The 8th Huada school on QCD @ CCNU, Wuhan, China

Foundations of GW from BNS merger

with application to nuclear/hadron physics

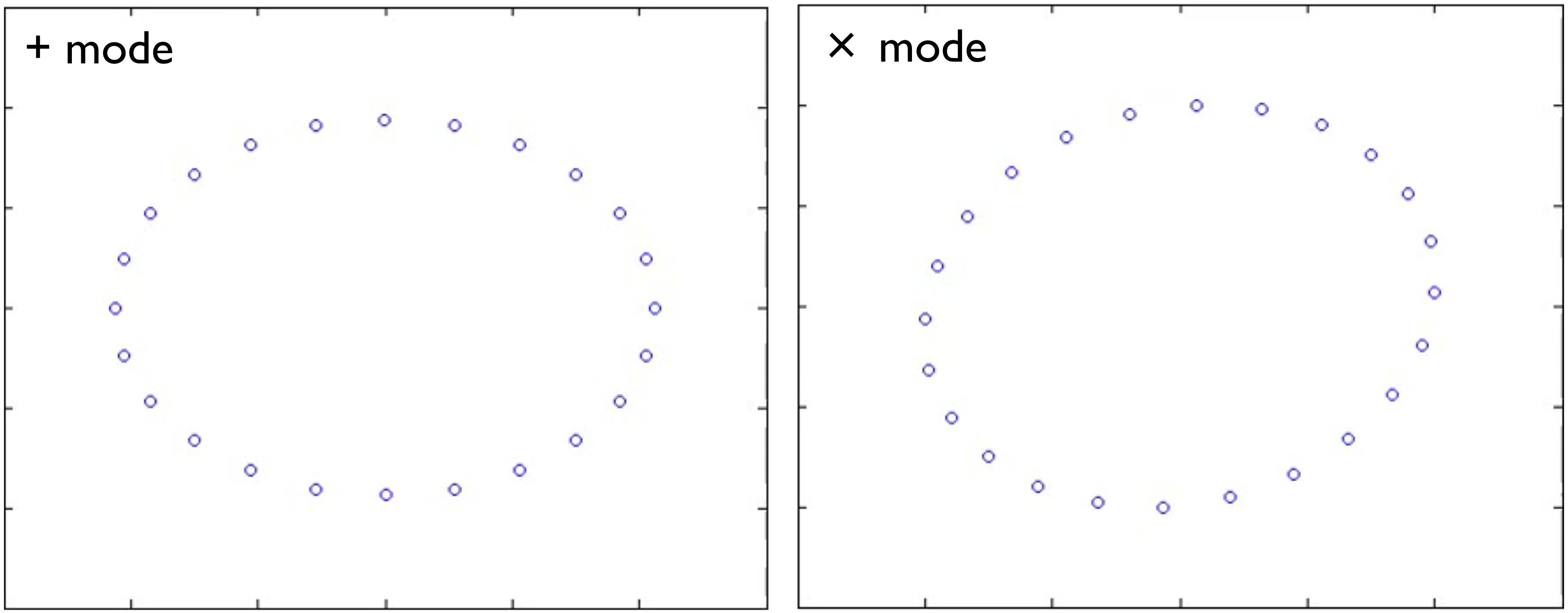
Yuichiro Sekiguchi (Toho University)

Contents

- Aim : Introduce physics of GW from NS-NS in a fundamental viewpoint
- Lecture 1: Linearized theory
  - GW propagation, TT gauge, polarization of GW (+, and × modes)
  - GW production, quadrupole formula
- Lecture 2: GW from binary system in circular orbit
  - the (point-particle) chirp signal, tidal deformability
  - Post-Newtonian GW and Numerical Relativity
- Lecture 3: Achievement in GW170817
  - Extraction of tidal deformability and its interpretation
  - Current constraint on EOS (combining with EM signals)
- Lecture 4: Future prospects
  - higher density regions, proving hadron-quark transition
  - Importance of numerical relativity

Summary of lecture 1

- We understand two polarization modes of GW propagation from fundamental viewpoints (using linearized Einstein's equaion)

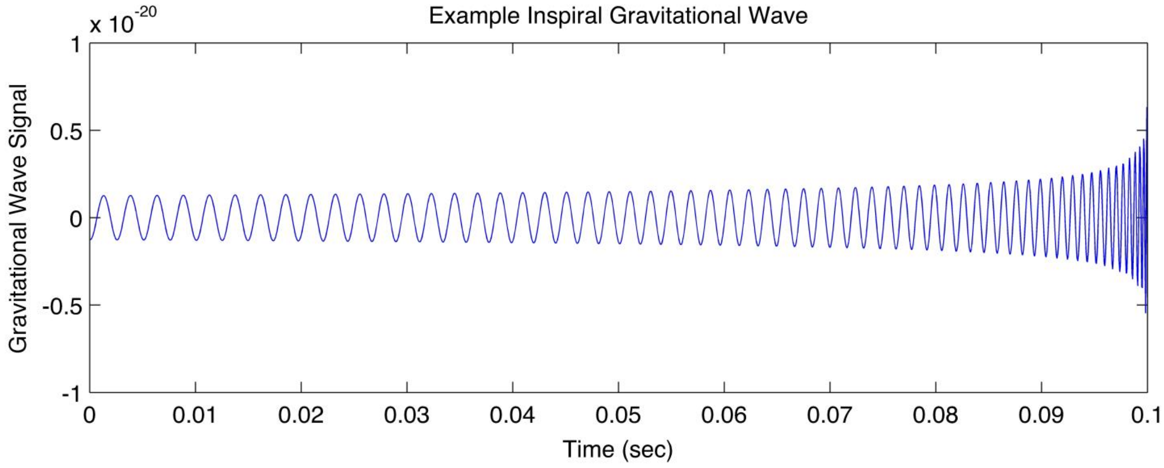


- We derived the energy flux formula $\frac{dP_{\rm gw}}{d\Omega} = \frac{r^2 c^3}{32\pi G}\langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle$
- We derived the quadrupole formula $h_{ij}^{TT,\,\rm quad} = \frac{1}{r}\frac{2G}{c^4}\ddot{Q}^{ij}(t - r/c)$

Lecture 2: GW from binary system in circular orbit

The goals of lecture 2

- Deriving the so-called chirp signal for point-particle GW



- Understanding Post-Newtonian (PN) expansion
- Understanding the concept of tidal deformability
- Deriving the result that the finite-size effect (tidal deformability) come in GW (via equation of motion) from 5PN order
  - Thus knowing point-particle GW at least up to 5PN order is essential in extracting information of tidal deformability

Lecture 2: GW from binary system in circular orbit

Derivation of the chirp signal

GWs from a binary in circular orbit (1)

- Let us consider GWs from a binary system with masses m_1 and m_2 in circular orbit of radius R

- In the center of mass frame, the relative coordinate $\vec{r} = \vec{r}_1 - \vec{r}_2$ are

$$x(t) = R\cos(\omega_s t + \pi/2) \qquad y(t) = R\sin(\omega_s t + \pi/2) \qquad z(t) = 0$$

- The 2nd mass moments $M^{ij} = \mu x^i x^j$ are (only for non-zero)

$$M^{11} = \frac{1}{2}\mu R^2(1 - \cos 2\omega_s t) \qquad M^{22} = \frac{1}{2}\mu R^2(1 + \cos 2\omega_s t) \qquad M^{12} = -\frac{1}{2}\mu R^2 \sin 2\omega_s t$$

  - where $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$ is the reduced mass

- Substituting the quadrupole formula, + and × modes of GWs seen from $n_i = (\sin\theta\sin\varphi, \sin\theta\cos\varphi, \cos\theta)$ are

$$h_{+}^{\text{quad}}(t;\theta,\varphi) = \frac{1}{r}\frac{4G}{c^4}\mu R^2 \omega_s^2 \cos\left(2(\omega_s t_{\text{ret}} + \varphi)\right) \frac{1+\cos^2\theta}{2}$$

$$h_{\times}^{\text{quad}}(t;\theta,\varphi) = \frac{1}{r}\frac{4G}{c^4}\mu R^2 \omega_s^2 \cos\left(2(\omega_s t_{\text{ret}} + \varphi)\right) \cos\theta$$

GWs from a binary in circular orbit (2)

- From Kepler's 3rd law, $\omega_s^2 = \frac{G(m_1+m_2)}{R^3}$ so that

$$h_+^{\text{quad}} = \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{\text{gw}}}{c}\right)^{2/3}\cos\left(2\pi f_{\text{gw}} t_{\text{ret}} + 2\varphi\right)\frac{1+\cos^2\theta}{2}$$

$$h_\times^{\text{quad}} = \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{\text{gw}}}{c}\right)^{2/3}\cos\left(2\pi f_{\text{gw}} t_{\text{ret}} + 2\varphi\right)\cos\theta$$

  - Here we use the chirp mass : $M_c = \mu^{3/5}(m_1+m_2)^{2/5} = \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}}$

  and $2\pi f_{\text{gw}} = \omega_{\text{gw}} = 2\omega_s$

- The radiated power are calculated by the energy flux formula

$$\frac{dP_{\text{gw}}}{d\Omega} = \frac{r^2 c^3}{16\pi G}\langle \dot{h}_+^2 + \dot{h}_\times^2\rangle = \frac{2}{\pi}\frac{c^5}{G}\left(\frac{GM_c\omega_{\text{gw}}}{2c^3}\right)^{10/3} g(\theta)$$

where R is eliminated using Kepler's law and $g(\theta) = \frac{(1+\sin^2\theta)^2}{4} + \cos^2\theta$

- Thus

$$P_{\text{gw}} = \frac{32}{5}\frac{c^5}{G}\left(\frac{GM_c\omega_{\text{gw}}}{2c^3}\right)^{10/3}$$

GWs from a binary in circular orbit (3)

- The binary system evolves according to the GW emission ($P_{\rm gw}$)
- The total energy of the system is

$$E = E_{\rm kin} + E_{\rm pot} = -\frac{Gm_1 m_2}{2R} = -\left(\frac{G^2 M_c^5 \omega_{\rm gw}^2}{32}\right)^{1/3}$$

- From the energy conservation $-\frac{dE}{dt} = P_{\rm gw}$, we have

$$\frac{d\omega_{\rm gw}}{dt} = \frac{12}{5} 2^{1/3} \left(\frac{GM_c}{c^3}\right)^{5/3} \omega_{\rm gw}^{11/3} \quad \text{or} \quad \frac{df_{\rm gw}}{dt} = \frac{96}{5} \pi^{8/3} \left(\frac{GM_c}{c^3}\right)^{5/3} f_{\rm gw}^{11/3}$$

- With a condition that $f_{\rm gw}$ diverges at the merger time $t_{\rm merge}$, we have

$$f_{\rm gw}(t) = \frac{1}{\pi}\left(\frac{5}{256}\frac{1}{t_{\rm merge} - t}\right)^{3/8} \left(\frac{GM_c}{c^3}\right)^{-5/8}$$

  - Of cause, in fact, the compact objects have finite size and the frequency never diverges in realistic systems
  - GW amplitude also increases with time and diverges at $t_{\rm merge}$

The chirp signal

- Substituting f_{gw} to h_+ and $h_\times$, we obtain the chirp signal as

$$h_+(t) = \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{\rm gw}(t_{\rm ret})}{c}\right)^{2/3}\cos\Phi(t_{\rm ret})\frac{1+\cos^2\theta}{2}$$

$$h_\times(t) = \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{\rm gw}(t_{\rm ret})}{c}\right)^{2/3}\sin\Phi(t_{\rm ret})\cos\theta$$

where $\Phi(t)$ is a solution of $d\Phi/dt = \omega_{\rm gw}$, given by

$$\Phi(t) = \Phi_0 - 2\left(\frac{5GM_c}{c^3}\right)^{-5/8}\left(t_{\rm merge} - t\right)^{5/8}$$

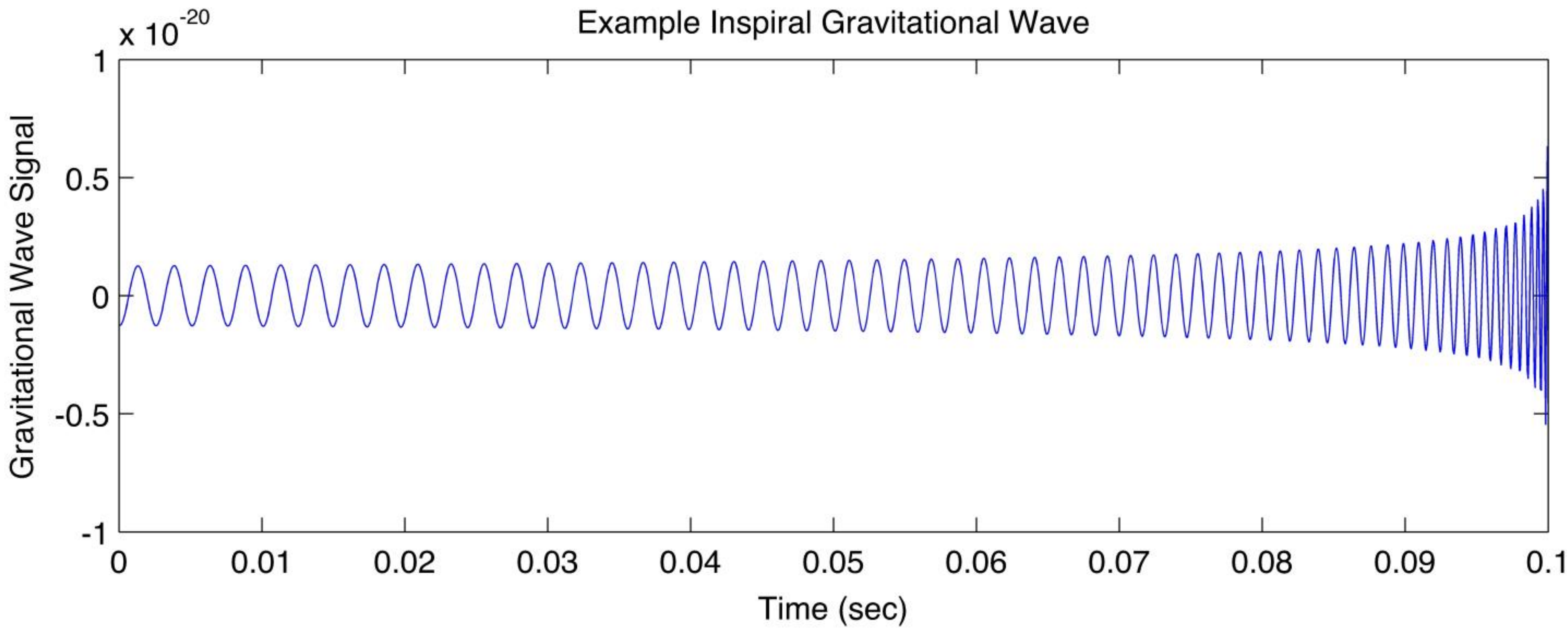
- Note that the chirp signal depends on (chirp) mass of the system and we can obtain information of mass from its analysis

The chirp signal

- Substituting f_{gw} to h_+ and $h_\times$, we obtain the chirp signal as

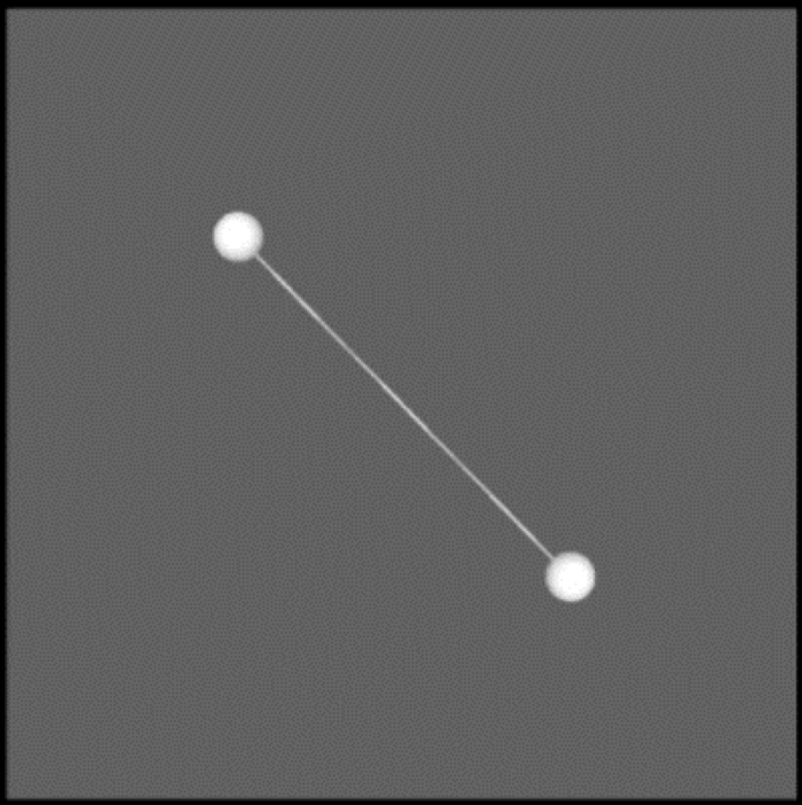
$$h_+(t) = \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{\mathrm{gw}}(t_{\mathrm{ret}})}{c}\right)^{2/3}\cos\Phi(t_{\mathrm{ret}})\frac{1+\cos^2\theta}{2}$$

$$h_\times(t) = \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{\mathrm{gw}}(t_{\mathrm{ret}})}{c}\right)^{2/3}\sin\Phi(t_{\mathrm{ret}})\cos\theta$$

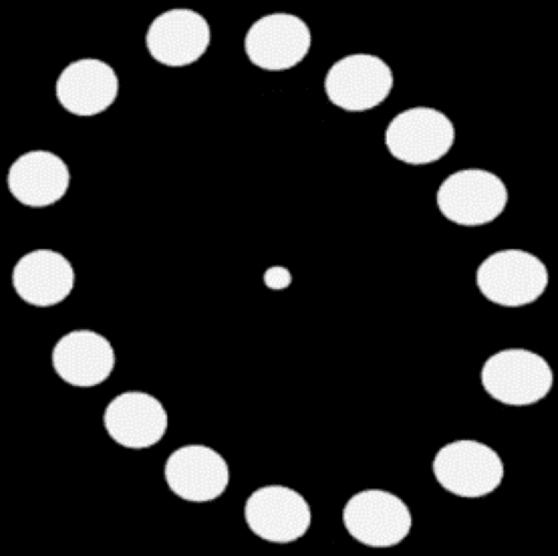


https://www.youtube.com/watch?v=Y6tSFk5ESAo

Orbiting compact objects: a gravitational-wave source



A ring of particles, at the viewer location, responds to the waves



For a typical NS-NS case (1)

- For a typical case with $m_1 = m_2 = 1.4M_\odot$ ($M_\odot \approx 2\times10^{33}$g : solar mass)

$$f_{\rm gw}(t) \approx 130\ {\rm Hz}\left(\frac{1\ {\rm sec}}{t_{\rm merge}-t}\right)^{3/8}\left(\frac{M_c}{1.21M_\odot}\right)^{-5/8}$$

  - or

$$t_{\rm merge}-t \approx 2\ {\rm sec}\left(\frac{f_{\rm gw}}{100\ {\rm Hz}}\right)^{-8/3}\left(\frac{M_c}{1.21M_\odot}\right)^{-5/8}$$

- Number of GW cycles N can be calculated by $dN = dt/T(t) = f_{\rm gw}(t)dt$

$$N = \int_{t_{\rm min}}^{t_{\rm max}} dt\, f_{\rm gw}(t) = \int_{f_{\rm min}}^{f_{\rm max}} df_{\rm gw}\frac{f_{\rm gw}}{\dot{f}_{\rm gw}} = \frac{1}{32\pi^{8/3}}\left(\frac{GM_c}{c^3}\right)^{-5/3}\left(f_{\rm min}^{-5/3}-f_{\rm max}^{-5/3}\right)$$

  - For a typical advanced GW detector : $f_{\rm min}\sim 10{\rm Hz}$, $f_{\rm max}\sim 1000{\rm Hz}$, and

$$N \approx 1.6\times10^4\left(\frac{f_{\rm min}}{10\ {\rm Hz}}\right)^{-5/3}\left(\frac{M_c}{1.21M_\odot}\right)^{-5/3}$$

For a typical NS-NS case (2)

- The merger time is ($T(t) = 2\pi/\omega_s = 2/f_{\rm gw}$)

$$t_{\rm merge} - t \approx 7\times10^6\ {\rm yr}\left(\frac{T}{1\ {\rm hr}}\right)^{8/3}\left(\frac{M_c}{1.21M_\odot}\right)^{-5/3} \approx 3\times10^{10}\ {\rm yr}\left(\frac{T}{1\ {\rm day}}\right)^{8/3}\left(\frac{M_c}{1.21M_\odot}\right)^{-5/3}$$

Name	$M_{\rm tot}$ $[M_\odot]$	$M_{\rm A}$ $[M_\odot]$	$M_{\rm B}$ $[M_\odot]$	q	$T_{\rm orb}$ [days]	R [light s]	$e_{\rm orb}$	D [kpc]	$f_{\rm s}$ [Hz]	$B_{\rm surf}$ [G]
J0453+1559 [8]	2.734	1.559	1.174	0.75	4.1	14	0.11	1.8	22	9.3E+09
J0737-3039 [9]	2.587	1.338	1.249	0.93	0.10	1.4	0.088	1.1	44	6.4E+09
J1518+4904 [10]	2.718	<1.766	>0.951	>0.54	8.6	20	0.25	0.7	24	9.6E+08
B1534+12 [11]	2.678	1.333	1.345	0.99	0.42	3.7	0.27	1.0	26	9.6E+09
J1753-2240 [12]	–	–	–	–	14	18	0.30	3.5	10	9.7E+09
J1756-2251 [13]	2.577	1.341	1.23	0.92	0.32	2.8	0.18	0.73	35	5.4E+09
J1807-2500B [14]	2.571	1.366	1.21	0.89	1.0	29	0.75	–	239	≤9.8E+08
J1811-1736 [15]	2.571	<1.478	>1.002	>0.68	19	35	0.83	5.9	9.6	9.8E+09
J1829+2456 [16]	2.59	<1.298	>1.273	>0.98	1.2	7.2	0.14	0.74	24	1.5E+09
J1906+0746 [17]	2.613	1.291	1.322	0.98	0.17	1.4	0.085	7.4	6.9	1.7E+12
J1913+1102 [18]	2.875	<1.84	>1.04	>0.56	0.21	1.8	0.090	13	1.1	2.1E+09
B1913+16 [19]	2.828	1.449	1.389	0.96	0.32	2.3	0.62	7.1	17	2.3E+10
J1930-1852 [20]	2.59	<1.199	>1.363	>0.88	45	87	0.40	2.3	5.4	6.0E+10
B2127+11C [21]	2.713	1.358	1.354	1.0	0.34	2.5	0.68	13	33	1.2E+10

For a typical NS-NS case (3)

- GW amplitude for a event at $100\ \mathrm{Mpc} \approx 3.1\times10^{19}\mathrm{km}$

$$h \sim \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{\mathrm{gw}}}{c}\right)^{2/3} \sim 10^{-24}\left(\frac{M_c}{M_\odot}\right)^{5/3}\left(\frac{f_{\mathrm{gw}}}{100\ \mathrm{Hz}}\right)^{2/3}\left(\frac{100\ \mathrm{Mpc}}{r}\right)$$

- Effective amplitude is enhanced ($\times\sqrt{N}$) by integration

$$h_{\mathrm{eff}} \sim h\sqrt{N} \sim 10^{-22}\left(\frac{M_c}{M_\odot}\right)^{5/6}\left(\frac{f_{\mathrm{gw}}}{100\ \mathrm{Hz}}\right)^{2/3}\left(\frac{100\ \mathrm{Mpc}}{r}\right)$$

  - For GW170817 (NS-NS) : $h_{\mathrm{eff}} \sim 10^{-22}(r = 40\mathrm{Mpc}, M_c \approx M_\odot)$
  - For GW150914 (Bh-BH) : $h_{\mathrm{eff}} \sim 10^{-21}\ (r = 400\mathrm{Mpc}, M_c \approx 60M_\odot)$

Lecture 2: GW from binary system in circular orbit

Very brief introduction of Post-Newtonian formalism

Post-Newtonian (PN) formalism (1)

- Post-Newtonian (PN) formalism is a method which can be applied to
  - slowly moving ($v/c \ll 1$)
  - weakly self-gravitating ($GM/c^2 d \ll 1$)

  GW sources like (early-phases of) compact-star binaries

- For self-gravitating systems, as a consequence of the virial theorem,

$$\left(\frac{v}{c}\right)^2 \sim \frac{GM}{c^2 d} = \frac{R_g}{d}$$

  so. we may perform perturbation expansion only in terms of v/c

- The correction in G^n or $(v/c)^{2n}$ is called n-th ordher PN correction or nPN order correction

Post-Newtonian (PN) formalism (2)

- In the PN formalism, we first define $\mathsf{h}^{\alpha\beta} \equiv \sqrt{-g}\, g^{\alpha\beta} - \eta^{\alpha\beta}$, and impose de Donder gauge condition $\partial_\beta \mathsf{h}^{\alpha\beta} = 0$
- In de Donder gauge, Einstein's equation exactly takes a form

$$\Box \mathsf{h}^{\alpha\beta} = \frac{16\pi G}{c^4}\tau^{\alpha\beta}$$ the relaxed Einstein's equation

where

$$\tau^{\alpha\beta} = (-g)T^{\alpha\beta} + \frac{c^4}{16\pi G}T_{\mathsf{h}}^{\alpha\beta}$$

$$T_{\mathsf{h}}^{\alpha\beta} = \frac{16\pi G}{c^4}(-g)\,t_{LL}^{\alpha\beta} + \left(\partial_\nu \mathsf{h}^{\alpha\mu}\partial_\mu \mathsf{h}^{\beta\nu} - \mathsf{h}^{\mu\nu}\partial_\mu\partial_\nu \mathsf{h}^{\alpha\beta}\right)$$

$$\frac{16\pi G}{c^4}(-g)\,t_{LL}^{\alpha\beta} = g_{\lambda\mu}g^{\nu\rho}\partial_\nu \mathsf{h}^{\alpha\lambda}\partial_\mu \mathsf{h}^{\beta\nu} + \frac{1}{2}g_{\lambda\mu}g^{\alpha\beta}\partial_\rho \mathsf{h}^{\lambda\nu}\partial_\nu \mathsf{h}^{\rho\mu} - g_{\mu\nu}\partial_\lambda \mathsf{h}^{\rho\mu}\left(g^{\lambda\alpha}\partial_\rho \mathsf{h}^{\beta\nu} + g^{\lambda\beta}\partial_\rho \mathsf{h}^{\alpha\nu}\right)$$

- Then, we expand metric in terms of $1/c$ as $\mathsf{h}^{\mu\nu} = \sum_{n=2} \frac{1}{c^n}\, {}^{(n)}\mathsf{h}^{\mu\nu}$ and perturbatively solve the relaxed Einstein's equation

Post-Newtonian (PN) formalism (3)

- Once we have the PN metric, we can derive the equation of motion from the action

$$S = -mc \int dt \sqrt{-g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}} = -mc^2 \int dt \sqrt{-g_{00} - 2g_{0i} \frac{v^i}{c} - g_{ij} \frac{v^i v^j}{c^2}}$$

- The equation of motion schematically takes the form

$$\frac{d^2 x^i}{dt^2} = \frac{Gm}{r^2} \left[\frac{x^i}{r} \left(1 + O\left((v/c)^2\right) + O\left((v/c)^4\right) + \cdots\right) + \frac{\boldsymbol{v}^i}{c} \left(O\left((v/c)^2\right) + O\left((v/c)^4\right) + \cdots\right) \right]$$

- If we can solve the equation of motion, the energy momentum tensor is given by

$$T^{\mu\nu} = \frac{1}{\sqrt{-g}} \sum_a m_a \frac{d\tau_a}{dt} \frac{dx_a^\mu}{d\tau_a} \frac{dx_a^\nu}{d\tau_a} \delta^{(3)}\left(\boldsymbol{x} - \boldsymbol{x}_a(t)\right)$$

which becomes the source term for the relaxed Einstein's equation

Difficulty in PN formalism (1)

- Due to the nonlinearity of GR, h_{ab} itself becomes the source of GW
- Then, the Poisson integral does not have compact support
- As we have shown in Lecture 1, there is the correspondence "higher $v/c \Leftrightarrow$ higher multipole (x^l)"
- The Poisson integrals are necessarily divergent beyond some order
  - Divergent features start to appear from 2PN order (become inexorably divergent at 3PN)
- Some special treatment using a variant of the analytic continuation is necessary to obtain a particular solution
- For details of PN formalism, see
  - Blanchet, "Gravitational Radiation from Post-Newtonian Sources and Inspiralling Compact Binaries", Living Reviews in Relativity, 17, 2 (2014)

Difficulty in PN formalism (2)

- In PN formalism, we use approximation : $-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\nabla^2 \approx \left(1+O(v^2/c^2)\right)\nabla^2$
  - This means that retardation effects are assumed to be small corrections
- We are trying to reconstruct a retarded field from its expansions for small retardation $r/c \ll t$ $(v \sim r/t \ll c)$

$$h_{\mu\nu} = \frac{1}{r}F_{\mu\nu}(t-r/c) = \frac{1}{r}F_{\mu\nu}(t) + \frac{1}{c}\dot{F}_{\mu\nu}(t) + \frac{r}{2c^2}\ddot{F}_{\mu\nu}(t) + \frac{r^2}{6c^3}\dddot{F}_{\mu\nu}(t) + \cdots$$

- The higher-order-term coefficients blow up as $r \to \infty$
  - Mathematically, the PN expansion is an example of asymptotic expansion (singular perturbation theory)
- Just as in electrodynamics, it is convenient to distinguish between the 'near zone' and the 'far (wave) zone'.
  - The near zone and far zone are separated by an intermediated region at $r \sim \lambda_{gw}$

Difficulty in PN formalism (3)

- Since GWs carry away the energy from the system, the equation of motion is modified due to the back-reaction of GW emission
- As shown in the linearized theory, at the leading order, GW power is

$$P_{\rm gw} = \frac{32}{5}\frac{c^5}{G}\left(\frac{GM_c\omega_{\rm gw}}{2c^3}\right)^{10/3} \sim \frac{G\mu^2}{c^5} r^4 \omega_s^6 \sim \frac{Gm^2}{c^5}\frac{v^6}{r^2}$$

- The total energy of self-gravitating systems is $E = E_{\rm kin} + E_{\rm pot} \sim E_{\rm pot} \sim E_{\rm kin}$

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right) \sim P_{\rm gw} \quad \Longrightarrow \quad \frac{dv}{dt} \sim \frac{Gm}{r^2}\left(\frac{v}{c}\right)^5$$

- Thus the back-reaction of GWs comes in EOM from 2.5PN
- There are other effects like spin-orbit and spin-spin coupling, etc,
  - For more details, see Blanchet (2014)

Khan et al. PRD 93, 044008 (2016)

Taylor F2 waveform : 3.5PN GW

$$M = m_1 + m_2,$$
$$\eta = m_1 m_2 / M^2,$$
$$\delta = (m_1 - m_2)/M,$$
$$\chi_s = (\chi_1 + \chi_2)/2,$$
$$\chi_a = (\chi_1 - \chi_2)/2.$$

$$\phi_{\rm TF2} = 2\pi f t_c - \varphi_c - \pi/4 + \frac{3}{128\eta}(\pi f M)^{-5/3} \sum_{i=0}^{7} \varphi_i(\Xi)(\pi f M)^{i/3}$$

$$\varphi_0 = 1$$

$$\varphi_1 = 0$$

$$\varphi_2 = \frac{3715}{756} + \frac{55\eta}{9}$$

mass can be determined in 1PN

$$\varphi_3 = -16\pi + \frac{113\delta\chi_a}{3} + \left(\frac{113}{3} - \frac{76\eta}{3}\right)\chi_s$$

spin effect comes n 1.5PN

$$\varphi_4 = \frac{15293365}{508032} + \frac{27145\eta}{504} + \frac{3085\eta^2}{72} + \left(-\frac{405}{8} + 200\eta\right)\chi_a^2 - \frac{405}{4}\delta\chi_a\chi_s + \left(-\frac{405}{8} + \frac{5\eta}{2}\right)\chi_s^2$$

$$\varphi_5 = [1 + \log(\pi M f)]\left[\frac{38645\pi}{756} - \frac{65\pi\eta}{9} + \delta\left(-\frac{732985}{2268} - \frac{140\eta}{9}\right)\chi_a + \left(-\frac{732985}{2268} + \frac{24260\eta}{81} + \frac{340\eta^2}{9}\right)\chi_s\right]$$

$$\varphi_6 = \frac{11583231236531}{4694215680} - \frac{6848\gamma_E}{21} - \frac{640\pi^2}{3} + \left(-\frac{15737765635}{3048192} + \frac{2255\pi^2}{12}\right)\eta + \frac{76055\eta^2}{1728} - \frac{127825\eta^3}{1296}$$
$$- \frac{6848}{63}\log(64\pi M f) + \frac{2270}{3}\pi\delta\chi_a + \left(\frac{2270\pi}{3} - 520\pi\eta\right)\chi_s,$$

$$\varphi_7 = \frac{77096675\pi}{254016} + \frac{378515\pi\eta}{1512} - \frac{74045\pi\eta^2}{756} + \delta\left(-\frac{25150083775}{3048192} + \frac{26804935\eta}{6048} - \frac{1985\eta^2}{48}\right)\chi_a$$
$$+ \left(-\frac{25150083775}{3048192} + \frac{10566655595\eta}{762048} - \frac{1042165\eta^2}{3024} + \frac{5345\eta^3}{36}\right)\chi_s.$$

Lecture 2: GW from binary system in circular orbit

The tidal deformability appears in 5PN

Tidal deformability

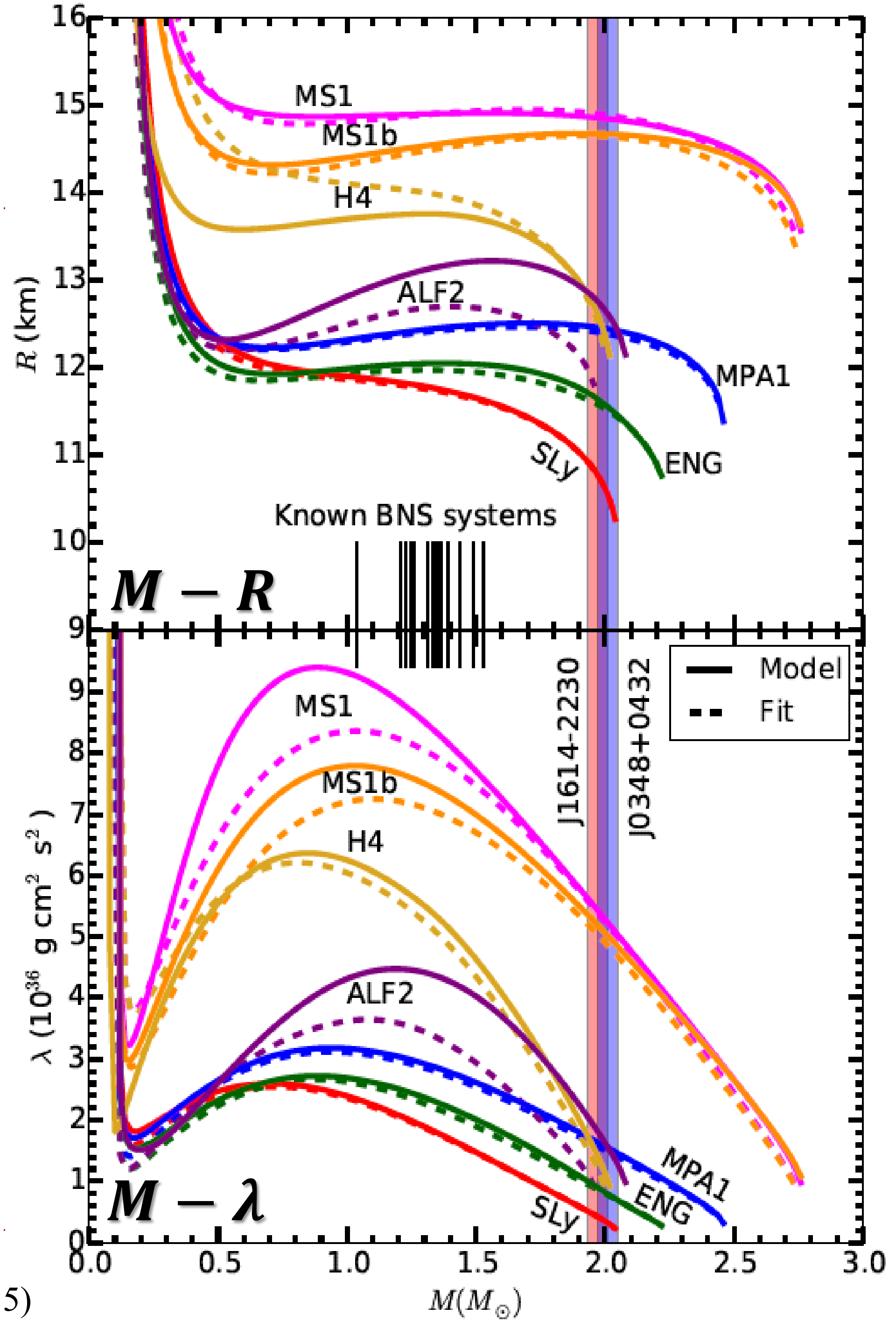
- Tidal Love number : λ
- Response of quadrupole moment Q_{ij} to external tidal field E_{ij}

$$Q_{ij} = -\lambda E_{ij}$$

- Stiffer NS EOS
  - ⇒ NS Gravity can be supported with less contraction
  - ⇒ larger NS radius
  - ⇒ larger λ
- We also use the non-dimensional tidal deformability : Λ

$$\lambda = \frac{C^5}{G} \Lambda R^5 \qquad C = \frac{GM}{c^2 R}$$

Compactness parameter



Lackey et al. PRD 91, 043002(2015)

When tidal deformation joins ? (1)

- Treating the compact objects as perfect fluids, we can take into account the effect of the tidal deformation (finite-size effect)

- The basic equations are the continuity and Euler equations

$$\partial_t \rho + \partial_i (\rho v^i) = 0$$

$$\rho \partial_t v^i + \rho v^j \partial_j (\rho v^i) = -\partial_i p + \rho \partial_i U$$

  - together with an equation of state : $p = p(\rho)$ and Poisson eq. $\nabla^2 U = -4\pi G \rho$

- The mass and center-of-mass coordinates of the a-th ($a = 1{,}2$) NS are

$$m_a = \int_{V_a} d^3 x \rho(t, \boldsymbol{x}) \qquad z_a^i(t) = \frac{1}{m_a} \int_{V_a} d^3 x \rho(t, \boldsymbol{x}) x^i$$

  - where V_a is the volume occupied by the a-th object

- Using the continuity and Euler equations, we get equations of motion

$$m_a \frac{d^2 z_a^i}{dt^2} = \int_{V_a} d^3 x \left(-\partial_i p + \rho \partial_i U \right)$$

When tidal deformation joins ? (2)

- Derivation

$$\frac{dz_a^i}{dt} = \frac{1}{m_a}\int_{V_a} d^3x\,(\partial_t\rho)x^i = -\frac{1}{m_a}\int_{V_a} d^3x\,\partial_k\left(\rho v^k\right)x^i$$
$$= -\frac{1}{m_a}\int_{V_a} d^3x\,\partial_k\left(\rho v^k x^i\right) + \frac{1}{m_a}\int_{V_a} d^3x\,\rho v^k \partial_k x^i = \frac{1}{m_a}\int_{V_a} d^3x\,\rho v^k \delta_k^i$$
$$= \frac{1}{m_a}\int_{V_a} d^3x\,\rho v^i$$

$$\frac{d^2z_a^i}{dt^2} = \frac{1}{m_a}\int_{V_a} d^3x\,\partial_t\left(\rho v^i\right) = \frac{1}{m_a}\int_{V_a} d^3x\left[-\partial_k(\rho v^i v^k) - \delta^{ij}\partial_j p + \rho\delta^{ij}\partial_j U\right]$$
$$= \frac{1}{m_a}\int_{V_a} d^3x\left[-\delta^{ij}\partial_j p + \rho\delta^{ij}\partial_j U\right] = \frac{1}{m_a}\int_{V_a} d^3x\left[-\partial_i p + \rho\partial_i U\right]$$

When tidal deformation joins ? (3)

- The gravitational potential can be split into "self-" and "external" parts as

$$U(t,\boldsymbol{x}) = G\int_{V_1} d^3x' \frac{\rho(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|} + G\int_{V_2} d^3x' \frac{\rho(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|} = U^{\rm self} + U^{\rm ext}$$

- The self-part vanishes because

$$F^{\rm self} = \int_{V_1} d^3x \left[-\partial_i p + \rho\partial_i U^{\rm self}\right] = \int_{V_1} d^3x\, \rho\partial_i U^{\rm self}$$

$$= G\int_{V_1} d^3x\, \rho(t,\boldsymbol{x}) \frac{\partial}{\partial x^i}\int_{V_1} d^3x' \frac{\rho(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}$$

$$= -G\int_{V_1} d^3x \int_{V_1} d^3x'\, (x-x')^i \frac{\rho(t,\boldsymbol{x})\rho(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|^3} = 0$$

  - The integrand is odd under the exchange $x \leftrightarrow x'$, while the domain is symmetric
  - Note that the two densities $\rho(t,x)$ and $\rho(t,x')$ are conceptually different
    - $\rho(t,x)$: passive gravitational mass density
    - $\rho(t,x')$: active gravitational mass density
  - The vanishing of the self-force is rooted in the equality of two gravitational mass

When tidal deformation joins ? (4)

- Thus the equation of motion only contains external gravitational force
- Dependence on the tidal deformability is obtained from a multipole expansion of the external force
- We introduce a coordinate $\boldsymbol{y} = \boldsymbol{x} - \boldsymbol{z}_1(t)$ around primary star $\boldsymbol{z}_1$ and expand the external field as

$$\partial_i U^{\text{ext}}(t, \boldsymbol{y} + \boldsymbol{z}_1) = \partial_i U^{\text{ext}}(t, \boldsymbol{z}_1) + y^j \partial_j \partial_i U^{\text{ext}}(t, \boldsymbol{z}_1) + \frac{1}{2} y^k y^j \partial_k \partial_j \partial_i U^{\text{ext}}(t, \boldsymbol{z}_1) + \cdots$$

- Then, equations of motion are

$$m_1 \frac{d^2 z_1^i}{dt^2} = \int_{V_1} d^3y\, \rho(t, \boldsymbol{x}) \partial_i U^{\text{ext}}(t, \boldsymbol{y} + \boldsymbol{z}_1)$$

$$= \int_{V_1} d^3y\, \rho(t, \boldsymbol{x}) \left[\partial_i U^{\text{ext}}(t, \boldsymbol{z}_1) + y^j \partial_j \partial_i U^{\text{ext}}(t, \boldsymbol{z}_1) + \frac{1}{2} y^k y^j \partial_k \partial_j \partial_i U^{\text{ext}}(t, \boldsymbol{z}_1) + \cdots \right]$$

$$= m_1 \partial_i U^{\text{ext}}(t, \boldsymbol{z}_1) + I_1^j \partial_j \partial_i U^{\text{ext}}(t, \boldsymbol{z}_1) + \frac{1}{2} I_1^{jk} \partial_k \partial_j \partial_i U^{\text{ext}}(t, \boldsymbol{z}_1) + \cdots$$

where $$I_1^j = \int_{V_1} d^3y\, \rho(t, \boldsymbol{x}) y^i, \qquad I_1^{jk} = \int_{V_1} d^3y\, \rho(t, \boldsymbol{x}) y^i y^k$$

When tidal deformation joins ? (5)

- Note that the term with dipole moment vanishes as follows

$$I_1^j = \int_{V_1} d^3y\, \rho(t,\boldsymbol{x}) y^i = \int_{V_1} d^3x\, \rho(t,\boldsymbol{x})(x - z_1\ )^i = \int_{V_1} d^3x\, \rho(t,\boldsymbol{x}) x^i - \int_{V_1} d^3x\, \rho(t,\boldsymbol{x}) z_1^i$$

$$= \int_{V_1} d^3x\, \rho(t,\boldsymbol{x}) x^i - m_1 z_1^i = \int_{V_1} d^3x\, \rho(t,\boldsymbol{x}) x^i - m_1 \frac{1}{m_1} \int_{V_1} d^3x\, \rho(t,\boldsymbol{x}) x^i = 0$$

- The term with I_1^{jk} can be rewritten with quadrupole moment Q_1^{jk} because $\delta^{jk}\partial_k\partial_j\partial_i U^{\text{ext}}(t,\boldsymbol{z}_1) = \partial_i \nabla^2 U^{\text{ext}}(t,\boldsymbol{z}_1) = \partial_i\big(-4\pi G \rho_2(t,\boldsymbol{z}_1)\big) = 0$
  - Spatial derivative with respect to the primary-star coordinate of density of the secondary star vanishes
- Then,

$$I_1^{jk}\partial_k\partial_j\partial_i U^{\text{ext}}(t,\boldsymbol{z}_1) = \left(I_1^{jk} - \frac{1}{3}\delta^{jk} I_1^{ll}\right)\partial_k\partial_j\partial_i U^{\text{ext}}(t,\boldsymbol{z}_1)$$

$$= Q_1^{jk}\partial_k\partial_j\partial_i U^{\text{ext}}(t,\boldsymbol{z}_1)$$

When tidal deformation joins ? (6)

- To summarize, the equation of motion is

$$m_a \frac{d^2 z_a^i}{dt^2} = m_a \partial_i U^{\text{ext}}(t, \boldsymbol{z}_a) + \frac{1}{2} Q_a^{jk} \partial_i \partial_j \partial_k U^{\text{ext}}(t, \boldsymbol{z}_a) + \cdots = F^{\text{Newt}} + F^{\text{induced}}$$

  - If there is non-vanishing quadrupole moment (tidal deformation) induced by the companion's tidal force, we will have the 2nd term

- The quadrupole moment induced by tidal force is $Q_a^{jk} \sim \varepsilon M R^2$, with typical ellipticity

$$\varepsilon \sim \frac{F_{\text{tidal}}}{F_{\text{self}}} \sim \frac{GMR/d^3}{GM/R^2} \sim \Lambda \left(\frac{R}{d}\right)^3$$

We introduced the non-dim tidal deformability Λ

- Then,

$$F^{\text{induced}} \sim \frac{Q^{jk}}{M} \partial^2 F^{\text{Newt}} \sim \frac{\varepsilon M R^2}{M} \frac{F^{\text{Newt}}}{d^2} \sim \Lambda \left(\frac{R}{d}\right)^5 F^{\text{Newt}} \sim \Lambda \left(\frac{v}{c}\right)^{10} F^{\text{Newt}}$$

where we use $\frac{GM}{d} \sim v^2$ (by Virial theorem) and $R \sim \frac{GM}{c^2}$ for compact object

- Thus, tidal effect joins at **5PN order !**

Tidal deformability

- Tidal Love number : λ
- Response of quadrupole moment Q_{ij} to external tidal field E_{ij}

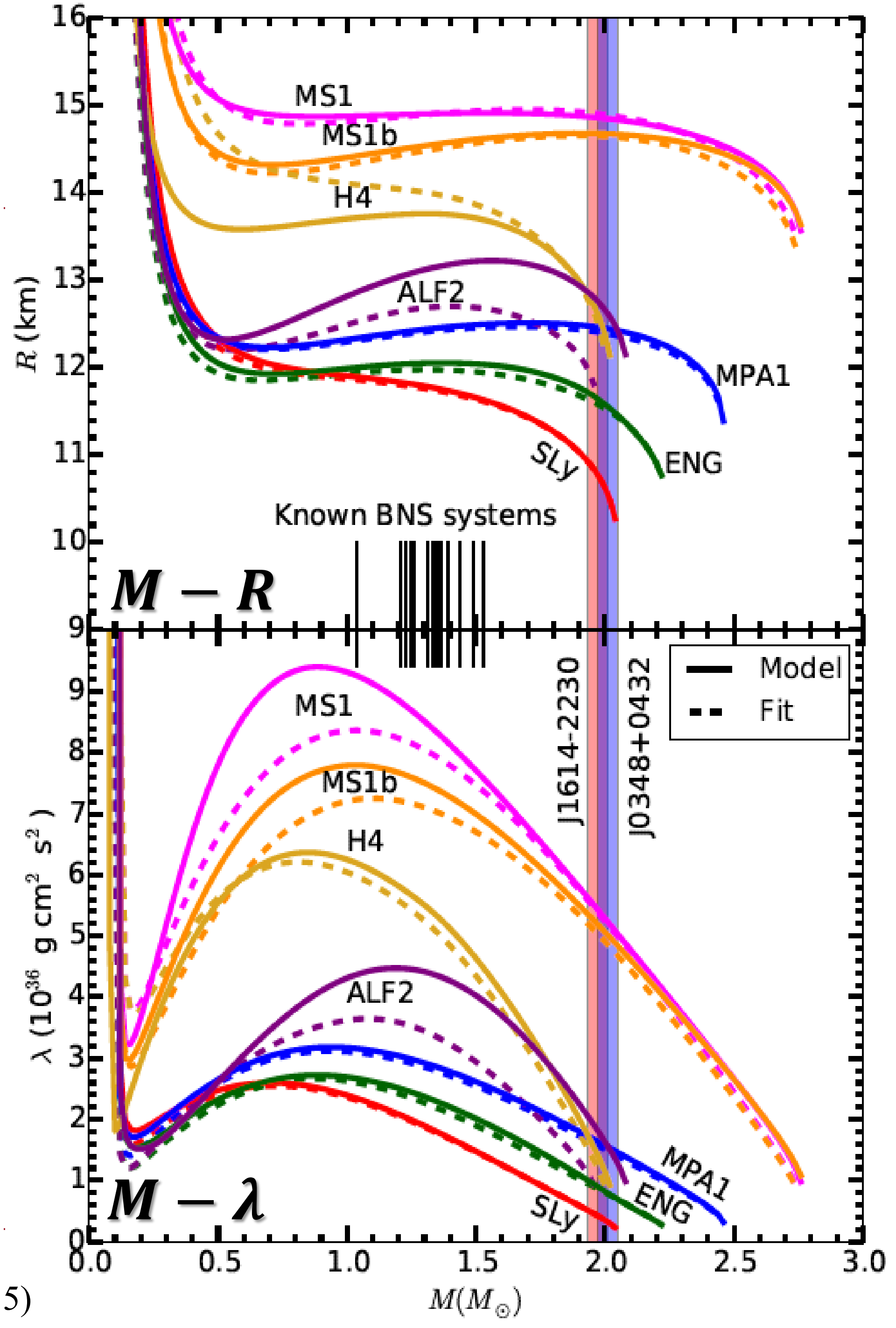
$$Q_{ij} = -\lambda E_{ij}$$

- Stiffer NS EOS
  - $\Rightarrow$ NS Gravity can be supported with less contraction
  - $\Rightarrow$ larger NS radius
  - $\Rightarrow$ larger λ

- We also use the non-dimensional tidal deformability : Λ

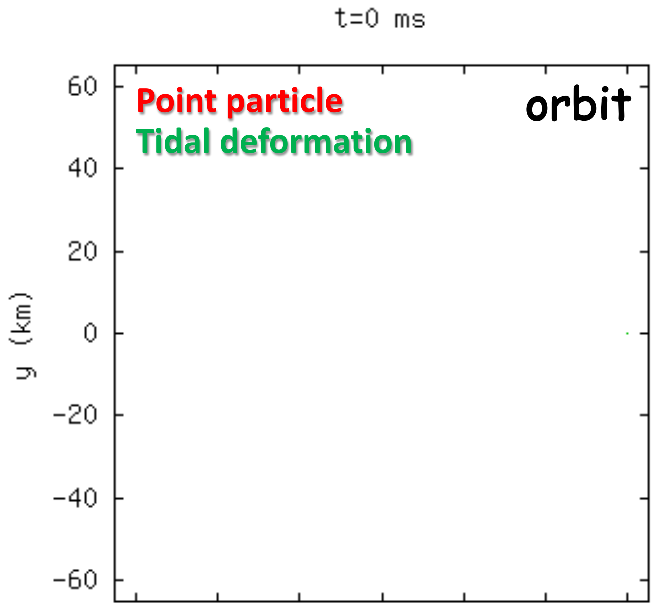
$$\lambda = \frac{C^5}{G}\Lambda R^5 \qquad C = \frac{GM}{c^2 R}$$

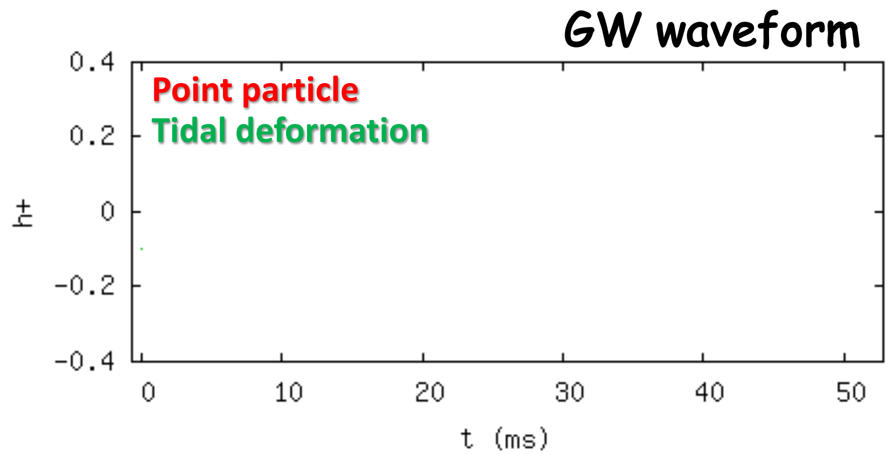
Compactness parameter



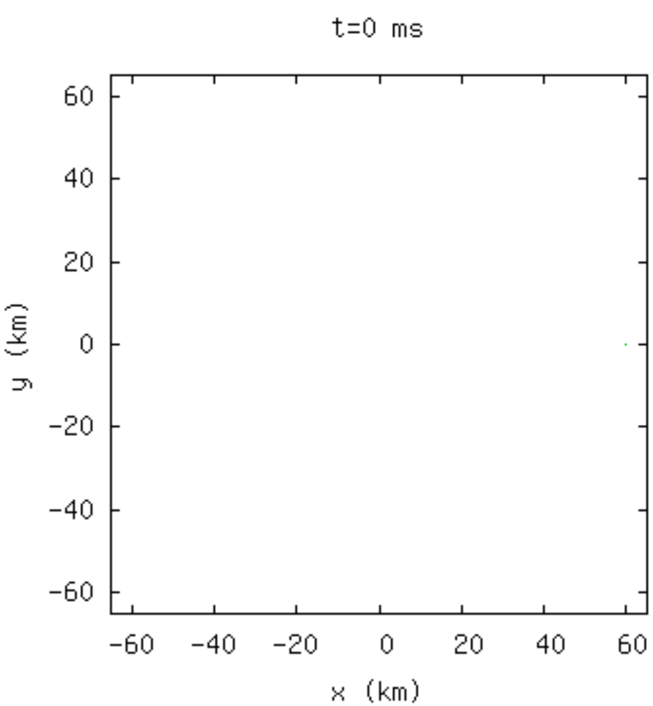
Lackey et al. PRD 91, 043002(2015)

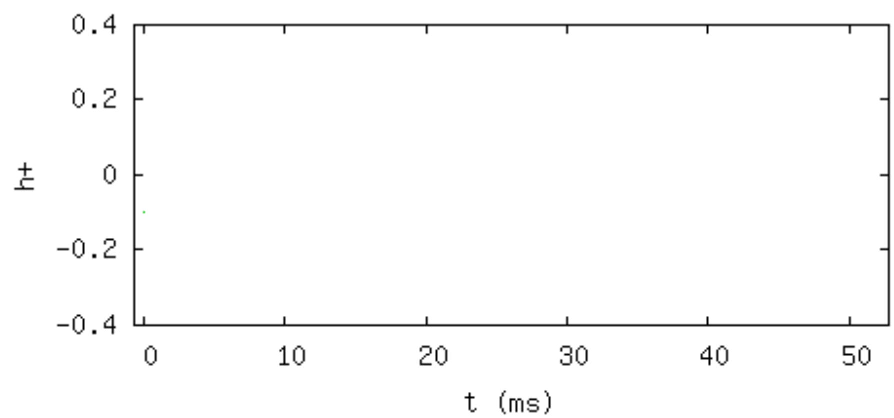
Soft EOS (Smaller NS radius, smaller tidal deformability)
Effect of tidal deformation is not prominent





Stiff EOS (larger NS radius, larger tidal deformability)
Deviation from point particle approximation can be clearly seen





Tidal effects on GW phase

$$\phi_{\text{tidal}} = -\frac{3}{128\eta}\left(\frac{39}{2}\tilde{\Lambda}\right)(\pi f M)^{10/3}\left[1+\left(\frac{3115}{1248}-\frac{6595}{7098}\sqrt{1-4\eta}\frac{\delta\tilde{\Lambda}}{\tilde{\Lambda}}\right)(\pi f M)^{2/3}\right]$$

$$\tilde{\Lambda} = \frac{8}{13}\left[(1+7\eta-31\eta^2)(\Lambda_1+\Lambda_2)-\sqrt{1-4\eta}(1+9\eta-11\eta^2)(\Lambda_1-\Lambda_2)\right]$$

$$\delta\tilde{\Lambda} = \frac{1}{2}\left[\sqrt{1-4\eta}\left(1-\frac{13272}{1319}\eta+\frac{8944}{1319}\eta^2\right)(\Lambda_1+\Lambda_2)-\left(1-\frac{15910}{1319}\eta+\frac{32850}{1319}\eta^2+\frac{3380}{1319}\eta^3\right)(\Lambda_1-\Lambda_2)\right]$$

Higher order *point-particle* GW is necessary

- To extract information of the tidal deformation correctly, we need point-particle GW waveform accurate at least in 5PN order
- Otherwise the tidal deformability Λ may be overestimated
  - Taking into account the PN corrections, in general leads to faster phase evolution due to the stronger gravity in GR
  - Tidal effects will also results in faster phase evolution, because the energy should be consumed in exciting the tidal modes, which can be regarded as an additional cooling source
- However, there is no well-established point-particle GW waveform higher order than 4.5PN Messina & Nagar PRD 96, 049907 (2017)
  - But see Messina et al. (2019) 1904.09558 for recent study for 5.5PN
  - It will be necessary to fully take into account the relativistic effects using Numerical Relativity (Lecture 4)

Importance of higher PN terms

- Comparison of BH-BH GW (no effect of tidal deformability) between PN waveform and a numerical-relativity-calibrated (phenomenological) waveform
- According to Messina et al. (2019), the 5.5PN correction significantly improves the accuracy of point-particle approximation
- We will see impact of the higher order corrections on Λ and importance of numerical relativity in Lec. 3,4