - m_K^2)(s_2 - m_K^2)}{m_1^2 - im_1\Gamma_1}) \int \frac{d^4q}{(2\pi)^4} \frac{1}{D_1 D_2 D_3} \\
-\;& i\frac{(s_1 - m_K^2)(s_2 - m_K^2)}{m_1^2 - im_1\Gamma_1}) \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 D_2 D_3} - i(1 + \frac{s_1 - m_K^2}{m_1^2 - im_1\Gamma_1}) \int \frac{d^4q}{(2\pi)^4} \frac{1}{D_1 D_3} \\
-\;& i(1 + \frac{s_2 - m_K^2}{m_1^2 - im_1\Gamma_1}) \int \frac{d^4q}{(2\pi)^4} \frac{1}{D_1 D_2} + i \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 D_1} - i\frac{m_K^2 - s_1}{m_1^2 - im_1\Gamma_1} \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 D_3} - i\frac{m_K^2 - s_2}{m_1^2 - im_1\Gamma_1} \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 D_2} \\
+\;& i \int \frac{d^4q}{(2\pi)^4} \frac{1}{D_2 D_3}
\end{aligned}
$$

$$
M(\Gamma) \equiv \int \frac{d^4q}{(2\pi)^4} \frac{1}{(q^2 - m_1^2 + im_1\Gamma_1)((q - p_2)^2 - m_2^2)((q - p_2 - p_3)^2 - m_3^2)}
$$

If $\Gamma_1 \neq 0$ we seek for an explicit expression of $\Im M(\Gamma_1)$, which goes to $\Im M_{cut}$ as $\Gamma_1 \to 0$

Cutkosky rule

$\Im M_{cut} =$



$$
\begin{aligned}
a_1 &= \frac{-s_2 + \frac{(s_1 + m^2 - m_k^2)(s_1 + s_3 - s_2)}{2s_1}}{2\sqrt{[\frac{(s_1 + m_k^2 - m^2)^2}{4s_1} - m_k^2][-s_2 + \frac{(s_1 + s_3 - s_2)^2}{4s_1}]}} \\
a_2 &= \frac{-m^2 + m_k^2 + s_2 + \frac{1}{2}(s_3 + s_2 - s_1)}{2\sqrt{[-s_2 + \frac{(s_3 + s_2 - s_1)^2}{4s_3}][-m_k^2 + \frac{s_3}{4}]}}, \\
p_\pi^{(s_1)} &\equiv \frac{\lambda[s_1, s_2, s_3]^{1/2}}{2\sqrt{s_1}}
\end{aligned}
$$

$$
\Im M(0) = \Im M_{cut} = \frac{1}{32\pi\sqrt{s_1} p_\pi^{(s_1)}} \ln \left| \frac{(a_1 + 1)(a_2 + 1)}{(a_1 - 1)(a_2 - 1)} \right|
$$

Ref[8] $$M = \frac{1}{16\pi^2}\int_0^1 dy \frac{1}{N_1(y)}[\ln u_1(y) - \ln u_1(y_0^{(1)})] - \frac{1}{16\pi^2}\int_0^1 dy \frac{1}{N_2(y)}[\ln u_2(y) - \ln u_2(y_0^{(2)})] + \frac{1}{16\pi^2}\int_0^1 dy \frac{1}{N_3(y)}[\ln u_3(y) - \ln u_3(y_0^{(3)})]$$

$$\begin{aligned} u_1(y) &\equiv by^2 + (c+e)y + a + d + f = s_3 y^2 + (m_2^2 - m_3^2 - s_3)y + m_3^2 \\ u_2(y) &\equiv (a+b+c)y^2 + (e+d)y + f = s_2 y^2 + (m_2^2 - m_1^2 - s_2)y + m_1^2 \\ u_3(y) &\equiv ay^2 + dy + f = s_1 y^2 + (m_3^2 - m_1^2 - s_1)y + m_1^2 \ , \end{aligned}$$

$y_0^{(i)}$: solution to $N_i(y) = 0$

$y_j^{(i)}(j = 1,2)$: solution to $u_i(y) = 0$

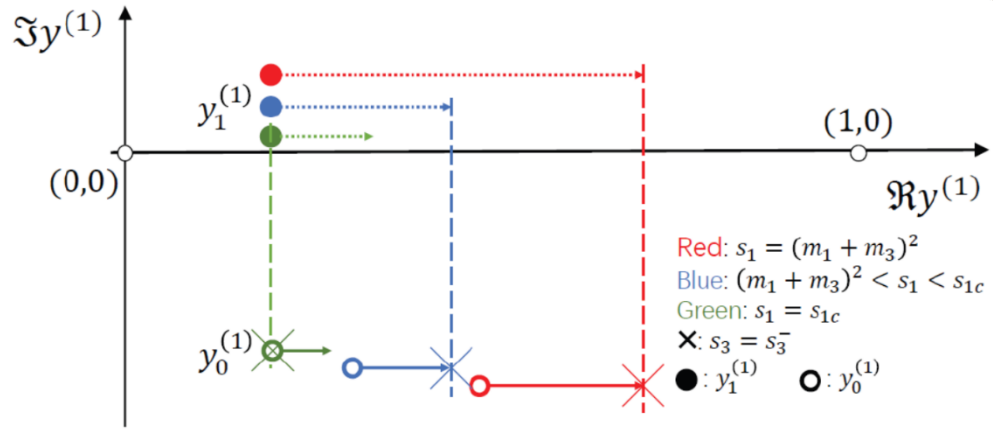
$\Rightarrow$ $\{y_0^{(1)}, y_1^{(1)}, y_2^{(1)}\}, \{y_0^{(2)}, y_1^{(2)}, y_2^{(2)}\}, \{y_0^{(2)}, y_1^{(2)}, y_2^{(2)}\}$

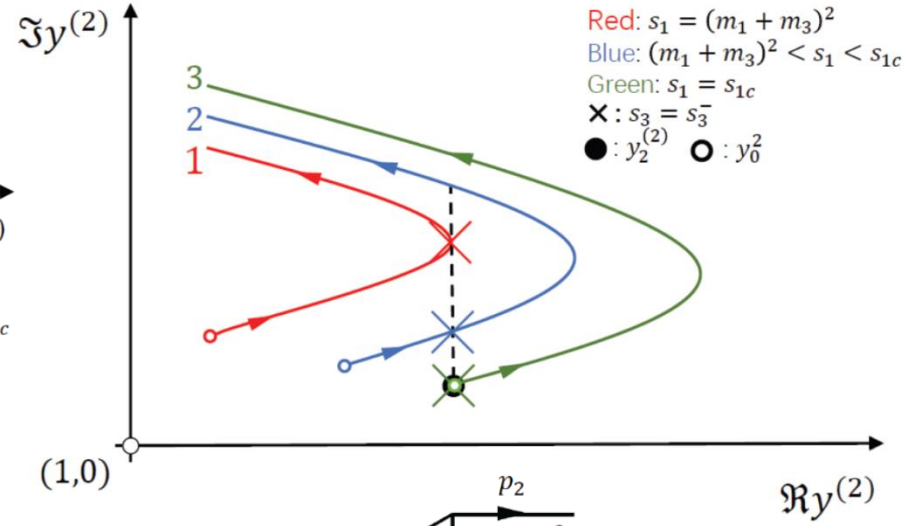
$$\boxed{\begin{aligned} M &= \frac{1}{16\pi^2}\frac{1}{c+2b\alpha}(S^{(1)} - S^{(2)} + S^{(3)}), \\ S^{(i)} &= \Sigma_{j=1}^2 R_j^{(i)} + \sigma^{(i)}, \\ R_j^{(i)} &= Sp(z_{j1}^{(i)}) - Sp(z_{j2}^{(i)}) + T_j^{(i)} \equiv W_j^{(i)} + T_j^{(i)}, \\ z_{1k}^{(i)} &= \frac{y_k^{(i)} - 1}{y_k^{(i)} - y_0^{(i)}}, \quad z_{2k}^{(i)} = \frac{y_k^{(i)}}{y_k^{(i)} - y_0^{(i)}}, \end{aligned}}$$

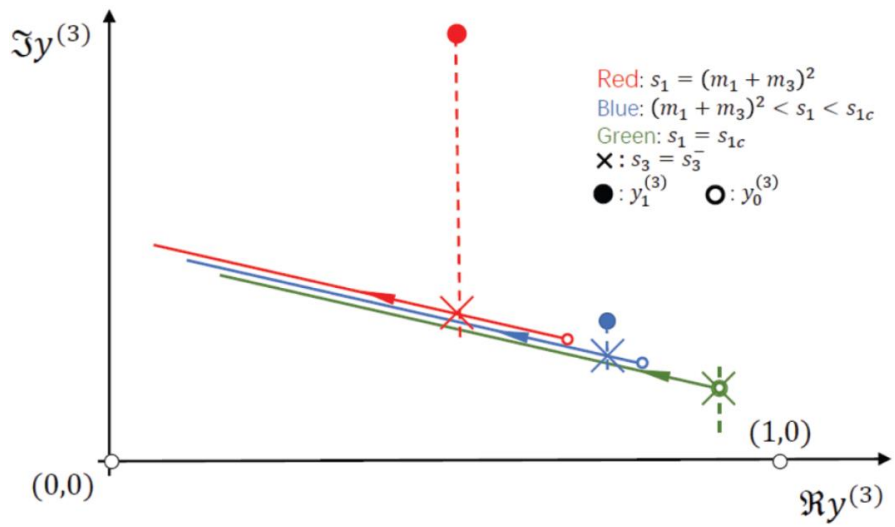
$$Sp(z) = -\int_0^1 \frac{\ln(1-zt)}{t} dt$$

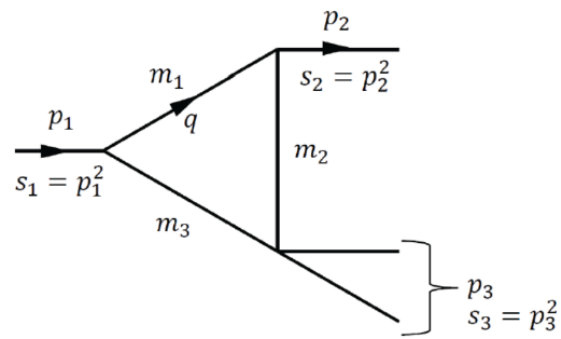
1. The condition for TS can be identified by the motion of $y_j^{(i)}$ in the complex plane
2. The locations of $y_j^{(i)}$ in the complex plane determine the absorptive part of M

[8] G. ’t Hooft and M. J. G. Veltman, NP B153, 365









$$M = \frac{1}{16\pi^2(c + 2b\alpha)}\left(S^{(1)} - S^{(2)} + S^{(3)}\right)$$

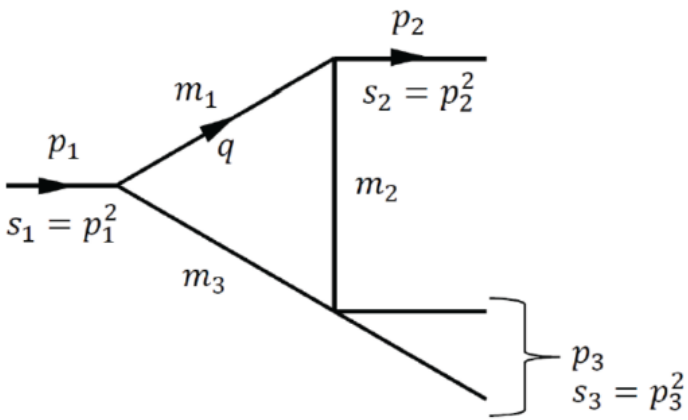
Fig.3. The motion of $y_j^{(i)}$ in the complex planes. Different colors represent different values of s_1. The lines with arrows are the trajectories as s_3 varies from the normal threshold to s_{3c}.

TS happens when

$$\left.\begin{array}{l} y_0^{(1)} \text{ meets } y_1^{(1)} \text{ at the cross} \\ y_0^{(2)} \text{ meets } y_2^{(2)} \text{ at the cross} \\ y_0^{(3)} \text{ meets } y_1^{(3)} \text{ at the cross} \end{array}\right\} \text{Simultaneously}$$

Extract the absorptive part

$$M = \frac{1}{16\pi^2}\frac{1}{c+2b\alpha}(S^{(1)} - S^{(2)} + S^{(3)}),$$
$$S^{(i)} = \Sigma_{j=1}^{2} R_j^{(i)} + \sigma^{(i)},$$
$$R_j^{(i)} = Sp(z_{j1}^{(i)}) - Sp(z_{j2}^{(i)}) + T_j^{(i)} \equiv W_j^{(i)} + T_j^{(i)},$$
$$z_{1k}^{(i)} = \frac{y_k^{(i)} - 1}{y_k^{(i)} - y_0^{(i)}}, \quad z_{2k}^{(i)} = \frac{y_k^{(i)}}{y_k^{(i)} - y_0^{(i)}},$$



Small-width approximation

$$\Im Sp(a \pm ib) = \pm\pi\Theta(a-1)\ln a \mp \int_0^1 \frac{dt}{t}\arctan\frac{bt}{at-1} \simeq \pm\pi\Theta(a-1)\ln a$$

Near-threshold approximation

The energy region where TS occurs is close to $\sqrt{s_3} = m_2 + m_3$

$$\Im M(\Gamma) = \frac{1}{16\pi^2\lambda[s_1,s_2,s_3]^{\frac{1}{2}}}(\Im W_{tot} + \Im T_{tot} + \Im\sigma_{tot})$$

$$\Im M_{cut} = \frac{1}{32\pi\sqrt{s_1}p_\pi^{(s_1)}}\ln\left|\frac{(a_1+1)(a_2+1)}{(a_1-1)(a_2-1)}\right|$$

1. We verified $\Im M(0) = \Im M_{cut}$

2. $\Im M(\Gamma) = \Im M_{cut}|_{m_1^2 \to m_1^2 - i\, m_1\Gamma} \equiv \Im M'_{cut}$

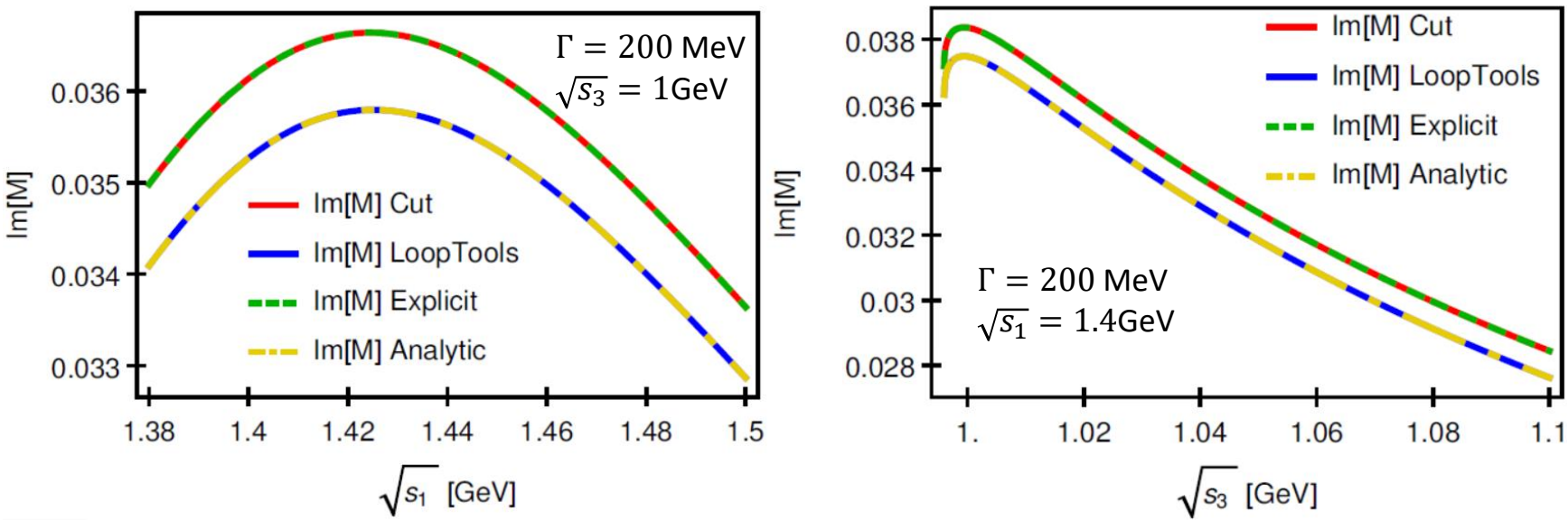


Fig.4. The absorptive part of M calculated using different formulas, with $\Gamma = 200$MeV. Left: $\sqrt{s_3}$ is fixed at 1GeV. Right: $\sqrt{s_1}$ is fixed at 1.4GeV.

- The analytic Γ dependence of $\Im M$ can be obtained by looking into $\Im M'_{cut}$, which is much simpler.

$$\Im M'_{cut} = \frac{1}{32\pi\sqrt{s_1}p_\pi^{(s_1)}} \ln\left|\frac{(a_1+1)(a_2+1)}{(a_1-1)(a_2-1)}\right|\Bigg|_{m_1^2 \to m_1^2 - i\, m_1\Gamma}$$

Expand $\Im M'_{cut}$ with respect to Γ and set $s_1 = s_1^-$

$$\max \Im M = \frac{1}{32\pi\sqrt{s_1}p_\pi^{(s_1)}} \ln\frac{\beta^2}{\Gamma^2}$$

$$\beta^2 = \frac{1}{\sqrt{s_1}m^2}\frac{32 s_1 (p_K^{(s_1)})^3 p_\pi^{(s_1)} p_\pi^{(s_3)} p_K^{(s_3)}}{E_K^{(s_1)}(2E_f^{(s_1)}E_K^{(s_1)} - s_3) - 2E_f^{(s_1)}(p_K^{(s_1)})^2}$$

$$p_\pi^{(s_1)} \equiv \frac{\lambda[s_1,s_2,s_3]^{1/2}}{2\sqrt{s_1}},\ p_\pi^{(s_3)} = \frac{\lambda[s_3,s_2,s_1]^{1/2}}{2\sqrt{s_3}},\ p_K^{(s_1)} \equiv \frac{\lambda[s_1,m^2,m_k^2]^{1/2}}{2\sqrt{s_1}},$$

$$p_K^{(s_3)} \equiv \sqrt{\frac{s_3}{4} - m_k^2},\ E_K^{(s_1)} \equiv \frac{s_1 + m_k^2 - m^2}{2\sqrt{s_1}},\ E_f^{(s_1)} \equiv \frac{s_1 + s_3 - s_2}{2\sqrt{s_1}},\ s_2 = m_\pi^2$$

Typically $\beta \ll m_1$, and $\frac{\Gamma}{\beta} \sim O(1)$

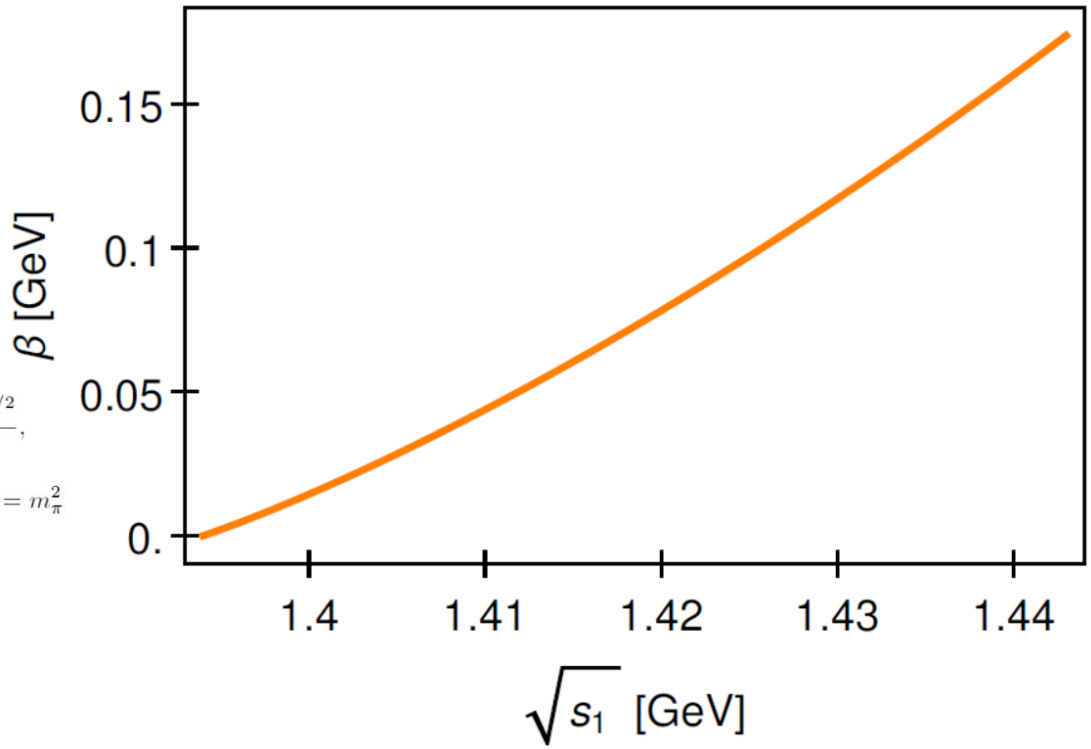


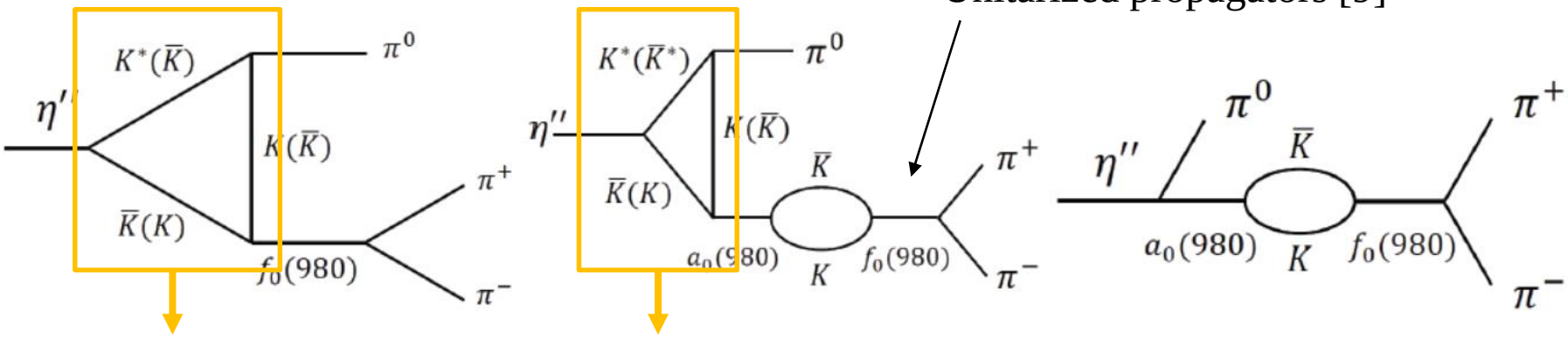
Fig.5. The energy dependence of the new scale β.

Analysis on $\eta(1405/1475)$ decays

$\eta'' \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0$

$\Gamma_{K^*} = 50\text{MeV}$

Unitarized propagators [9]



destructive

constructive

Table.1. List of couplings needed in our calculation.

Parameters	Values
g_{VPP}	4.52
$g_{f_0K^+K^-}$	5.92GeV
$g_{f_0\pi^+\pi^-}$	2.96GeV
$g_{a_0K^+K^-}$	2.24GeV
$g_{a_0\eta\pi}$	3.03GeV
$g_{\eta''K^*\bar{K}}$	~ 3.6 Ref[3]
$g_{\eta''a_0\pi}$	?
$\tilde{g}_{\eta''a_0\pi}$	?
$\{\phi_a, \phi_b\}$	?

From $\phi \to K\bar{K}$ width

From KLOE experiment [10,11]

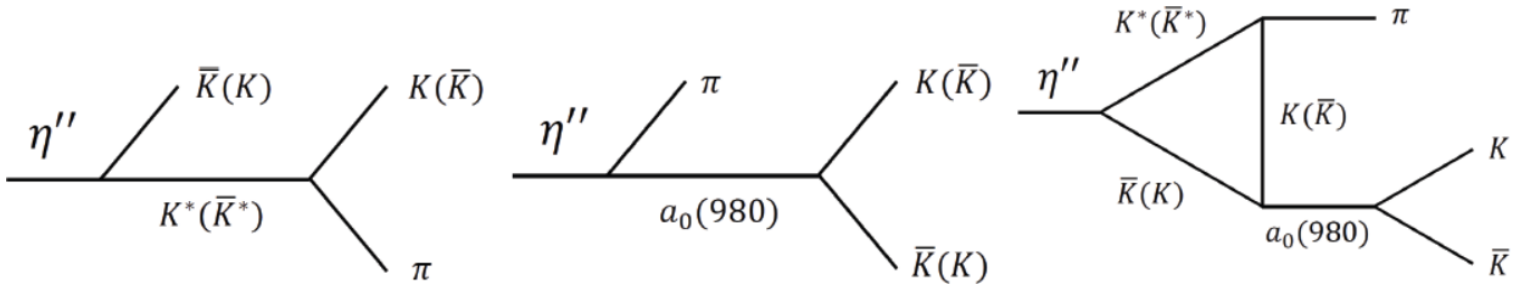
[10] A. Aloisio et al. [KLOE Col.], PL B 536, 209 (2002)

[11] A. Aloisio et al. [KLOE Col.], PL B537, 21 (2002)

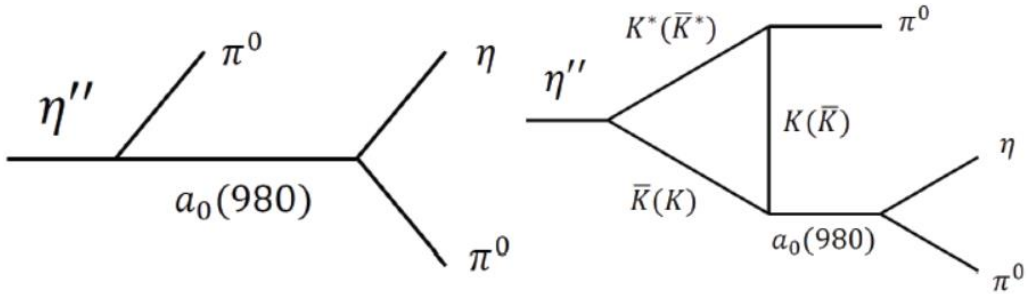
Need an analysis on $\eta'' \to a_0(980)\pi \to \eta\pi\pi$ and $\eta'' \to \bar{K}K\pi$ channels

[3] Xiao-Gang Wu, Xiao-Hai Liu, Qiang Zhao, Bing-Song Zou, PR D87, 014023 (2013); [9] N. N. Achasov and A. V. Kiselev, PR D70, 111901 (2004)

$$\eta'' \to \bar{K}K\pi$$



$$\eta'' \to a_0\pi \to \eta\pi\pi$$



Form factor $F(q^2)$ arises naturally in an effective theory:

$$I = \int \frac{d^4q}{(2\pi)^4} \frac{(2p_1 - q)_\mu \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right)(q - 2p_2)_\nu F(q^2)}{(q^2 - m_1^2)((q-p_2)^2 - m_2^2)((q-p_2-p_3)^2 - m_3^2)} \qquad F(q^2) = \prod_{i=1}^{3} \frac{\Lambda_i^2 - m_i^2}{\Lambda_i^2 - p_{i(q)}^2}$$

$\Lambda_i = m_i + \alpha\Lambda_{QCD}$, $\alpha = 1{\sim}2$ and $\Lambda_{QCD} = 250$MeV.

The result should be stable as α varies. $F(q^2)$ is safe when the physical result mostly arises from the region where 1) $p_i^2 = m_i^2$ or 2) p_i is space like.

Experiment Constraints:

$$\frac{BR(J/\Psi \to \gamma\eta'' \to \gamma f_0\pi \to \gamma\pi^+\pi^-\pi^0)}{BR(J/\Psi \to \gamma\eta'' \to \gamma a_0\pi \to \gamma\eta\pi^0\pi^0)} = \frac{(1.50 \pm 0.11 \pm 0.11) \times 10^{-5}}{(8.40 \pm 1.75) \times 10^{-5}} = (17.9 \pm 4.2)\%$$

$$\frac{BR(J/\Psi \to \gamma\eta'' \to \gamma f_0\pi \to \gamma\pi^+\pi^-\pi^0)}{BR(J/\Psi \to \gamma\eta'' \to \gamma K\bar{K}\pi)} = \frac{(1.50 \pm 0.11 \pm 0.11) \times 10^{-5}}{(2.8 \pm 0.6) \times 10^{-3}} = (0.53 \pm 0.13)\%$$

$$\frac{BR(J/\Psi \to \gamma\eta'' \to \gamma a_0\pi \to \gamma f_0\pi \to \gamma\pi^+\pi^-\pi^0)}{BR(J/\Psi \to \gamma\eta'' \to \gamma a_0\pi \to \gamma\eta\pi^0\pi^0)} \sim 1\%$$

$$a_0\pi \ll K\bar{K}\pi$$

$$a_0\pi \ll K^*\bar{K}$$

$$\Gamma_{\eta''\to a_0\pi\to\eta\pi\pi} + \Gamma_{\eta''\to K\bar{K}\pi} \sim 50\text{MeV}$$

Parameters	Values
g_{VPP}	4.52
$g_{f_0K^+K^-}$	5.92GeV
$g_{f_0\pi^+\pi^-}$	2.96GeV
$g_{a_0K^+K^-}$	2.24GeV
$g_{a_0\eta\pi}$	3.03GeV

	$\alpha = 2$	$\alpha = 1$
$g_{\eta''K^*\bar{K}}$	3.97	4.03
$g_{\eta''a_0\pi}$	0.72 GeV	0.37 GeV
$\tilde{g}_{\eta''a_0\pi}$	0.5 GeV	0.49 GeV
$\{\phi_a, \phi_b\}$	$\{60°, -40°\}$	$\{10°, -70°\}$

Table.2. List of couplings and their phases determined by matching with experiment data.

Channels	Partial width [MeV] $\alpha = 1$	Partial width [MeV] $\alpha = 2$
$\eta'' \to a_0\pi \to \eta\pi\pi(total)$	4.01	4.25
$\eta'' \to a_0\pi \to \eta\pi\pi\ (tree)$	2.24	8.17
$\eta'' \to K^*\bar{K} \to a_0\pi \to \eta\pi\pi(tri)$	5.27	15.5
$\eta'' \to K\bar{K}\pi(total)$	46.0	45.7
$\eta'' \to K^*\bar{K} \to K\bar{K}\pi(tree)$	50.2	48.7
$\eta'' \to a_0\pi \to K\bar{K}\pi(tree + tri)$	0.79	1.51
$\eta'' \to a_0\pi \to K\bar{K}\pi(tree)$	0.28	1.10
$\eta'' \to K^*\bar{K} \to a_0\pi \to K\bar{K}\pi(tri)$	1.34	3.27
$\eta'' \to K^*\bar{K} \to f_0\pi$ $\to \pi^+\pi^-\pi^0$(tri)	0.241	0.242
$\eta'' \to a_0\pi \to K\bar{K} \to f_0\pi$ $\to \pi^+\pi^-\pi^0(tri + tree)$	0.0047	0.00413

Table.3. Partial widths of $\eta'' \to a_0\pi \to \eta\pi\pi$, $\eta'' \to K\bar{K}\pi$ and $\eta'' \to f_0\pi \to \pi^+\pi^-\pi^0$ by the parameters given in Table.2.

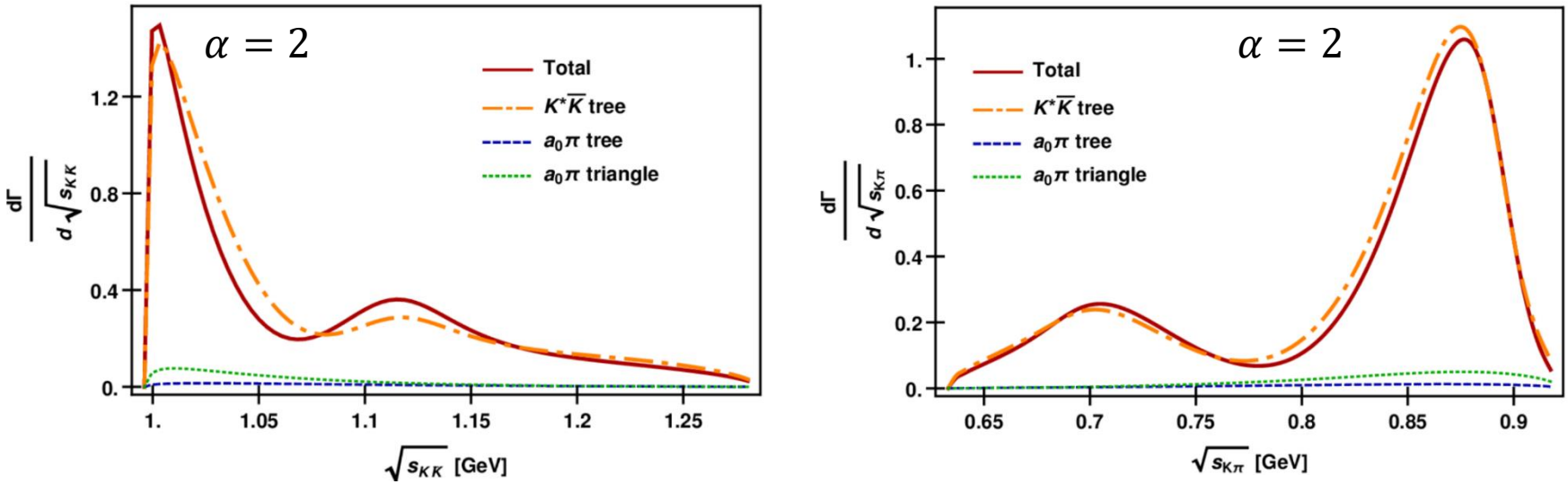


Fig.6. $K\bar{K}$ (left panel) and $K\pi$ (right panel) invariant mass spectra (red solid) at $\sqrt{s_1} = 1.42$ GeV.

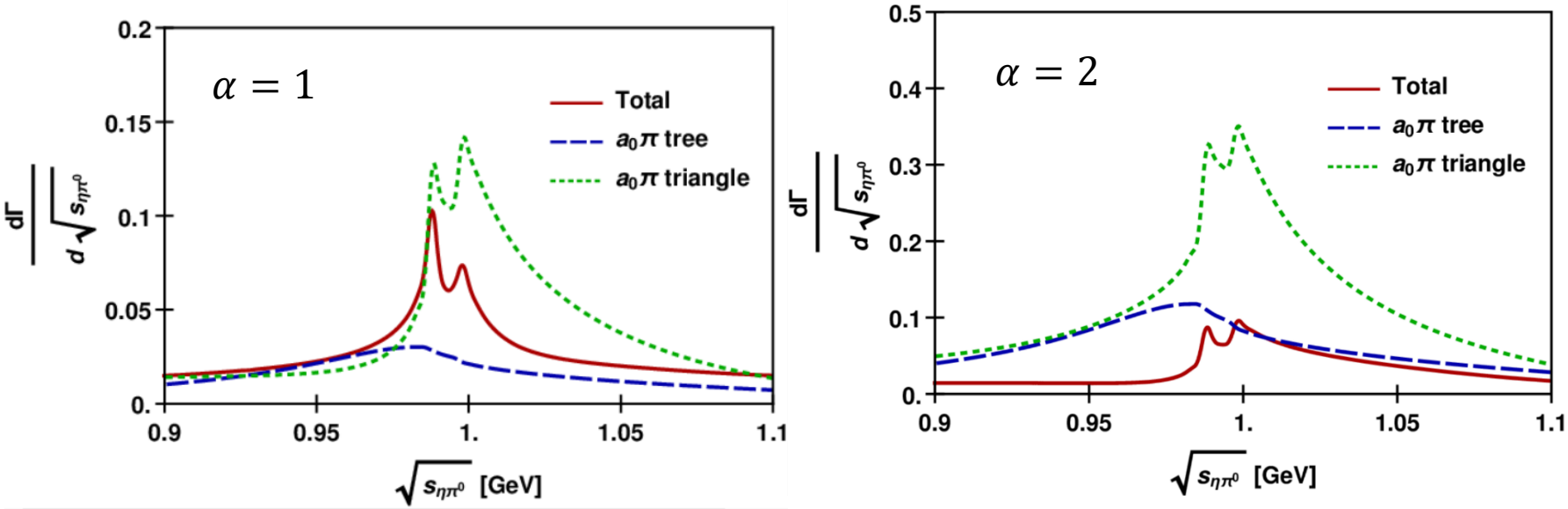


Fig.7. $\eta\pi^0$ invariant mass spectrum (red solid) at $\alpha = 1$ (left) and $\alpha = 2$ (right). The double-peak structure arises from the mass difference between the charged and the neutral loops.

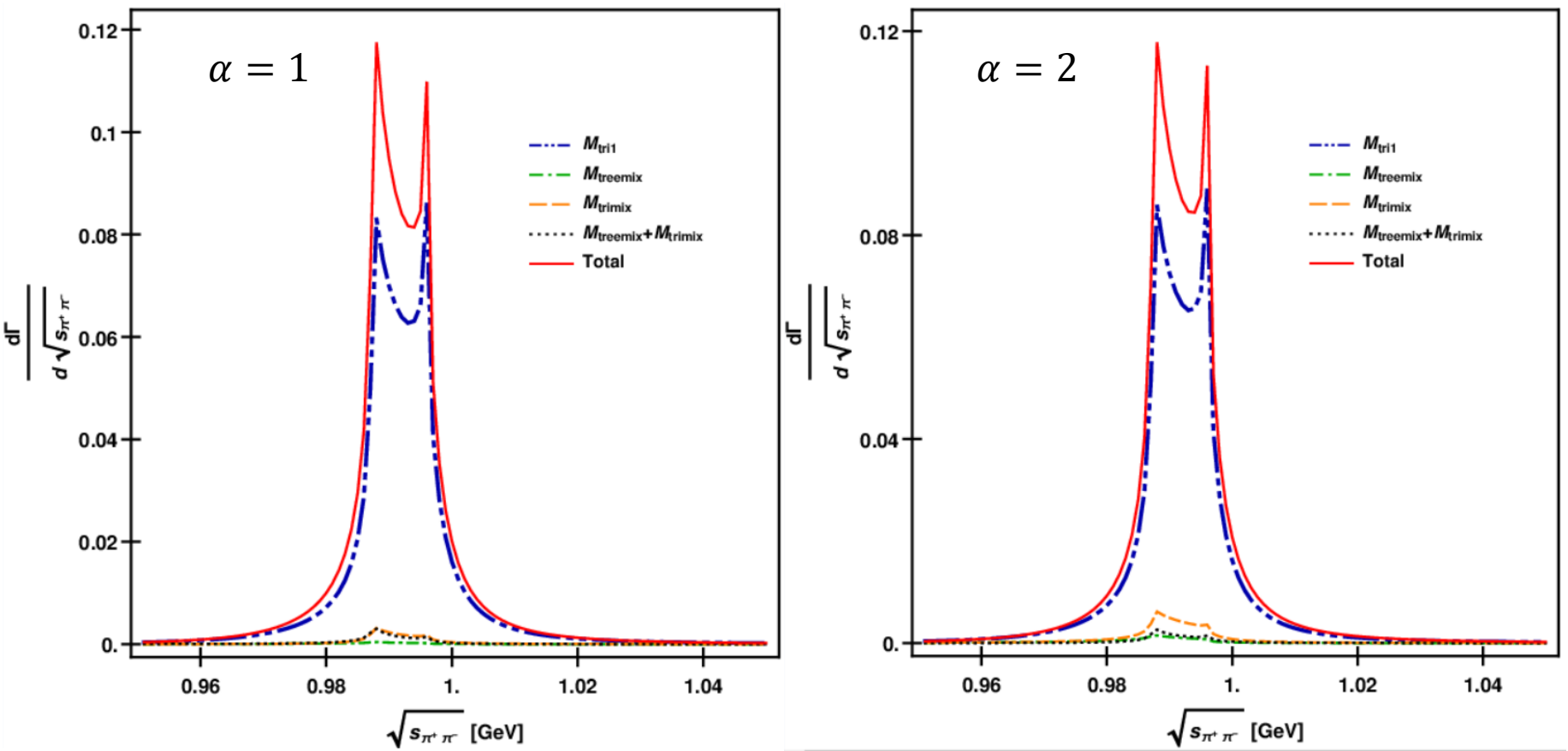
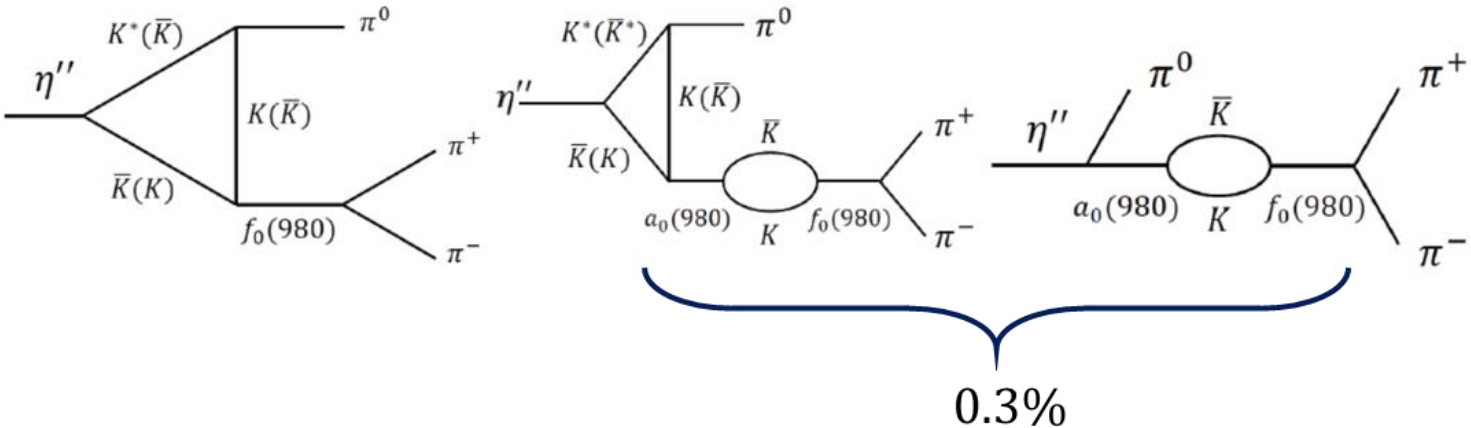


Fig.8. $\pi^+\pi^-$ invariant mass spectrum at $\alpha = 1$ (left) and $\alpha = 2$ (right).



Conclusions

1. The large suppression from the seemingly small Γ_{K^*} can be understood as being due to a small scale $\beta \sim O(\Gamma_{K^*})$.
2. The condition for the occurrence of the TS can be demonstrated by the motion of $y_j^{(i)}$.
3. The triangle mechanism can also enhance the $a_0\pi$ production from η''.
4. The small $\eta'' \to a_0\pi \to \eta\pi\pi$ branching ratio is due to the cancellation between the triangle and the tree diagrams.
5. Taken into account the non-zero Γ_{K^*}, the triangle singularity and the accompanying cancellation between charged and neutral loops is still the dominant contribution to the large isospin violation in J/Ψ $\to \gamma\eta'' \to \gamma f_0(980)\pi \to \gamma\pi^+\pi^-\pi^0$.

References

[1] M. Ablikim *et al.* [BESIII Collaboration], Phys. Rev. Lett. 108, 182001 (2012)

[2] Jia-Jun Wu, Xiao-Hai Liu, Qiang Zhao and Bing-Song Zou, Phys. Rev. Lett. 108, 081803 (2012)

[3] Xiao-Gang Wu, Jia-Jun Wu, Qiang Zhao, and Bing-Song Zou, Phys. Rev. D 87, 014023 (2013)

[4] L. D. Landau, Nucl. Phys. 13, 181 (1959)

[5] G. Bonnevay, I. J. R. Aitchison and J. S. Dowker, Nuovo Cim. 21, 3569 (1961)

[6] R. E. Cutkosky, J. Math. Phys. 1, 429 (1960)

[7] N. N. Achasov, A. A. Kozhevnikov and G. N. Shestakov, Phys. Rev. D 92, 036003 (2015)

[8] G. 't Hooft and M. J. G. Veltman, Nucl. Phys. B 153,365(1961)

[9] N. N. Achasov and A. V. Kiselev, Phys. Rev. D 70, 111901 (2004)

Thank you

Explicit absorptive part

$$\Im M(\Gamma) = \frac{1}{16\pi^2 \lambda[s_1, s_2, s_3]^{\frac{1}{2}}} (\Im W_{tot} + \Im T_{tot} + \Im \sigma_{tot})$$

The expressions for $\Im W_{tot}, \Im T_{tot}$ and $\Im \sigma_{tot}$ are real but different for each energy region.

Energy	$s_1 < s_1^-$	$s_1^- < s_1 < s_{1c}$	$s_{1c} < s_1 < s_1^+$	$s_1^+ < s_1$
$\Im W_{tot}$	$\pi \ln \left\lvert \frac{z_{11}^{(1)} z_{12}^{(1)} z_{21}^{(2)} z_{22}^{(2)} z_{11}^{(3)}}{z_{11}^{(2)} z_{12}^{(2)} z_{12}^{(3)}} \right\rvert$	$\pi \ln \left\lvert \frac{z_{21}^{(1)} z_{12}^{(1)} z_{21}^{(2)} z_{21}^{(3)}}{z_{11}^{(2)} z_{12}^{(3)}} \right\rvert$	$\pi \ln \left\lvert \frac{z_{21}^{(1)} z_{12}^{(1)} z_{21}^{(2)}}{z_{11}^{(2)} z_{21}^{(3)} z_{12}^{(3)}} \right\rvert$	$\pi \ln \left\lvert \frac{z_{21}^{(1)} z_{22}^{(1)} z_{21}^{(2)} z_{12}^{(2)}}{z_{11}^{(2)} z_{22}^{(2)} z_{11}^{(3)} z_{12}^{(3)}} \right\rvert$

Energy	$s_1 < s_1^-$	$s_1^- < s_1 < s_{1c}$	$s_{1c} < s_1 < s_1^+$	$s_1^+ < s_1$
$\Im T_{tot}$	$\pi \ln \left\lvert \frac{r_{31}^{(1)} r_{41}^{(2)} r_{42}^{(2)} (r_{32}^{(2)})^2 r_{31}^{(3)} r_{22}^{(3)}}{r_{21}^{(1)} r_{22}^{(1)} r_{32}^{(1)} r_{21}^{(2)} r_{22}^{(2)} (r_{12}^{(2)})^2 r_{21}^{(3)} r_{32}^{(3)}} \right\rvert$	$\pi \ln \left\lvert \frac{r_{11}^{(1)} r_{32}^{(2)} r_{41}^{(2)} r_{11}^{(3)} r_{22}^{(3)}}{r_{22}^{(1)} r_{32}^{(1)} r_{41}^{(1)} r_{12}^{(2)} r_{21}^{(2)} r_{32}^{(3)} r_{41}^{(3)}} \right\rvert$	$\pi \ln \left\lvert \frac{r_{11}^{(1)} r_{32}^{(2)} r_{41}^{(2)} r_{22}^{(3)} (r_{31}^{(3)})^2 r_{41}^{(3)}}{r_{22}^{(1)} r_{32}^{(1)} r_{41}^{(1)} r_{12}^{(2)} r_{21}^{(2)} r_{11}^{(3)} r_{32}^{(3)}} \right\rvert$	$\pi \ln \left\lvert \frac{r_{11}^{(1)} r_{12}^{(1)} r_{22}^{(2)} r_{41}^{(2)} r_{21}^{(3)} r_{22}^{(3)} r_{31}^{(3)}}{(r_{32}^{(1)})^2 r_{41}^{(1)} r_{42}^{(1)} r_{21}^{(2)} r_{42}^{(2)} r_{32}^{(3)}} \right\rvert$

$$\Im \sigma_{tot} \equiv \Im \sigma^{(1)} - \Im \sigma^{(2)} + \Im \sigma^{(3)} = -2\pi \Theta(s_{1c} - s_1) \ln \left| \frac{y_0^{(1)} - 1}{y_0^{(1)}} \right| - 2\pi \Theta(s_1 - s_1^+) \ln \left| \frac{y_0^{(2)} - 1}{y_0^{(2)}} \right|$$

The parameters $z_{jk}^{(i)}$ and $r_{jk}^{(i)}$ are functions of $\{y_j^{(i)}\}$