三体体系在有限体积内的能移
The Energy Shift of 3-body System in the Finite Volume

吴佳俊 (中国科学院大学)

Collaborators: M. Doring, H.-W. Hammer, M. Mai, Ulf-G. Meiβner, J.-Y. Pang, and A. Rusetsky

Phys.Rev. D97 (2018) no.11, 114508

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Outline

- Motivation
- 3-body system in the infinite volume
- 3-body quantization condition in the finite volume
- The energy shift in the finite volume
- Summary and Outlook

Motivation

Experimental Observations ← **QCD theory**

Motivation

Experimental Observations ← **Non-Perturbation** — **QCD theory**

Motivation

Non-Perturbation

Experimental Observations ← **QCD theory**

↓ ↓

S matrix elements ← **Lattice QCD Energy levels**

Motivation

Non-Perturbation

Experimental Observations ← **QCD theory**

↓ ↓

S matrix elements ← **Lattice QCD Energy levels**

For the 2-body system, Lüscher's method,
Hamiltonian Effective Field Theory, …

Motivation

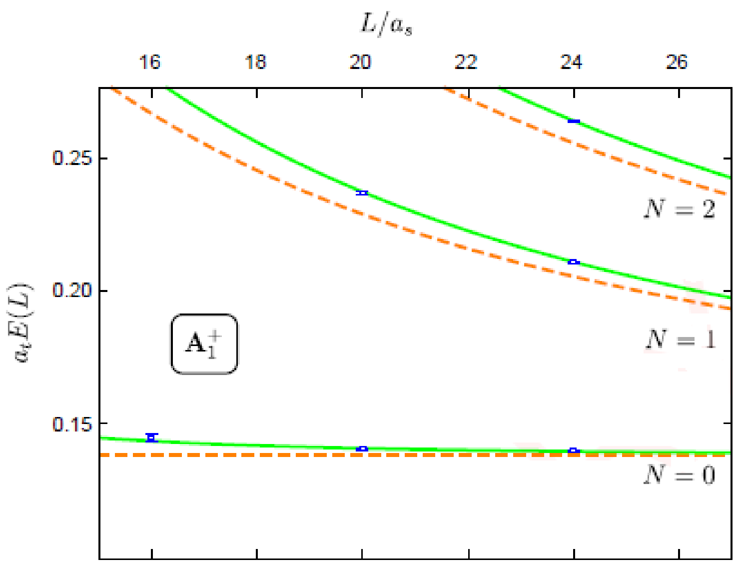
Non-Perturbation

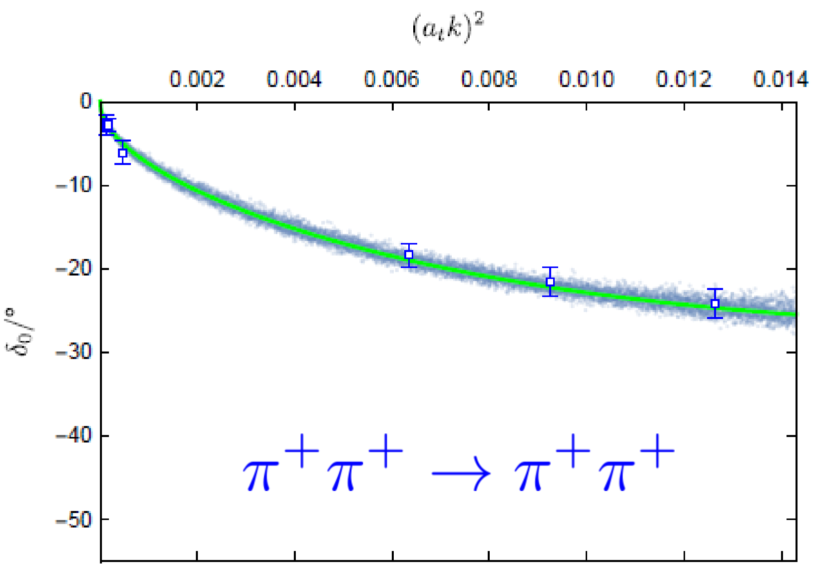
Experimental Observations ← **QCD theory**

↓ ↓

S matrix elements ← **Lattice QCD Energy levels**

For the 2-body system, Lüscher's method, Hamiltonian Effective Field Theory, …





Motivation

Non-Perturbation

Experimental Observations ← **QCD theory**

↓ ↓

S matrix elements ← **Lattice QCD Energy levels**

For the 3-body system, it becomes very complicated,

1. **There are two free momenta**
2. **How to define the S matrix of three body**
3. **How to deal with the divergent of three body re-scattering**

But the 3-body system is extremely important to describe low energy resonances, such as $\eta \to 3\pi$, $\omega \to 3\pi$, $N^*(1440) \to N\pi\pi$.

Motivation

- History

K. Ploejaeva and A. Rusetsky, EPJA 48(2012) 67 "Three particles in a finite volume"

By S matrix

M. Hansen and S. Sharpe, PRD 90(2014) 116003
Relativistic, model independent, three-particle quantization condition
PRD 92(2015) 114509
Expressing the three-particle finite-volume spectrum in terms of the three-to-three scattering amplitude

Quantization Condition

R. Briceno and Z. Davoudi, PRD 87 (2013) 094507
Three-particle scattering amplitudes from a finite volume formalism

Dimer formalism
Quantization Condition

P. Guo, PRD 95 (2017) 054508
One spatial dimensional finite volume three-body interaction for a short-range potential

Quantization Condition
In 1+1 dimensional case

S. Kreuzer and H.-W. Hammer, PLB 694(2011) 424; The triton in a finite volume
EPJA 43(2010) 229; There-boson bound states in finite volume with EFT
S. Kreuzer and H.-W. Grießhammer, PLB 673 (2009) 260 Efimov physics in a finite volume
EPJA 48 (2012) 93 Three-boson bound states in finite volume with Eft

Dimer formalism
Numerical solution

The quantization condition complicated, not well suited for the analysis of the lattice data.

Motivation

- For the large boxes (small momentum step), the energy spectrum can be calculated, using non-relativistic EFT in a finite volume.

3-body System in the infinite volume

- Non-Relativistic Effective Field Theory (NREFT)

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3$$

$$p \cot \delta_0 = -\frac{1}{a_0} + \frac{r_0}{2} p^2$$

$$p^5 \cot \delta_2 = -\frac{1}{a_2}$$

$$\mathcal{L}_1 = \psi^\dagger \left(i\partial_0 - \frac{\nabla^2}{2m} \right) \psi$$

$$\mathcal{L}_2 = -\frac{C_0}{2} \psi^\dagger \psi^\dagger \psi \psi - \frac{C_2}{4} \left(\psi^\dagger \nabla^2 \psi^\dagger \psi \psi + h.c. \right) + \cdots$$

$$-\frac{C_2'}{4} \left(\vec{\nabla} \psi^\dagger \cdot \vec{\nabla} \psi \vec{\nabla} \psi^\dagger \cdot \vec{\nabla} \psi - 3 \psi^\dagger \nabla^2 \psi^\dagger \nabla^2 \psi \psi + h.c. \right) + \cdots$$

$$\mathcal{L}_3 = -\frac{D_0}{6} \psi^\dagger \psi^\dagger \psi^\dagger \psi \psi \psi - \frac{D_2}{12} \left(\psi^\dagger \psi^\dagger \nabla^2 \psi^\dagger \psi \psi \psi + h.c. \right) + \cdots$$

$$-\frac{D_2'}{4} \left(\psi^\dagger \vec{\nabla} \psi^\dagger \cdot \vec{\nabla} \psi \vec{\nabla} \psi^\dagger \cdot \vec{\nabla} \psi \psi - 3 \psi^\dagger \psi^\dagger \nabla^2 \psi^\dagger \nabla^2 \psi \psi \psi + h.c. \right) + \cdots$$

3-body System in the infinite volume

S-wave

$$\mathcal{L}_1 = \psi^\dagger \left(i\partial_0 - \frac{\nabla^2}{2m} \right) \psi$$

$$\mathcal{L}_2 = -\frac{C_0}{2}\psi^\dagger\psi^\dagger\psi\psi - \frac{C_2}{4}\left(\psi^\dagger\nabla^2\psi^\dagger\psi\psi + h.c.\right) + \cdots$$

$$\mathcal{L}_3 = -\frac{D_0}{6}\psi^\dagger\psi^\dagger\psi^\dagger\psi\psi\psi - \frac{D_2}{12}\left(\psi^\dagger\psi^\dagger\nabla^2\psi^\dagger\psi\psi\psi + h.c.\right) + \cdots$$

Dimer picture

$$(C_0,\, C_2,\, D_0,\, D_2)$$

$$\mathcal{L}_1 = \psi^\dagger \left(i\partial_0 - \frac{\nabla^2}{2m} \right) \psi$$

$$\sim (\sigma,\, f_1,\, h_0,\, h_2)$$

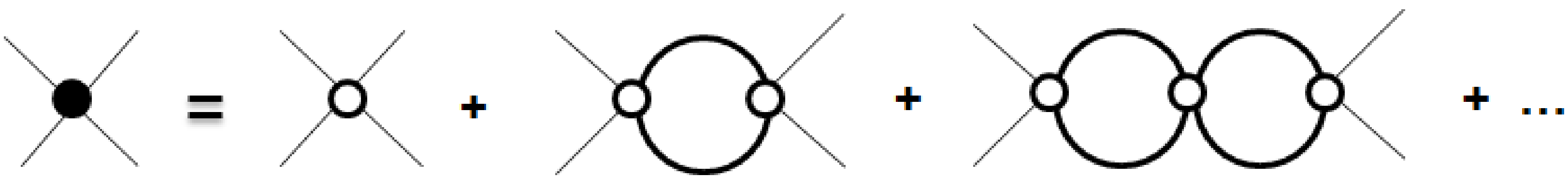
$$\mathcal{L}_2 = \sigma T^\dagger T + \frac{1}{2}\left[T^\dagger\left(\psi\psi + f_1\psi\nabla^2\psi + \cdots\right) + h.c.\right]$$

$$\mathcal{L}_3 = h_0 T^\dagger T\psi^\dagger\psi + h_2 T^\dagger T\left(\psi^\dagger\nabla^2\psi + h.c.\right) + \cdots$$

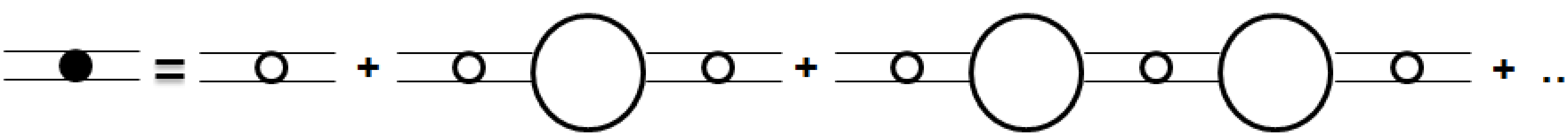
3-body System in the infinite volume

Dimer picture

$$\mathcal{L}_2 = -\frac{C_0}{2}\psi^\dagger\psi^\dagger\psi\psi - \frac{C_2}{4}\left(\psi^\dagger\nabla^2\psi^\dagger\psi\psi + h.c.\right) + \cdots$$



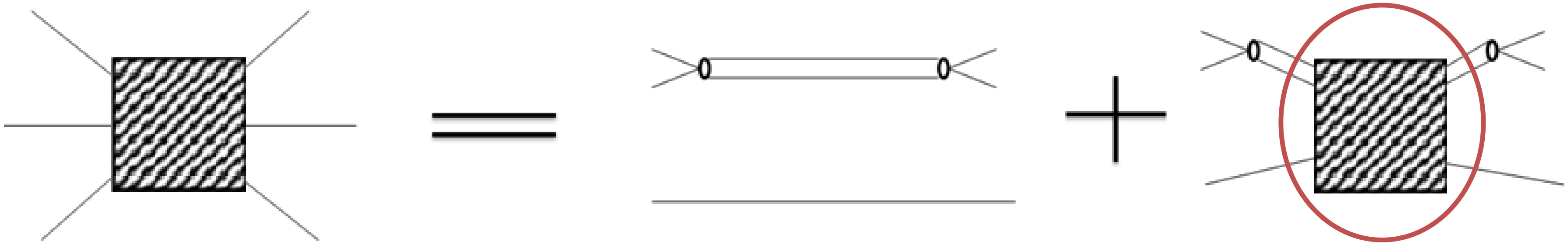
$$\mathcal{L}_2 = \sigma T^\dagger T + \left[T^\dagger\left(\psi\psi + f_1\psi\nabla\psi + \cdots\right) + h.c.\right]$$



$$\tau(\vec{k}, E) = \frac{1}{k^* cot\delta_0(k^*) + ik^*}$$

3-body System in the infinite volume

Dimer Particle scattering VS 3 Particles scattering



3-body System in the infinite volume

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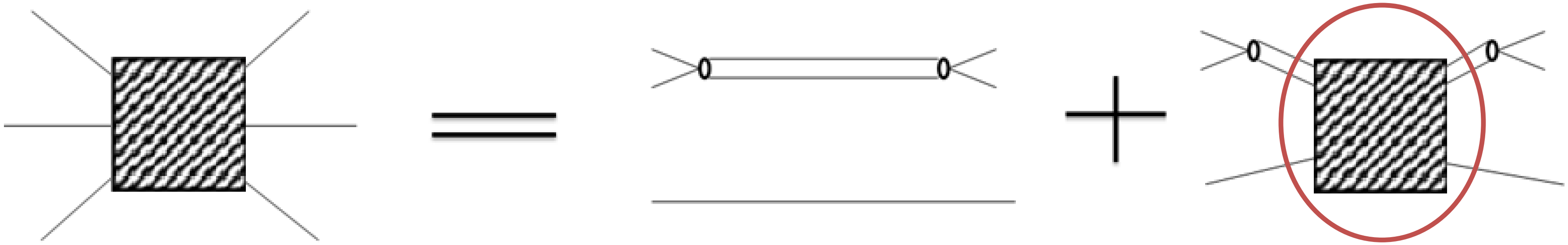
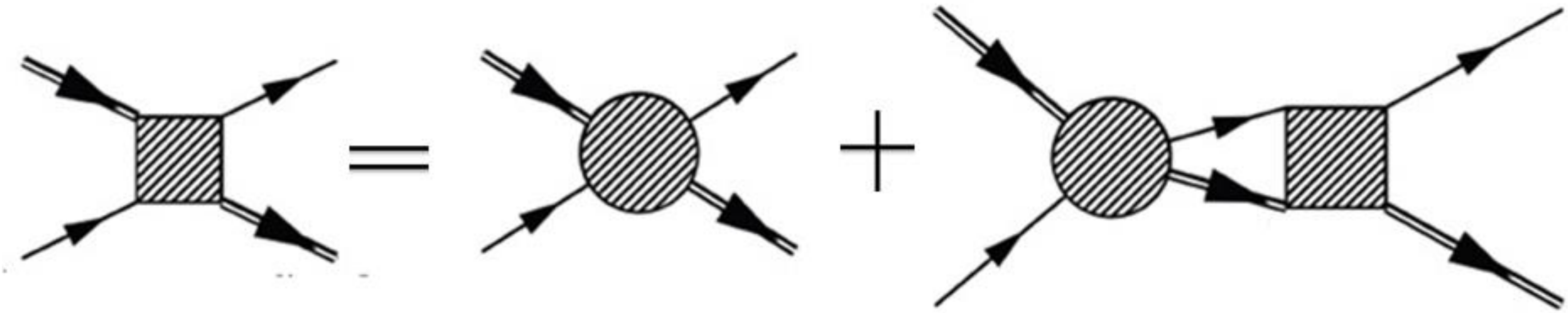
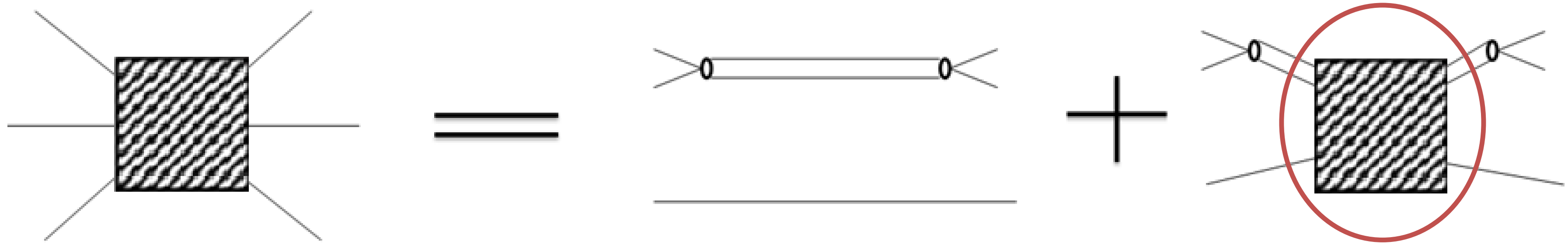
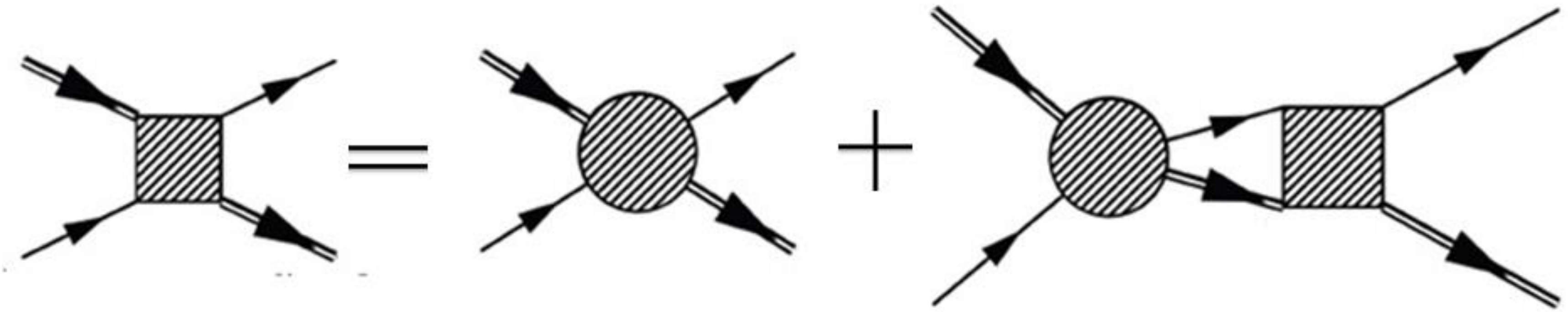


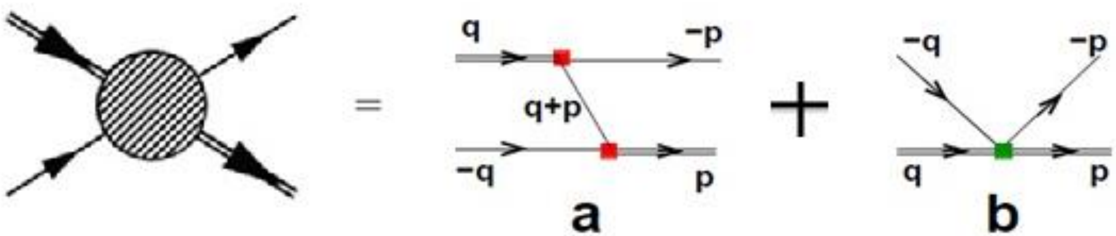


3-body System in the infinite volume

Dimer Particle scattering VS 3 Particles scattering







A toy Model

S-wave & O(p) & Non-Relativistic

$$\mathcal{L}_1 = \psi^\dagger \left(i\partial_0 - \frac{\nabla^2}{2m} \right) \psi \quad \mathcal{L}_2 = \sigma T^\dagger T + \left[T^\dagger \psi\psi + h.c. \right] \quad \mathcal{L}_3 = h_0 T^\dagger T \psi^\dagger \psi$$

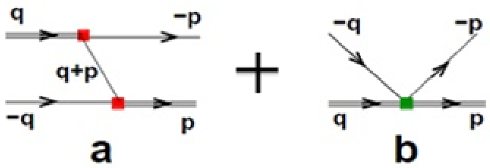
A toy Model

S-wave & O(p) & Non-Relativistic

$$\mathcal{L}_1 = \psi^\dagger \left(i\partial_0 - \frac{\nabla^2}{2m} \right) \psi \quad \mathcal{L}_2 = \sigma T^\dagger T + \left[T^\dagger \psi\psi + h.c. \right] \quad \mathcal{L}_3 = h_0 T^\dagger T \psi^\dagger \psi$$

Scattering equation

$$\mathcal{M}(\vec{p}, \vec{q}, E) = Z(\vec{p}, \vec{q}, E) + 8\pi \int_0^\Lambda \frac{d^3\vec{k}}{8\pi^3} Z(\vec{p}, \vec{k}, E)\tau(\vec{k}, E)\mathcal{M}(\vec{k}, \vec{q}, E)$$



$$\tau(\vec{k}, E) = \frac{1}{-\frac{1}{a_0} + \sqrt{\frac{3}{4}k^2 - mE}}$$

A toy Model

S-wave & O(p) & Non-Relativistic

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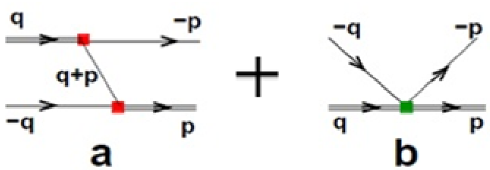
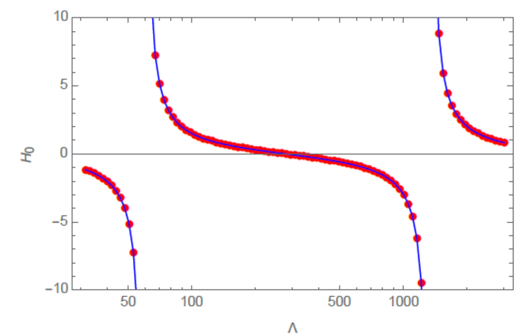


$$\tau(\vec{k},E) = \frac{1}{-\frac{1}{a_0} + \sqrt{\frac{3}{4}k^2 - mE}}$$

Physical Input $a_0 = m = 1$, $B_3(3p) = 10$ $\Longrightarrow$ $H_0(\Lambda)$ vs Λ



$$\mathcal{F}(\vec{p},B_3) = \int_0^\Lambda d^3\vec{k} Z(\vec{p},\vec{k},B_3)\tau(\vec{k},B_3)\mathcal{F}(\vec{k},B_3)$$

A toy Model

S-wave & O(p) & Non-Relativistic

$$\mathcal{L}_1 = \psi^\dagger \left(i\partial_0 - \frac{\nabla^2}{2m} \right) \psi \quad \mathcal{L}_2 = \sigma T^\dagger T + \left[T^\dagger \psi\psi + h.c. \right] \quad \mathcal{L}_3 = h_0 T^\dagger T \psi^\dagger \psi$$

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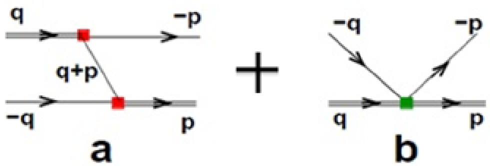

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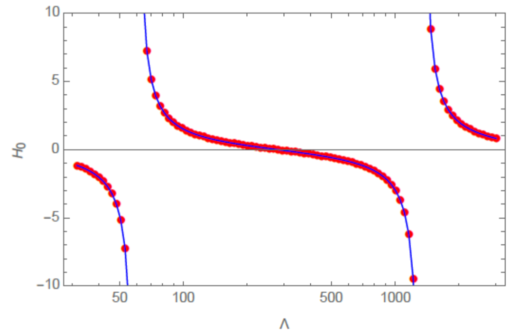

$$\mathcal{F}(\vec{p},B_3) = \int_0^\Lambda d^3\vec{k} Z(\vec{p},\vec{k},B_3)\tau(\vec{k},B_3)\mathcal{F}(\vec{k},B_3)$$

$\mathcal{M}(\vec{p},\vec{q},E)$ is Λ independent

A toy Model

S-wave & O(p) & Non-Relativistic

$$\mathcal{L}_1 = \psi^\dagger \left(i\partial_0 - \frac{\nabla^2}{2m} \right) \psi \quad \mathcal{L}_2 = \sigma T^\dagger T + \left[T^\dagger \psi\psi + h.c. \right] \quad \mathcal{L}_3 = h_0 T^\dagger T \psi^\dagger \psi$$

Scattering equation

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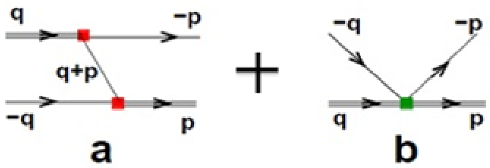

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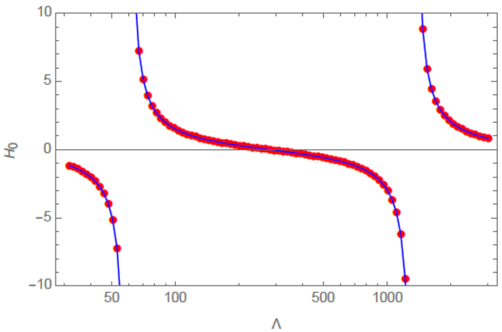


$$\mathcal{F}(\vec{p}, B_3) = \int_0^\Lambda d^3\vec{k} Z(\vec{p}, \vec{k}, B_3)\tau(\vec{k}, B_3)\mathcal{F}(\vec{k}, B_3)$$

$$\mathcal{M}(\vec{p}, \vec{q}, E) \text{ is } \Lambda \text{ independent}$$

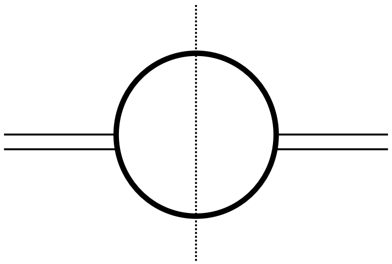
P.F. Bedaque, H.-W. Hammer, and U. van Kolck NPA 646 444 (1999)
The three-boson system with short-range interactions

3-body System in the finite volume

- Infinite Volume -> Box
- Momentum space,

  continuum -> discrete $(2\pi/\mathrm{L})\,\vec{n}$, $\vec{n} = (n_1, n_2, n_3)$
- Propagator of dimer,

$$\tau(\vec{k}, E) = \frac{1}{-\frac{1}{a_0} + \sqrt{\frac{3}{4}k^2 - mE}}$$



$\Longrightarrow$

$$\tau_L(\vec{m}, E) = \frac{1}{-\frac{1}{a_0} - \frac{4\pi}{L^3}\sum_{\vec{l}} \frac{1}{\frac{4\pi^2}{L^2}\left(m^2 + l^2 + \vec{k}\cdot\vec{l}\right) - mE}}$$

3-body System in the finite volume

- Scattering equation,

$$\mathcal{M}(\vec{p},\vec{q},E) = Z(\vec{p},\vec{q},E) + 8\pi\int_0^{\Lambda}\frac{d^3\vec{k}}{8\pi^3}Z(\vec{p},\vec{k},E)\tau(\vec{k},E)\mathcal{M}(\vec{k},\vec{q},E)$$

$$\Longrightarrow$$

$$\mathcal{M}(\frac{2\pi}{L}\vec{m},\frac{2\pi}{L}\vec{n},E) = Z(\frac{2\pi}{L}\vec{m},\frac{2\pi}{L}\vec{n},E)+\frac{8\pi}{L^3}\sum_{\vec{l}\in\mathcal{Z}^3}^{\frac{L\Lambda}{2\pi}} Z(\frac{2\pi}{L}\vec{m},\frac{2\pi}{L}\vec{l},E)\tau_L(\vec{l},E)\mathcal{M}(\frac{2\pi}{L}\vec{l},\frac{2\pi}{L}\vec{n},E)$$

$$\mathcal{M}_L(\vec{m},\vec{n},E) = Z_L(\vec{m},\vec{n},E) + \frac{8\pi}{L^3}\sum_{\vec{l}\in\mathcal{Z}^3}^{\frac{L\Lambda}{2\pi}} Z_L(\vec{m},\vec{l},E)\tau_L(\vec{l},E)\mathcal{M}_L(\vec{l},\vec{n},E)$$

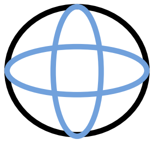
- Quantization condition:

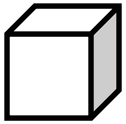
$$\det(\tau_L^{-1} - \frac{8\pi}{L^3}Z_L) = 0$$

3-body System in the finite volume

Symmetry

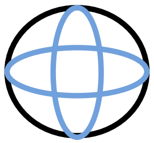
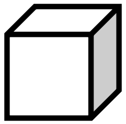
- Infinite Volume -> SO(3) group
- Box Volume -> O_h group





3-body System in the finite volume

Symmetry

- Infinite Volume -> SO(3) group 
- Box Volume -> O_h group 

Infinite Volume, $\mathcal{M}(\vec{p}, \vec{q}, E) = \mathcal{M}(\hat{R}\vec{p}, \hat{R}\vec{q}, E)$

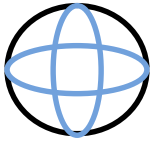
In the finite Volume, $\hat{R}_{o_h} \in O_h$,

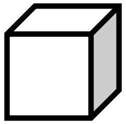
$$\mathcal{M}_L(\vec{n}, \vec{m}, E) = \mathcal{M}_L(\hat{R}_{O_h}\vec{n}, \hat{R}_{O_h}\vec{m}, E)$$

3-body System in the finite volume

Symmetry

- Infinite Volume -> SO(3) group
- Box Volume -> O_h group



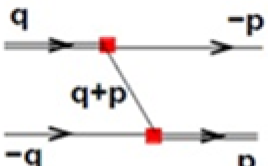


Infinite Volume, $\mathcal{M}(\vec{p},\vec{q},E)=\mathcal{M}(\hat{R}\vec{p},\hat{R}\vec{q},E)$

In the finite Volume, $\hat{R}_{o_h} \in O_h$,

$$\mathcal{M}_L(\vec{n},\vec{m},E)=\mathcal{M}_L(\hat{R}_{O_h}\vec{n},\hat{R}_{O_h}\vec{m},E)$$

$\mathbf{0}^+ = \mathbf{A}_1^+$,
$\mathbf{1}^- = \mathbf{T}_1^-$,
$\mathbf{2}^+ = \mathbf{E}^+ \oplus \mathbf{T}_2^+$,
$\mathbf{3}^- = \mathbf{A}_2^- \oplus \mathbf{T}_1^- \oplus \mathbf{T}_2^-$,
$\mathbf{4}^+ = \mathbf{A}_1^+ \oplus \mathbf{E}^+ \oplus \mathbf{T}_1^+ \oplus \mathbf{T}_2^+$,



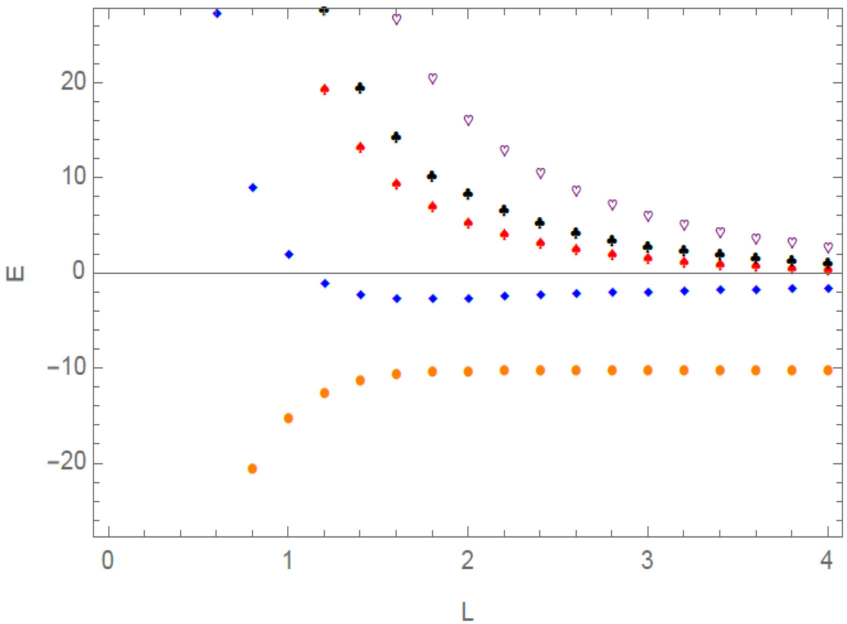
$$\det(\tau_L^{-1}-\frac{8\pi}{L^3}Z_L)=0$$

$$\det\left(\theta^{-1}(s)\tau_L^{-1}(r,E)\delta_{rs}-\frac{8\pi}{L^3}\frac{1}{G}Z_L^{A_1^+}(r,s,E)\right)=0$$

3-body System in the finite volume

- Results of Toy model and discussion

$$\det\left(\theta^{-1}(s)\tau_L^{-1}(r,E)\delta_{rs} - \frac{8\pi}{L^3}\frac{1}{G}Z_L^{A_1^+}(r,s,E)\right) = 0$$



Purple: excited state of dimer-particle

Black: ground state of three particle

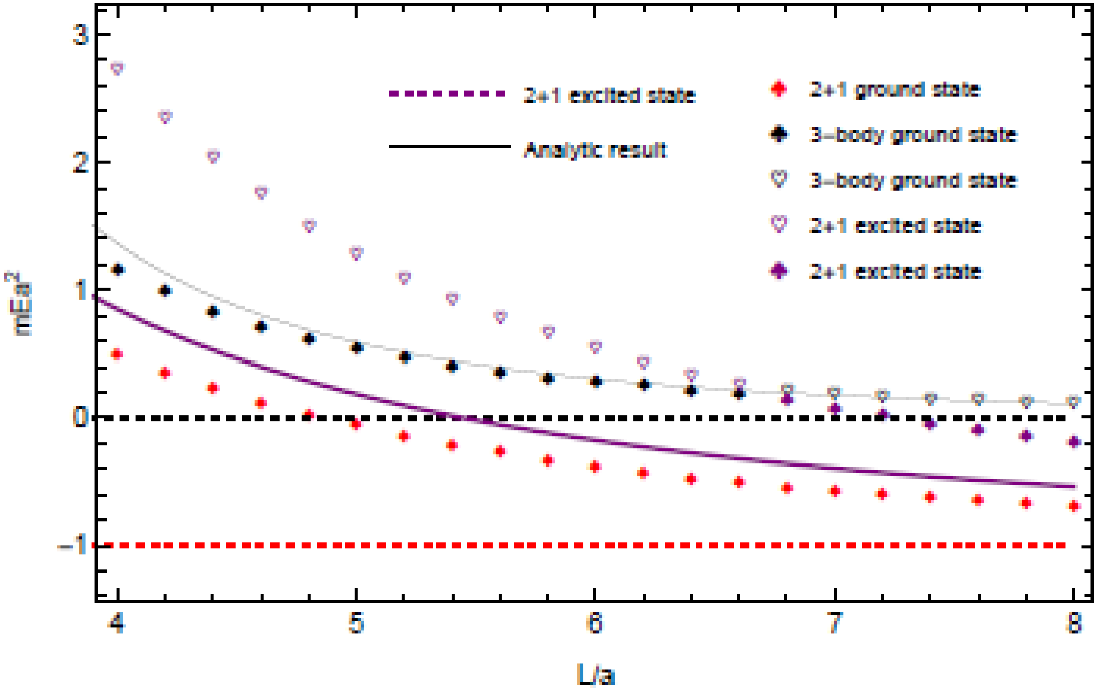
Red: ground state of dimer-particle scattering state

Blue: dimer – particle bound state

Orange: three body bound state

3-body System in the finite volume

- A interesting crossing between black and purple lines at larger size.



Purple: excite state of dimer-particle

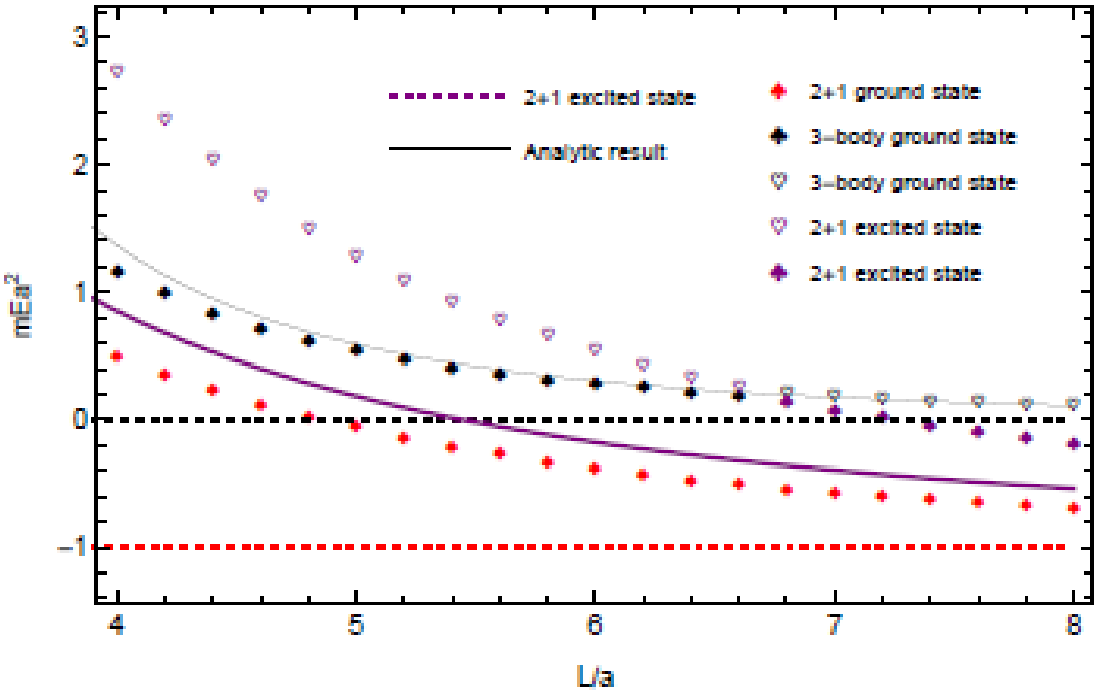
Black: ground state of three particle

Red: ground state of dimer-particle scattering state

$$\Delta E = c_1 \frac{a}{L^3} + c_2 \frac{a^2}{L^4} + c_3 \frac{a^3}{L^5} + \mathcal{O}(L^{-6})$$

c_1, c_2, c_3 are all fixed parameters as shown later.

Energy shifts in the finite volume



In this section, we will discuss this thin line and try to give the analytical expression of this line up to $O(L^{-6})$.

$$\Delta E = c_1 \frac{a}{L^3} + c_2 \frac{a^2}{L^4} + c_3 \frac{a^3}{L^5} + c_4 \frac{a^4}{L^6} + \mathcal{O}(L^{-7})$$

Energy shifts in the finite volume

- Previous work :

PHYSICAL REVIEW D **96,** 054515 (2017)

Testing the threshold expansion for three-particle energies at fourth order in ϕ^4 theory

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(Received 24 July 2017; published 26 September 2017)

$$\Delta E_{3,\mathrm{thr}} = \frac{12\pi a}{mL^3}\left\{1 - \left(\frac{a}{\pi L}\right)\mathcal{I} + \left(\frac{a}{\pi L}\right)^2(\mathcal{I}^2 + \mathcal{J}) + \frac{64\pi^2 a^2 \mathcal{C}_3}{mL^3} + \frac{3\pi a}{m^2L^3} + \frac{6\pi r a^2}{L^3} + \left(\frac{a}{\pi L}\right)^3[-\mathcal{I}^3 + \mathcal{I}\mathcal{J} + 15\mathcal{K} + c_L \log(N_{\mathrm{cut}}) + \mathcal{C}_F + \mathcal{C}_4 + \mathcal{C}_5]\right\} - \frac{\mathcal{M}_{3,\mathrm{thr}}}{48m^3L^6} + \mathcal{O}(L^{-7}), \quad (16)$$

$$N_{\mathrm{cut}} = mL/(2\pi), \; c_L = 16\pi^3(\sqrt{3} - 4\pi/3)$$

and evaluated in Ref. [8]. The new amplitude entering at $\mathcal{O}(1/L^6)$ is the divergence-free three-to-three threshold amplitude $\mathcal{M}_{3,\mathrm{thr}}$, which begins at $\mathcal{O}(\lambda^2)$ in perturbation theory. The numerical values of $\mathcal{C}_3$, $\mathcal{C}_4$, and $\mathcal{C}_5$ depend on the choice of UV cutoff, but this dependence cancels with that of $\mathcal{M}_{3,\mathrm{thr}}$. This cancellation is necessary because $\Delta E_{3,\mathrm{thr}}$ is a physical quantity.

Ground State of three body scattering, i.e., $0 < mE < L^{-2}$

$$\det\left(\theta^{-1}(s)\tau_L^{-1}(r,E)\delta_{rs} - \frac{8\pi}{L^3}\frac{1}{G}Z_L^{A_1^+}(r,s,E)\right) = 0$$

Analytical Method

$$mE = \left(\frac{2\pi}{L}\right)^2\left(\frac{x_1}{L/a} + \frac{x_2}{(L/a)^2} + \frac{x_3}{(L/a)^3} + \frac{x_4}{(L/a)^4}\right)$$

Ground State of three body scattering, i.e., $0 < mE < L^{-2}$

$$\det\left(1-\tilde{Z}_L(r,s)\tilde{\tau}(s)\right)=0$$

$$\begin{vmatrix} 1-\tilde{Z}(1,1)\tilde{\tau}(1) & -\tilde{Z}(1,2)\tilde{\tau}(2) & \cdots & -\tilde{Z}(1,N)\tilde{\tau}(N) \\ -\tilde{Z}(2,1)\tilde{\tau}(1) & 1-\tilde{Z}(2,2)\tilde{\tau}(2) & \cdots & -\tilde{Z}(2,N)\tilde{\tau}(N) \\ \vdots & \vdots & \ddots & \vdots \\ -\tilde{Z}(N,1)\tilde{\tau}(1) & -\tilde{Z}(N,2)\tilde{\tau}(2) & \cdots & 1-\tilde{Z}(N,N)\tilde{\tau}(N) \end{vmatrix}=0$$

Analytical Method

$$mE=\left(\frac{2\pi}{L}\right)^2\left(\frac{x_1}{L/a}+\frac{x_2}{(L/a)^2}+\frac{x_3}{(L/a)^3}+\frac{x_4}{(L/a)^4}\right)$$

Ground State of three body scattering, i.e., $0 < mE < L^{-2}$

$$\det\left(1 - \tilde{Z}_L(r,s)\tilde{\tau}(s)\right) = 0$$

$$\begin{vmatrix} 1-\tilde{Z}(1,1)\tilde{\tau}(1) & -\tilde{Z}(1,2)\tilde{\tau}(2) & \cdots & -\tilde{Z}(1,N)\tilde{\tau}(N) \\ -\tilde{Z}(2,1)\tilde{\tau}(1) & 1-\tilde{Z}(2,2)\tilde{\tau}(2) & \cdots & -\tilde{Z}(2,N)\tilde{\tau}(N) \\ \vdots & \vdots & \ddots & \vdots \\ -\tilde{Z}(N,1)\tilde{\tau}(1) & -\tilde{Z}(N,2)\tilde{\tau}(2) & \cdots & 1-\tilde{Z}(N,N)\tilde{\tau}(N) \end{vmatrix} = 0$$

Analytical Method

Perturbative analysis of each elements of matrix for "L" is unavailable

$$N \propto \frac{L\Lambda}{2\pi}$$

$$mE = \left(\frac{2\pi}{L}\right)^2 \left(\frac{x_1}{L/a} + \frac{x_2}{(L/a)^2} + \frac{x_3}{(L/a)^3} + \frac{x_4}{(L/a)^4}\right)$$

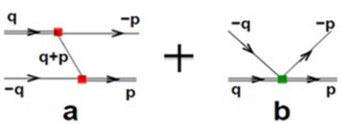
$$
\begin{vmatrix}
1-\tilde{Z}(1,1)\tilde{\tau}(1) & -\tilde{Z}(1,2)\tilde{\tau}(2) & \cdots & -\tilde{Z}(1,N)\tilde{\tau}(N) \\
-\tilde{Z}(2,1)\tilde{\tau}(1) & 1-\tilde{Z}(2,2)\tilde{\tau}(2) & \cdots & -\tilde{Z}(2,N)\tilde{\tau}(N) \\
\vdots & \vdots & \ddots & \vdots \\
-\tilde{Z}(N,1)\tilde{\tau}(1) & -\tilde{Z}(N,2)\tilde{\tau}(2) & \cdots & 1-\tilde{Z}(N,N)\tilde{\tau}(N)
\end{vmatrix} = 0
$$

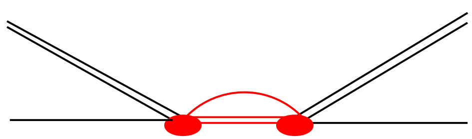
$$
1-\tilde{Z}(1,1)\tilde{\tau}(1) = 1 - \frac{3a_0}{\pi x_1} + \mathcal{O}(L^{-1})
$$

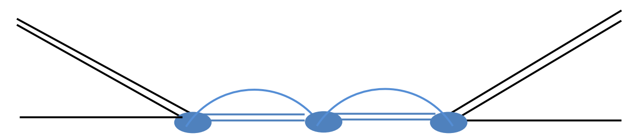
$$
\Longrightarrow \quad x_1 = \frac{3a_0}{\pi}
$$

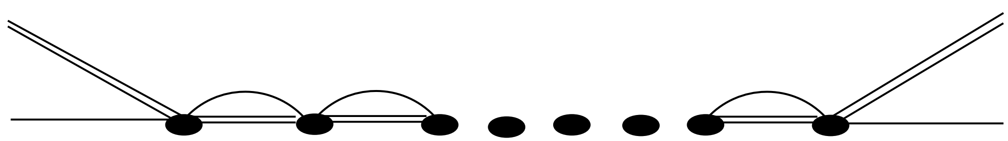
$$
\tilde{\tau}^{-1}(1) - \tilde{Z}(1,1) = \sum_{j_1 \neq 1} \tilde{Z}(1,j_1)\,\tilde{\tau}(j_1)\,\tilde{Z}(j_1,1) + \sum_{j_1,j_2 \neq 1} \tilde{Z}(1,j_1)\,\tilde{\tau}(j_1)\,\tilde{Z}(j_1,j_2)\,\tilde{\tau}(j_2)\,\tilde{Z}(j_2,1)
$$

$$
+ \sum_{j_1,j_2,j_3 \neq 1} \tilde{Z}(1,j_1)\tilde{\tau}(j_1)\tilde{Z}(j_1,j_2)\tilde{\tau}(j_2)\tilde{Z}(j_2,j_3)\tilde{\tau}(j_3)\tilde{Z}(j_3,1) + \cdots
$$









$$x_1 = \frac{3a_0}{\pi} \quad \Longrightarrow \quad mE = \frac{12\pi a}{L^3}\left(1 + \frac{x_1}{L/a} + \frac{x_2}{(L/a)^2} + \frac{x_3' Log[L/a]}{(L/a)^3} + \frac{x_3}{(L/a)^3}\right)$$

$$\tilde{\tau}^{-1}(1) - \tilde{Z}(1,1) = \sum_{j_1 \neq 1} \tilde{Z}(1,j_1)\,\tilde{\tau}(j_1)\,\tilde{Z}(j_1,1) + \sum_{j_1, j_2 \neq 1} \tilde{Z}(1,j_1)\,\tilde{\tau}(j_1)\,\tilde{Z}(j_1,j_2)\,\tilde{\tau}(j_2)\,\tilde{Z}(j_2,1)$$
$$+ \sum_{j_1,j_2,j_3 \neq 1} \tilde{Z}(1,j_1)\tilde{\tau}(j_1)\tilde{Z}(j_1,j_2)\tilde{\tau}(j_2)\tilde{Z}(j_2,j_3)\tilde{\tau}(j_3)\tilde{Z}(j_3,1) + \cdots$$

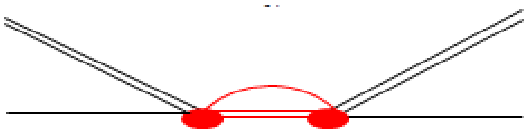
$$x_1 = \frac{3a_0}{\pi} \quad \Longrightarrow \quad mE = \frac{12\pi a}{L^3}\left(1 + \frac{x_1}{L/a} + \frac{x_2}{(L/a)^2} + \frac{x_3' Log[L/a]}{(L/a)^3} + \frac{x_3}{(L/a)^3}\right)$$

$$\tilde{\tau}^{-1}(1) - \tilde{Z}(1,1) = \sum_{j_1 \neq 1} \tilde{Z}(1,j_1)\,\tilde{\tau}(j_1)\,\tilde{Z}(j_1,1) + \sum_{j_1,j_2 \neq 1} \tilde{Z}(1,j_1)\,\tilde{\tau}(j_1)\,\tilde{Z}(j_1,j_2)\,\tilde{\tau}(j_2)\,\tilde{Z}(j_2,1)$$

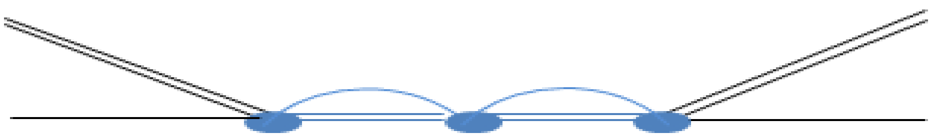
$$+ \sum_{j_1,j_2,j_3 \neq 1} \tilde{Z}(1,j_1)\tilde{\tau}(j_1)\tilde{Z}(j_1,j_2)\tilde{\tau}(j_2)\tilde{Z}(j_2,j_3)\tilde{\tau}(j_3)\tilde{Z}(j_3,1) + \cdots$$

$$\tilde{\tau}^{-1}(1) - \tilde{Z}(1,1) = \frac{1}{L/a}\left(-x_1 - \frac{\mathbb{I}(\vec{0})}{\pi}\right) + \frac{1}{(L/a)^2}\left(-x_2 + x_1^2 - \frac{3\mathbb{J}(\vec{0})}{\pi^2}\right) + \frac{-x_3' \log[L/a]}{(L/a)^3}$$

$$+ \frac{1}{(L/a)^3}\left(-x_3 + 2x_1x_2 - x_1^3 + \frac{6\pi r}{a} - \frac{1}{\pi^3}\left(3x_1\pi\mathbb{J}(\vec{0}) + 9\mathbb{K}(\vec{0})\right)\right) - \frac{1}{(L/a)^3}\left(\frac{8\pi H_0(\Lambda)}{a^2\Lambda^2}\right) + \mathbb{O}(L^{-3-\epsilon})$$



$$= -\frac{4\mathbb{J}(\vec{0})}{(\pi L/a)^2} - \frac{16\sqrt{3}\log(L/a)}{(L/a)^3} + \frac{1}{(\pi L/a)^3}(X_0(\Lambda) + C_2 - 24\mathbb{K}(0)) + \mathbb{O}(L^{-3-\epsilon})$$



$$= \frac{64\pi\log(L/a)}{3(L/a)^3} + \frac{1}{(\pi L/a)^3}X_1(\Lambda) + \mathbb{O}(L^{-3-\epsilon})$$

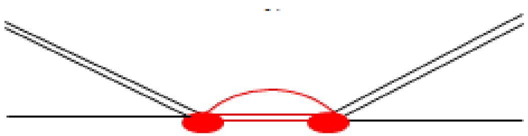
$$= X_2(\Lambda)\frac{1}{(\pi L/a)^3} + \mathbb{O}(L^{-3-\epsilon})$$

$$x_1 = \frac{3a_0}{\pi} \Longrightarrow mE = \frac{12\pi a}{L^3}\left(1+\frac{x_1}{L/a}+\frac{x_2}{(L/a)^2}+\frac{x_3' Log[L/a]}{(L/a)^3}+\frac{x_3}{(L/a)^3}\right)$$

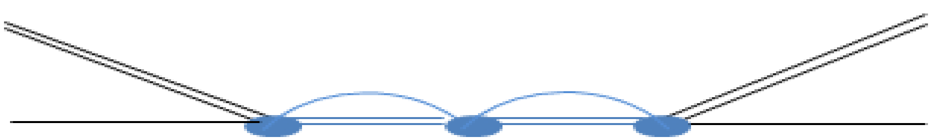
$$\tilde{\tau}^{-1}(1)-\tilde{Z}(1,1) = \sum_{j_1\neq 1}\tilde{Z}(1,j_1)\,\tilde{\tau}(j_1)\,\tilde{Z}(j_1,1)+\sum_{j_1,j_2\neq 1}\tilde{Z}(1,j_1)\,\tilde{\tau}(j_1)\,\tilde{Z}(j_1,j_2)\,\tilde{\tau}(j_2)\,\tilde{Z}(j_2,1)$$

$$+\sum_{j_1,j_2,j_3\neq 1}\tilde{Z}(1,j_1)\tilde{\tau}(j_1)\tilde{Z}(j_1,j_2)\tilde{\tau}(j_2)\tilde{Z}(j_2,j_3)\tilde{\tau}(j_3)\tilde{Z}(j_3,1)+\cdots$$

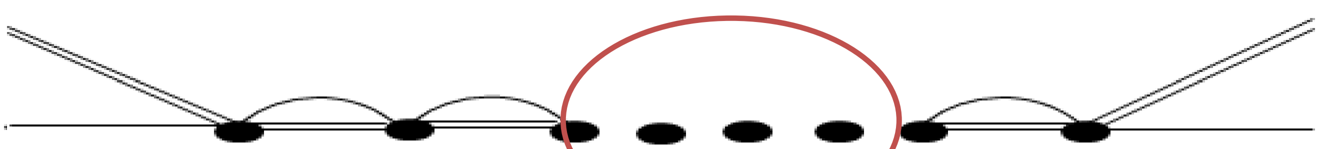
$$\tilde{\tau}^{-1}(1)-\tilde{Z}(1,1) = \frac{1}{L/a}\left(-x_1-\frac{\mathbb{I}(\vec{0})}{\pi}\right)+\frac{1}{(L/a)^2}\left(-x_2+x_1^2-\frac{3\mathbb{J}(\vec{0})}{\pi^2}\right)+\frac{-x_3'\log[L/a]}{(L/a)^3}$$
$$+\frac{1}{(L/a)^3}\left(-x_3+2x_1x_2-x_1^3+\frac{6\pi r}{a}-\frac{1}{\pi^3}\left(3x_1\pi\mathbb{J}(\vec{0})+9\mathbb{K}(\vec{0})\right)\right)-\frac{1}{(L/a)^3}\left(\frac{8\pi H_0(\Lambda)}{a^2\Lambda^2}\right)+\mathbb{O}(L^{-3-\epsilon})$$



$$= -\frac{4\mathbb{J}(\vec{0})}{(\pi L/a)^2}-\frac{16\sqrt{3}\log(L/a)}{(L/a)^3}+\frac{1}{(\pi L/a)^3}(X_0(\Lambda)+C_2-24\mathbb{K}(0))+\mathbb{O}(L^{-3-\epsilon})$$



$$= \frac{64\pi\log(L/a)}{3(L/a)^3}+\frac{1}{(\pi L/a)^3}X_1(\Lambda)+\mathbb{O}(L^{-3-\epsilon})$$



$$= X_2(\Lambda)\frac{1}{(\pi L/a)^3}+\mathbb{O}(L^{-3-\epsilon})$$

Non-perturbation

Energy shift of Ground state of 3-body scattering state

$$mE = \frac{12\pi a}{L^3}\left(1 + \frac{x_1}{L/a} + \frac{x_2}{(L/a)^2} + \frac{x_3' \log[L/a]}{(L/a)^3} + \frac{x_3}{(L/a)^3} + \cdots\right)$$

$$x_3 = \mathbf{Q}\, X(\Lambda) + \mathbf{R}$$

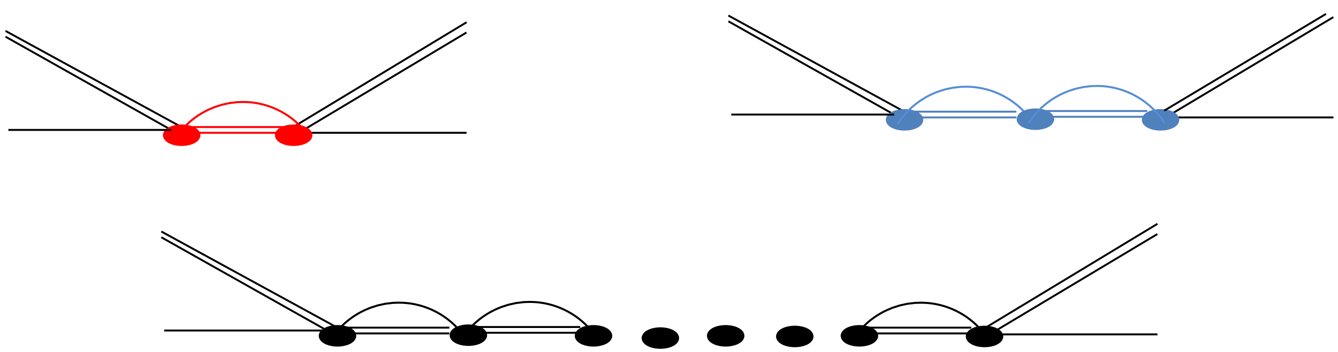
Cutoff Dependence

$$X(\Lambda) = \frac{H_0(\Lambda)}{\Lambda^2} + \sum_{\vec{i}\in\mathbb{Z}^3}^{[1,N]} \frac{8\pi}{L^3} z(\vec{0},\vec{i},0)\tau(\vec{i},0)M^L(\vec{i},\vec{0}),$$

$$M^L(\vec{j},\vec{0}) = z(\vec{j},\vec{0},0) + \sum_{\vec{i}\in\mathbb{Z}^3}^{[1,N]} \frac{8\pi}{L^3} z(\vec{j},\vec{i},0)\tau(\vec{i},0)M^L(\vec{i},\vec{0}).$$

$$z(\vec{i},\vec{j},n) = \frac{L^2}{4\pi^2}\frac{1}{\vec{i}^2+\vec{j}^2+\vec{i}\cdot\vec{j}-n} + \frac{H_0}{\Lambda^2}$$

$$\tau(\vec{i},n) = -\frac{1}{a} + \frac{\pi}{L}\sqrt{3\vec{i}^2-4n}$$

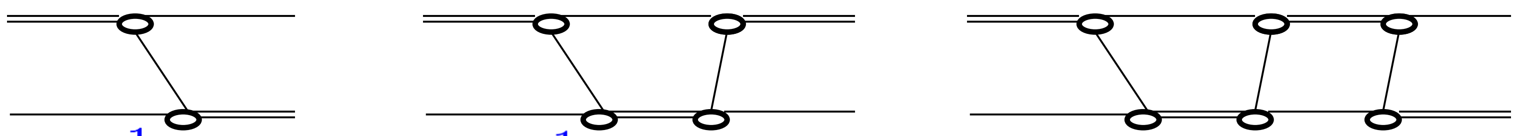


X_3 : Λ independence

$$X(\Lambda) = \frac{H_0(\Lambda)}{\Lambda^2} + \sum_{\vec{i}\in\mathbb{Z}^3}^{[1,N]} \frac{8\pi}{L^3} z(\vec{0},\vec{i},0)\tau(\vec{i},0)M^L(\vec{i},\vec{0}), \quad \sim \quad M(\vec{0},\vec{0},0)$$

Divergent From

$$M^L(\vec{j},\vec{0}) = z(\vec{j},\vec{0},0) + \sum_{\vec{i}\in\mathbb{Z}^3}^{[1,N]} \frac{8\pi}{L^3} z(\vec{j},\vec{i},0)\tau(\vec{i},0)M^L(\vec{i},\vec{0}).$$



$\frac{1}{\epsilon^2}$ — Tree Diagram

$\frac{1}{\epsilon}$ & $\log[\epsilon]$ — One loop Diagram,

$\log[\epsilon]$ — Two loops Diagram.

$$\mathcal{A}_b = \lim_{\epsilon\to 0^+}\left(M_\epsilon(\vec{0},\vec{0},0) + \frac{4a}{\pi\epsilon} + \left(\frac{8}{3} - \frac{2\sqrt{3}}{\pi}\right) a^2 \log\left(\frac{a\epsilon}{2\pi}\right)\right) - a^2\frac{112\zeta(3)}{9}$$

$$M_\epsilon(\vec{0},\vec{0},0) \equiv \frac{H_0(\Lambda)}{\Lambda^2} + 8\pi\int_\epsilon^\Lambda \frac{d^3\vec{k}}{(2\pi)^3} Z(\vec{0},\vec{k},0)\tau(\vec{k},0)M(\vec{k},\vec{0},0)$$

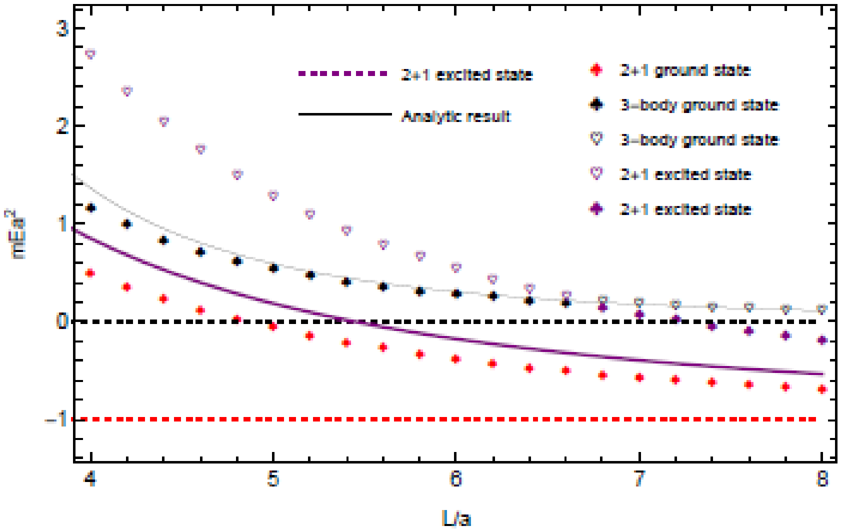
$$X(\Lambda) = \mathbf{Q}'\mathcal{A} + \mathbf{R}'$$

X_3 : Λ independence

$$x_3 = +\frac{1}{\pi^3}\left(-\mathbb{I}^3(0) + \mathbb{I}(0)\mathbb{J}(\vec{0}) + 15\mathbb{K}(\vec{0}) + C_2\right) - \frac{8\pi}{a^2}\mathcal{A} - 8\pi\left(X_1 - \frac{\sqrt{3}\mathbb{I}_3}{2\pi^2}\right)$$

amplitude $\mathcal{M}_{3,\mathrm{thr}}$, which begins at $\mathcal{O}(\lambda^2)$ in perturbation theory. The numerical values of $\mathcal{C}_3$, $\mathcal{C}_4$, and $\mathcal{C}_5$ depend on the choice of UV cutoff, but this dependence cancels with that of $\mathcal{M}_{3,\mathrm{thr}}$. This cancellation is necessary because $\Delta E_{3,\mathrm{thr}}$ is a physical quantity.

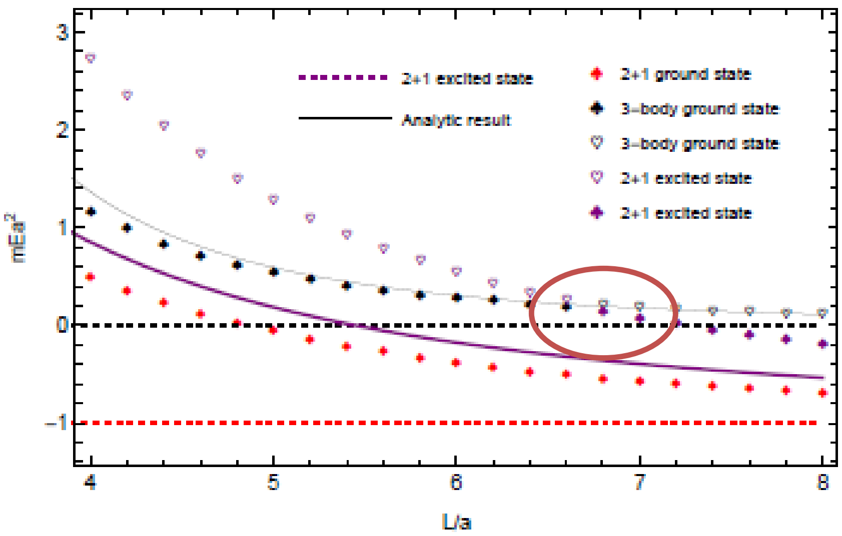
Energy Shift of 3-body Ground state

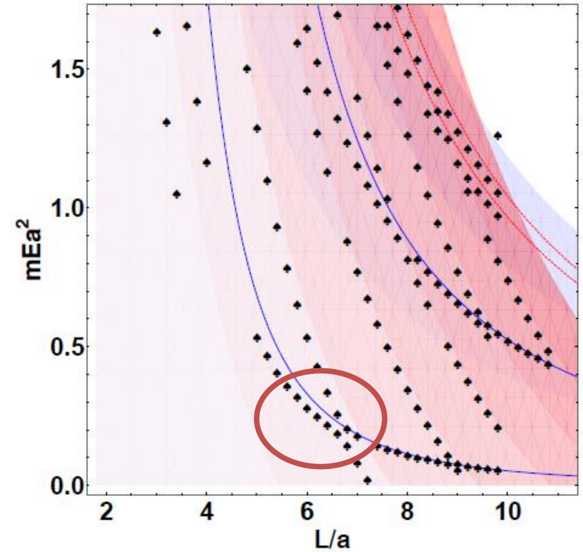


Ground State: $\kappa^2 = \frac{g_0}{L}\left(1 + \frac{g_1}{L} + \frac{g_2}{L^2} + \frac{g_3}{L^3}\ln\frac{mL}{2\pi} + \frac{g_4}{L^3} + \cdots\right)$

where $g_4 = \left(-5.159159617 + 6\pi\left(\frac{r}{a}\right) - 8\pi\left(\frac{\hat{\mathscr{M}}}{a^2}\right)\right)a^3$

More Energy Shift





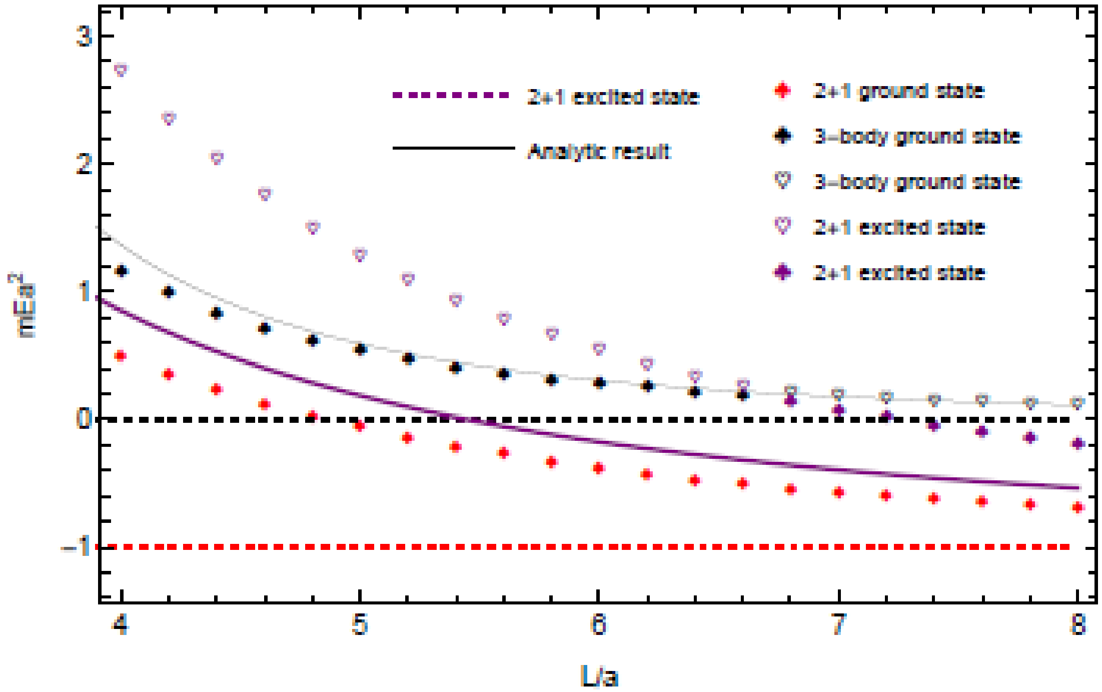
$$\text{Ground State: } \kappa^2 = \frac{g_0}{L}\left(1 + \frac{g_1}{L} + \frac{g_2}{L^2} + \frac{g_3}{L^3}\ln\frac{mL}{2\pi} + \frac{g_4}{L^3} + \cdots\right)$$

$$\text{Excited State: } \kappa^2 - 1 = \frac{h_0}{L}\left(1 + \frac{h_1}{L} + \frac{h_2}{L^2} + \frac{h_3}{L^3}\ln\frac{mL}{2\pi} + \frac{h_4}{L^3} + \cdots\right),$$

$$\text{where } g_4 = \left(-5.159159617 + 6\pi\left(\frac{r}{a}\right) - 8\pi\left(\frac{\hat{\mathscr{M}}}{a^2}\right)\right)a^3$$

$$h_4 = \left(-172.001650 + 83.745841\left(\frac{r}{a}\right) - \frac{27}{5}\times 8\pi\left(\frac{\hat{\mathscr{M}}}{a^2}\right)\right)a^3$$

More Energy Shift



Purple: excite state of dimer-particle

Black: ground state of three particle

Red: ground state of dimer-particle scattering state

$$E_{1d}(L) - E_{1d}^{\text{free}}(L) = \frac{\mathcal{A}_d}{L^3} + O\left(L^{-4}\right)$$

Ad denotes the particle-dimer scattering amplitude at threshold

Summary

- In the infinite volume, we build the scattering equation of dimer particle amplitude to describe the 3-body system based on the non-relativistic effective field theory.

- By considering the symmetry and the discrete momentum in the box, the quantization condition for the energy level of 3-body system is derived.

- Through this quantization condition, we derived the lowest energy shift of the 3-body scattering state, and prove it is cutoff independence.

Outlook

- Non-relativistic -> relativistic
- S-wave dimer -> high partial wave dimer

- To apply this method to study real 3-body system, such as 3- pions, it still needs more efforts.

THANKS VERY MUCH

3-body System in the finite volume

There are 48 rotation operators $\hat{g} \in O_h$,

(x,y,z) => (x,y,z) , (x,y,-z), (x,-y,z), (x,-y,-z), (-x,y,z) , (-x,y,-z), (-x,-y,z), (-x,-y,-z)

(y,x,z) , (y,x,-z), (y,-x,z), (y,-x,-z), (-y,x,z) , (-y,x,-z), (-y,-x,z), (-y,-x,-z)

……

=> 8 x 6 = 48 vectors

Irreducible Representation Γ, $A_1^{\pm},\ A_2^{\pm}\ E^{\pm},\ T_1^{\pm},\ T_2^{\pm}$

$\vec{n}$ is on the shell s, and $\vec{n}_0$ is the special vector on the shell s

$$f(\vec{n}) = f(\hat{g}\vec{n}_0) = \sum_{\Gamma}\sum_{j} T_{ij}^{\Gamma}(\hat{g}) f_j^{\Gamma,i}(\vec{n}_0(s))$$

$$\frac{G}{D_\Gamma} f_j^{\Gamma,i}(\vec{n}_0(s)) = \sum_{g\in O_h} T_{ij}^{\Gamma\,*}(\hat{g}) f(\hat{g}\vec{n}_0)$$

3-body System in the finite volume

$$Z_L(\vec{n},\vec{m},E) = Z_L(\hat{g}\vec{n},\hat{g}\vec{m},E)$$

$$Z^{\Gamma}_{L\,ij}(r,s,E) = \sum_{g\in O_h} T^{\Gamma\,*}_{ji}(\hat{g}) Z_L(\hat{g}\vec{n}_0(r),\vec{m}_0(s),E)$$

- Quantization condition: $\det(\tau_L^{-1} - \frac{8\pi}{L^3} Z_L) = 0$

$$\det\left(\theta^{-1}(s)\tau_L^{-1}(r,E)\delta_{rs}\delta_{ij} - \frac{8\pi}{L^3}\frac{1}{G} Z^{\Gamma}_{L\,ij}(r,s,E)\right) = 0$$

$$\sum_{\vec{n}\in\mathcal{Z}^3} f(\vec{n}) = \sum_{s}\sum_{g\in O_h} \frac{\theta(s)}{G} f(\hat{g}\vec{n}_0(s))$$

3-body System in the finite volume

- Results of Toy model && $\Gamma = A_1^+$

$$\det\left(\theta^{-1}(s)\tau_L^{-1}(r,E)\delta_{rs}\delta_{ij} - \frac{8\pi}{L^3}\frac{1}{G}Z^{\Gamma}_{L\,ij}(r,s,E)\right) = 0$$

$$\Longrightarrow$$

$$\det\left(\theta^{-1}(s)\tau_L^{-1}(r,E)\delta_{rs} - \frac{8\pi}{L^3}\frac{1}{G}Z_L^{A_1^+}(r,s,E)\right) = 0$$

$$\tau_L^{-1}(r,E) = -\frac{1}{a_0} - \frac{4\pi}{L^3}\sum_{\vec{l}}\frac{1}{\frac{4\pi^2}{L^2}\left(\vec{n}_0^2(r) + \vec{l}^2 + \vec{n}_0(r)\cdot\vec{l}\right) - mE}$$

$$Z_L^{A_1^+}(r,s,E) = \sum_{g\in O_h} Z_L(\hat{g}\vec{n}_0(r), \vec{n}_0(s), E)$$

Energy shift of Ground state of 3-body scattering state

$$mE = \frac{12\pi a}{L^3}\left(1 + \frac{x_1}{L/a} + \frac{x_2}{(L/a)^2} + \frac{x_3' \log[L/a]}{(L/a)^3} + \frac{x_3}{(L/a)^3} + \cdots\right)$$

$$x_1 = -\frac{\mathbb{I}(0)}{\pi} \quad x_2 = \frac{\mathbb{I}^2(0) + \mathbb{J}^2(0)}{\pi^2} \quad x_3' = 16\sqrt{3} - 64\pi/3$$

$$x_3 = +\frac{6\pi r}{a} + \frac{1}{\pi^3}\left(-\mathbb{I}^3(0) + \mathbb{I}(0)\mathbb{J}(\vec{0}) + 15\mathbb{K}(\vec{0}) + C_2\right)$$

$$-\frac{1}{\pi^3}(X_0(\Lambda) + X_1(\Lambda) + X_2(\Lambda)) - \frac{8\pi H_0(\Lambda)}{a^2\Lambda^2}$$

$$-(L/a)^3\left(X(\Lambda)\frac{8\pi a}{L^3} + \frac{4\mathbb{J}(\vec{0})}{(\pi L/a)^2} + \frac{(16\sqrt{3} - 64\pi/3)\log(L/a)}{(L/a)^3}\right)$$

$$X(\Lambda) = \frac{H_0(\Lambda)}{\Lambda^2} + \sum_{\vec{i}\in\mathbb{Z}^3}^{[1,N]} \frac{8\pi}{L^3} z(\vec{0},\vec{i},0)\tau(\vec{i},0)M^L(\vec{i},\vec{0}),$$

Cutoff dependence

$$M^L(\vec{j},\vec{0}) = z(\vec{j},\vec{0},0) + \sum_{\vec{i}\in\mathbb{Z}^3}^{[1,N]} \frac{8\pi}{L^3} z(\vec{j},\vec{i},0)\tau(\vec{i},0)M^L(\vec{i},\vec{0}).$$