Models for Dalitz plot analyses

Alessandro Pilloni

Indiana University Gateway Center, Beijing, April 3rd, 2019
Recipes to build an amplitude

M. Mikhasenko, AP, J. Nys et al. (JPAC), EPJC78, 3, 229
AP, J. Nys, M. Mikhasenko et al. (JPAC), EPJC78, 9, 727

The literature abounds with discussions on the optimal approach to construct the amplitudes for the hadronic reactions

- **Helicity formalism**
  
  Jacob, Wick, Annals Phys. 7, 404 (1959)

- **Covariant tensor formalisms**
  
  Chung, PRD48, 1225 (1993)
  Chung, Friedrich, PRD78, 074027 (2008)
  Anisovich, Sarantsev, EPJA30, 427 (2006)

The common lore is that the former one is nonrelativistic, especially when expressed in terms of LS couplings and the latter takes into account the proper relativistic corrections
How helicity formalism works

- Helicity formalism enforces the constraints about rotational invariance
- It allows us to fix the angular dependence of the amplitude
- What about energy dependence?

Example: $B \rightarrow \psi K^* \rightarrow \pi K$

Each set of angles is defined in a different reference frame

\[
\mathcal{M}^{K^*}_{\Delta \lambda \mu} \equiv \sum_n \sum_{\lambda_{K^*}} \sum_{\lambda_\psi} \mathcal{H}^{B \rightarrow K^*_n \psi}_{\lambda_{K^*}, \lambda_\psi} \delta_{\lambda_{K^*}, \lambda_\psi}
\]

\[
\mathcal{H}^{K^*_n \rightarrow K \pi}_{\lambda_{K^*}, 0} D^{J_{K^*_n}}_{\lambda_{K^*}, 0} (\phi_{K^*}, \theta_{K^*}, 0)^* \\
R_{K^*_n (m_{K^*_n})} D^{1}_{\lambda_\psi, \Delta \mu} (\phi_\mu, \theta_\psi, 0)^*
\]

Each set of angles is defined in a different reference frame
How tensor formalism works

The method is based on the construction of explicitly covariant expressions.

- To describe the decay $a \rightarrow bc$, we first consider the polarization tensor of each particle, $\varepsilon_{\mu_1...\mu_j}^i(p_i)$
- We combine the polarizations of $b$ and $c$ into a “total spin” tensor $S_{\mu_1...\mu_S}(\varepsilon_b, \varepsilon_c)$
- Using the decay momentum, we build a tensor $L_{\mu_1...\mu_L}(p_{bc})$ to represent the orbital angular momentum of the $bc$ system, orthogonal to the total momentum of $p_a$
- We contract $S$ and $L$ with the polarization of $a$

$\text{Tensor } \times R_X(m)$ which contain resonances and form factors
What do we know?

- Energy dependence is not constrained by symmetry
- Still, there are some known properties one can enforce

\[ R_X(m) = B'_{LX}^\Lambda(p, p_0, d) \left( \frac{p}{M^0_{\Lambda_b}} \right)^{L_X^{0\Lambda_b}} \]

\[ BW(m| M_{0X}, \Gamma_{0X}) B'_{LX}^\Lambda(q, q_0, d) \left( \frac{q}{M_{0X}} \right)^{L_X} \]

- **Kinematical singularities**: e.g. barrier factors (known)
- **Left hand singularities** (need model, e.g. Blatt-Weisskopf)
- **Right hand singularities** = resonant content (Breit Wigner, K-matrix...)

A. Pilloni – Reaction theory and analysis methods
Kinematics

- Kinematical singularities appear because of the spin of the external particle involved.
- We can write the most general covariant parametrization of the amplitude as a tensor of external polarizations $\otimes$ scalar amplitudes.
- Scalar amplitudes must be kinematical singularities free.
- They can be matched to the helicity amplitudes.
- We can get the minimal energy dependent factor.
- Any other additional energy factor would be model-dependent.
To consider the effect of spin, let’s consider $B \rightarrow \psi \pi K$

We focus on the parity violating amplitude for the $K^*$ isobars, scattering kinematics

\[
p = \text{incoming 3-momentum in the COM} = \frac{\lambda_{12}^{1/2}}{2\sqrt{s}}
\]

\[
= \frac{\sqrt{[s - (m_1 + m_2)^2][s - (m_1 - m_2)^2]}}{2\sqrt{s}}
\]
$B \rightarrow \psi \pi K$

To consider the effect of spin, let’s consider $B \rightarrow \psi \pi K$
We focus on the parity violating amplitude for the $K^*$ isobars, scattering kinematics

$q = \text{outgoing 3-momentum in the COM} = \frac{\lambda^{1/2}_{34}}{2\sqrt{s}}$

$$q = \sqrt{[s - (m_3 + m_4)^2][s - (m_3 - m_4)^2]} \quad \frac{1}{2\sqrt{s}}$$
$B \to \psi \pi K$

To consider the effect of spin, let's consider $B \to \psi \pi K$

We focus on the parity violating amplitude for the $K^*$ isobars, scattering kinematics

$$z_s = \cos \text{ of the scatt. angle in the COM}$$

$$= \frac{s(t-u) + (m_1^2 - m_2^2)(m_3^2 - m_4^2)}{\lambda_{12}^{1/2} \lambda_{34}^{1/2}} = \text{polynomial}$$
Helicity amplitudes

\[ A_\lambda = \frac{1}{4\pi} \sum_{j=|\lambda|}^{\infty} (2j + 1) A^j_\lambda(s) \ d^j_{\lambda0}(z_s) \]

\[ d^j_{\lambda0}(z_s) = \hat{d}^j_{\lambda0}(z_s) \xi_{\lambda0}(z_s), \quad \xi_{\lambda0}(z_s) = \left(\sqrt{1 - z_s^2}\right)^\lambda \]

\( \hat{d}^j_{\lambda0}(z_s) \) is a polynomial of order \( j - |\lambda| \) in \( z_s \),

The kinematical singularities of \( A^j_\lambda(s) \) can be isolated by writing

\[ A^j_0 = \frac{m_1}{p\sqrt{s}} \ (pq)^j \ \hat{A}^j_0 \quad \text{for } j \geq 1, \]

\[ A^j_\pm = q \ (pq)^j^{-1} \ \hat{A}^j_\pm \quad \text{for } j \geq 1, \]

\[ A^0_0 = \frac{p\sqrt{s}}{m_1} \ \hat{A}^0_0 \quad \text{for } j = 0, \]
Identify covariants

Two helicity couplings → two independent covariant structures

Important: we are not imposing any intermediate isobar

\[ A_\lambda(s, t) = \varepsilon_\mu(\lambda, p_1) \left[ (p_3 - p_4)^\mu - \frac{m_3^2 - m_4^2}{s} (p_3 + p_4)^\mu \right] C(s, t) \]

\[ + \varepsilon_\mu(\lambda, p_1) (p_3 + p_4)^\mu B(s, t) \]

\[ C(s, t) = \frac{1}{4\pi \sqrt{2}} \sum_{j>0} (2j + 1)(pq)^{j-1} \hat{A}_+^j(s) \hat{d}_{10}^j(z_s) \]

\[ B(s, t) = \frac{1}{4\pi} \hat{A}_0^0 + \frac{1}{4\pi} \frac{4m_1^2}{\lambda_{12}} \sum_{j>0} (2j + 1)(pq)^j \left[ \hat{A}_0^j(s) \hat{d}_{00}^j(z_s) + \frac{s + m_1^2 - m_2^2}{\sqrt{2}m_1^2} \hat{A}_+^j(s) z_s \hat{d}_{10}^j(z_s) \right] \]

Everything looks fine but the \( \lambda_{12} \) in the denominator

The brackets must vanish at \( \lambda_{12} = 0 \Rightarrow s = s_\pm = (m_1 \pm m_2)^2 \), \( \hat{A}_+^j \) and \( \hat{A}_0^0 \) cannot be independent
General expression and comparison

\[ \hat{A}^j_+ = \langle j - 1, 0; 1, 1 | j, 1 \rangle g_j(s) + f_j(s) \]

\[ \hat{A}^j_0 = \langle j - 1, 0; 1, 0 | j, 0 \rangle \frac{s + m_1^2 - m_2^2}{2m_1^2} g_j'(s) + f_j'(s) \]

\[ g_j(s_\pm) = g_j'(s_\pm), \text{ and } f_j(s), f_j'(s) \sim O(s - s_\pm) \]

All these four functions are free of kinematic singularity.

Comparison with tensor formalisms \((j = 1)\)

\[ g_1 = g_1' = \frac{4\pi}{3} g_S, \quad f_1 = \frac{2\pi \lambda_{12}}{3s} g_D, \quad f_1' = -\frac{4\pi \lambda_{12}}{3s} \frac{s + m_1^2 - m_2^2}{m_1^2} g_D. \]

If the \(g_S, g_D\) are the usual Breit-Wigner, the \(g', f'\) are fine

There is no unique recipe to build the right amplitude, but one can ensure the right singularities to be respected
We consider the example of two intermediate $K^*(892)$ and $K^*(1410)$

We set $g_S(s) = 0$ and $g_D(s) =$ sum of Breit-Wigner

For the plot on the right we multiply the amplitudes by the Blatt-Weisskopf barrier factors
The rescattering with the bachelor particle is known to modify the isobar lineshape (Khuri-Treiman equations)

I. Danilkin et al. (JPAC) PRD91, 094029
3-body unitarity

\[ \text{2 Im} = \sum_{n} + \sum_{n \neq r} + \sum_{n \neq j} + \sum_{r \neq k} \delta_{jk} \]

M. Mai, B. Hu, M. Doring, AP, A. Szczepaniak EPJA53, 9, 177
A. Jackura, ..., AP, et al. (JPAC), EPJC79, 1, 56
Factorizable model

= \sum_n + \sum_{n,r \neq r} + \sum_{n \neq j} + \sum_{r \neq k} (1 - \delta_{jk})

If one neglects the disconnected diagrams, one can suppress the dependence on the 2-body invariant masses.

The unitarity equation is now algebraic and easier to handle.

\[ \text{Im} \hat{A}(s) = \hat{A}(s) \hat{A}(s) \mathbf{^\dagger} \int d\Phi_3 \left| \sum_j f(\sigma_j) \right|^2 \]

Integral over the Dalitz plot (aka quasi 2-body)

M. Mikhasenko, AP, et al. (JPAC), PRD98 (2018) 9, 096021
M. Mikhasenko et al. (JPAC) to appear
Factorizable model

\[ \sum_n + \sum_{n \neq r} + \sum_{n \neq j} + \sum_{r \neq k} (1 - \delta_{jk}) \]

\[ \text{Im} \hat{A}(s) = \hat{A}(s) \hat{A}(s)^\dagger \int d\Phi_3 \left| \sum_j f(\sigma_j) \right|^2 \]

M. Mikhasenko, AP, et al. (JPAC), PRD98 (2018) 9, 096021
M. Mikhasenko et al. (JPAC) to appear

The approximation can be relaxed, but one can still obtain a (complicated) factorized form

The quasi 2-body gets corrections from rescattering
Amplitude analysis for $Z_c(3900)$

One can test different parametrizations of the amplitude, which correspond to different singularities $\rightarrow$ different natures

\[ Y \xrightarrow{D} \bar{D} \]

Triangle rescattering, logarithmic branching point

\[ s \quad t \quad D^* \]

(anti)bound state, II/IV sheet pole («molecule»)

\[ Y \xrightarrow{J/\psi} \]

Resonance, III sheet pole («compact state»)

AP et al. (JPAC), PLB772, 200

Szczepaniak, PLB747, 410-416
Szczepaniak, PLB757, 61-64
Guo et al. PRD92, 071502

Tornqvist, Z.Phys. C61, 525
Swanson, Phys.Rept. 429
Hanhart et al. PRL111, 132003

Maiani et al., PRD71, 014028
Faccini et al., PRD87, 111102
Esposito et al., Phys.Rept. 668
Amplitude model

\[ f_i(s, t, u) = 16\pi \sum_{l=0}^{L_{\text{max}}} (2l + 1) \left( a^{(s)}_{l,i}(s) P_l(z_s) + a^{(t)}_{l,i}(t) P_l(z_t) + a^{(u)}_{l,i}(u) P_l(z_u) \right) \]

\[ f_{0,i}(s) = \frac{1}{32\pi} \int_{-1}^{1} dz_s f_i(s, t(s, z_s), u(s, z_s)) = a^{(s)}_{0,i} + \frac{1}{32\pi} \int_{-1}^{1} dz_s \left( a^{(t)}_{0,i}(t) + a^{(u)}_{0,i}(u) \right) \equiv a^{(s)}_{0,i} + b_{0,i}(s) \]

\[ f_{l,i}(s) = \frac{1}{32\pi} \int_{-1}^{1} dz_s P_l(z_s) \left( a^{(t)}_{0,i}(t) + a^{(u)}_{0,i}(u) \right) \equiv b_{l,i}(s) \quad \text{for } l > 0. \quad f_{0,i}(s) = b_{0,i}(s) + \sum_j t_{ij}(s) \frac{1}{\pi} \int_{s_j}^{s_{j+1}} ds' \frac{\rho_j(s') b_{0,j}(s')}{s' - s}, \]

\[ f_i(s, t, u) = 16\pi \left[ a^{(t)}_{0,i}(t) + a^{(u)}_{0,i}(u) + \sum_j t_{ij}(s) \left( c_j + \frac{s}{\pi} \int_{s_j}^{s_{j+1}} ds' \frac{\rho_j(s') b_{0,j}(s')}{s' (s' - s)} \right) \right], \]
Logarithmic branch points due to exchanges in the cross channels can simulate a resonant behavior, only in very special kinematical conditions (Coleman and Norton, Nuovo Cim. 38, 438). However, this effect cancels in Dalitz projections, no peaks (Schmid, Phys.Rev. 154, 1363).

...but the cancellation can be spread in different channels, you might still see peaks in other channels only!

\[ f_{0,i}(s) = b_{0,i}(s) + \frac{t_{ij}}{\pi} \int_{s_i}^{\infty} ds' \frac{\rho_j(s') b_{0,j}(s')}{s' - s} \]

Szczepaniak, PLB747, 410-416
Szczepaniak, PLB757, 61-64
Guo, Meissner, Wang, Yang PRD92, 071502
Testing scenarios

- We approximate all the particles to be scalar – this affects the value of couplings, which are not normalized anyway – but not the position of singularities. This also limits the number of free parameters.

\[ f_i(s, t, u) = 16\pi \left[ a_{0,i}^{(t)}(t) + a_{0,i}^{(u)}(u) + \sum_j t_{ij}(s) \left( c_j + \frac{s}{\pi} \int_{s_j}^{\infty} ds' \frac{\rho_j(s')b_{0,j}(s')}{s'(s' - s)} \right) \right], \]

The scattering matrix is parametrized as \((t^{-1})_{ij} = K_{ij} - i \rho_i \delta_{ij}\)

Four different scenarios considered:

- «III»: the K matrix is \(\frac{g_i g_j}{M^2 - s'}\), this generates a pole in the closest unphysical sheet, the rescattering integral is set to zero.
- «III+tr.»: same, but with the correct value of the rescattering integral.
- «IV+tr.»: the K matrix is constant, this generates a pole in the IV sheet.
- «tr.»: same, but the pole is pushed far away by adding a penalty in the \(\chi^2\).
Fit: III

- $E_{CM} = 4.26$ GeV
- $E_{CM} = 4.23$ GeV

A. Pilloni – Models for Dalitz plot analyses
Fit: III+tr.

A. Pilloni – Models for Dalitz plot analyses
Fit: IV+tr.
Fit: tr.

A. Pilloni – Models for Dalitz plot analyses
Fit summary

<table>
<thead>
<tr>
<th>Scenario</th>
<th>$\chi^2$</th>
<th>DOF</th>
<th>$\chi^2$/DOF</th>
</tr>
</thead>
<tbody>
<tr>
<td>III</td>
<td>644</td>
<td>532</td>
<td>1.21</td>
</tr>
<tr>
<td>III+tr.</td>
<td>642</td>
<td>532</td>
<td>1.21</td>
</tr>
<tr>
<td>IV+tr.</td>
<td>666</td>
<td>532</td>
<td>1.25</td>
</tr>
<tr>
<td>tr.</td>
<td>695</td>
<td>532</td>
<td>1.31</td>
</tr>
</tbody>
</table>

Naive loglikelihood ratio test give a $\sim 4\sigma$ significance of the scenario III+tr. over IV+tr., looking at plots it looks too much – better using some more solid test.
Pole extraction

<table>
<thead>
<tr>
<th>Scenario</th>
<th>III+tr.</th>
<th>IV+tr.</th>
<th>tr.</th>
</tr>
</thead>
<tbody>
<tr>
<td>III</td>
<td>$1.5\sigma$ ($1.5\sigma$)</td>
<td>$1.5\sigma$ ($2.7\sigma$)</td>
<td>“2.4\sigma” (”1.4\sigma”)</td>
</tr>
<tr>
<td>III+tr.</td>
<td>–</td>
<td>$1.5\sigma$ ($3.1\sigma$)</td>
<td>“2.6\sigma” (”1.3\sigma”)</td>
</tr>
<tr>
<td>IV+tr.</td>
<td>–</td>
<td>–</td>
<td>“2.1\sigma” (”0.9\sigma”)</td>
</tr>
</tbody>
</table>

Not conclusive at this stage
How to improve the model

Molnar, Danilkin, Vanderhaeghen, 1903.08458

Ex. $e^+ e^- \rightarrow \psi' \pi\pi$

Khuri-Treiman with spin, Omnès function for $\pi\pi$

\[
\mathcal{H}_{\lambda_1 \lambda_2}(s, t, u) \approx \sum_{J_{\text{even}}}^{J_{\text{max}}} (2J + 1) \\
\times \left\{ h^{(J),s}_{\lambda_1 \lambda_2}(s) d^{(J)}_{\Delta,0}(\theta_s) + h^{(J),t}_{\lambda_1 \lambda_2}(t) d^{(J)}_{\Delta,0}(\theta_t) + h^{(J),u}_{\lambda_1 \lambda_2}(u) d^{(J)}_{\Delta,0}(\theta_u) \right\}
\]

\[
\mathcal{H}_{\lambda_1 \lambda_2}(s, t, u) = h^{(0),t}_{\lambda_1 \lambda_2}(t) + h^{(0),u}_{\lambda_1 \lambda_2}(u) \\
+ \Omega^{(0)}(s) \left\{ a + b \left( s - \frac{s^2}{\pi} \int \frac{ds'}{s'^2} \frac{\text{Disc} \left( \Omega^{(0)} \right)^{-1}(s') h^{(0),L}_{\lambda_1 \lambda_2}(s')}{s' - s} \right) \right\}
\]
How to improve the model

Molnar, Danilkin, Vanderhaeghen, 1903.08458

$Z_c(3900)$

$q = 4.226 \text{ GeV}$

$q = 4.258 \text{ GeV}$

$Z_c(3900)$
How to improve the model

Molnar, Danilkin, Vanderhaeghen, 1903.08458

No $Z_c$

Broad $Z_c'$
Conclusions

• The study of kinematic singularities provides the «minimal» energy dependence of partial waves
• Isobar model remains the most effective method to describe Dalitz plots
• 3-body effects can be introduced in a flexible way, depending on the dominant effect expected

Thank you