The 8th Huada school on QCD @ CCNU, Wuhan, China

Foundations of GW from BNS merger

with application to nuclear/hadron physics

Yuichiro Sekiguchi (Toho University)

Introduction

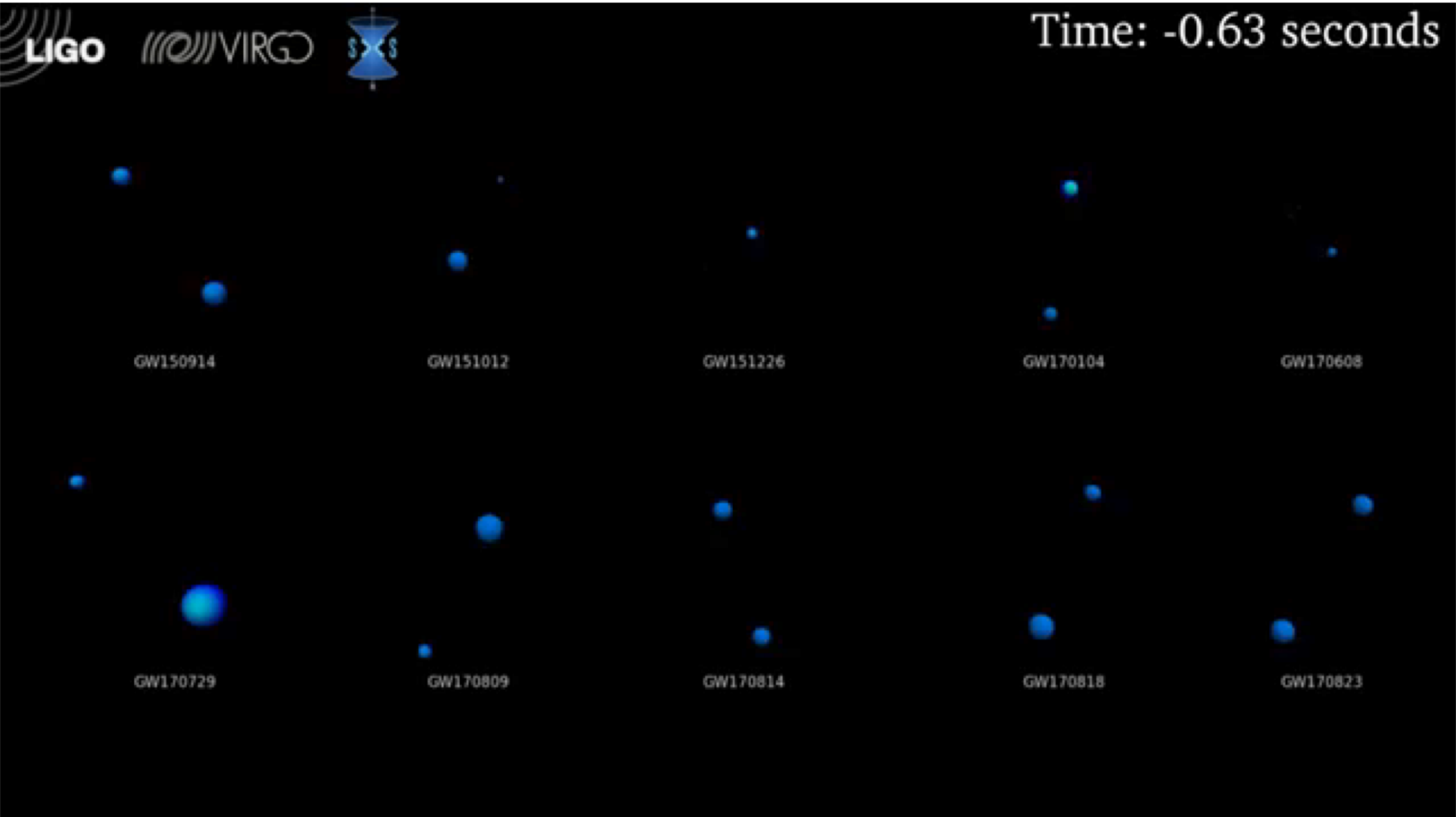
- Era of Gravitational-wave astronomy has come !

- Qualitatively new information provided by GW enabled us
  - Test of Einstein's theory of general relativity as the theory of gravity especially in strong field regimes
  - Test of cosmology via independent estimates of Hubble constant
  - Exploring the physics of dense nuclear matter using GW from binary neutron star (BNS) mergers

- Also, together with observations of electromagnetic signals,
  - The origin of short-hard gamma-ray bursts
  - The origin of heavy elements, like gold, lanthanides and actinoides
  - Constraining the maximum mass of neutron star by inferring the remnant of BNS merger

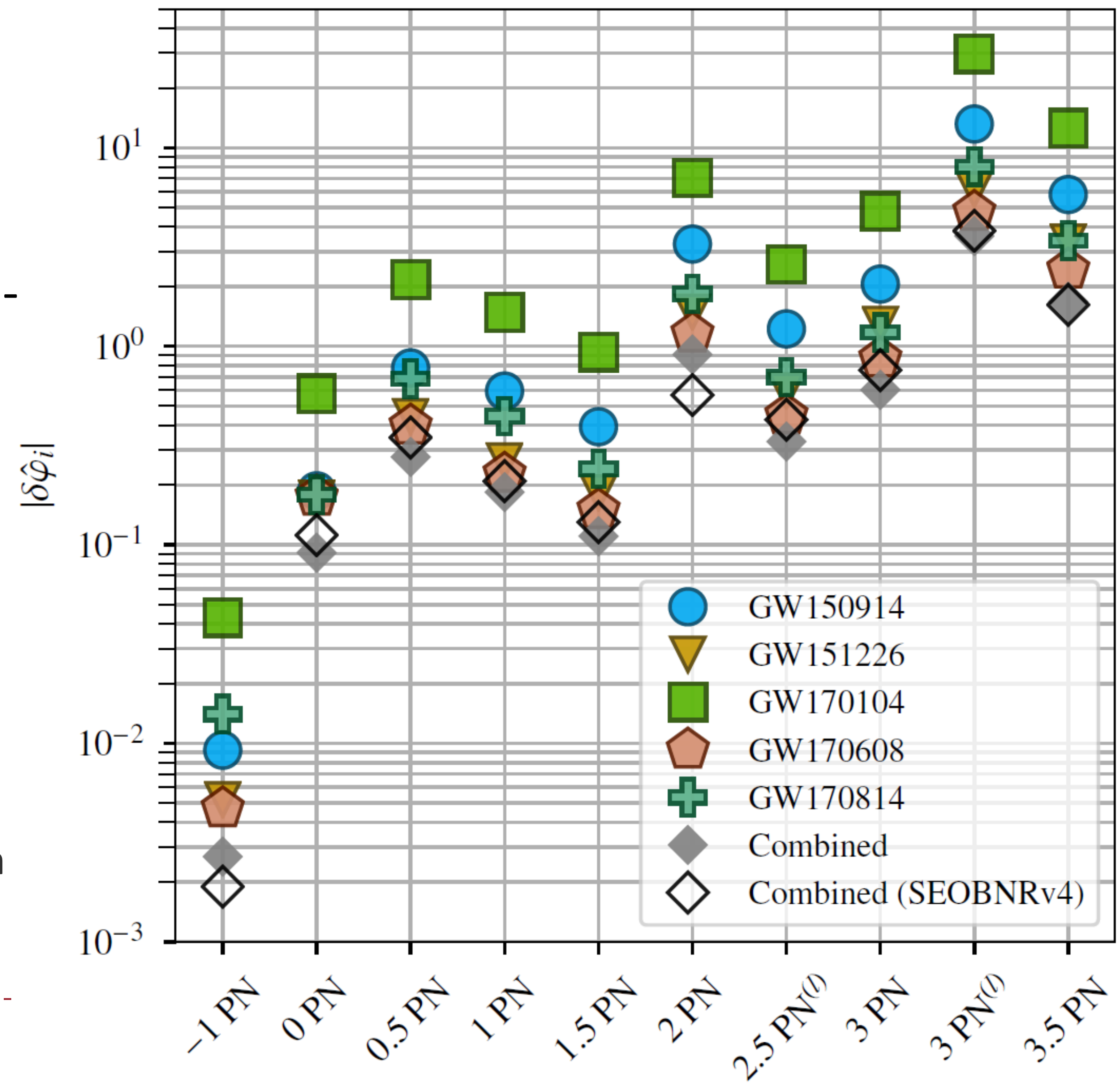
Tests of general relativity by BH-BH (1)

- 10 BH-BH mergers + 3 candidates (S190408an, S190412m, S190503bf)



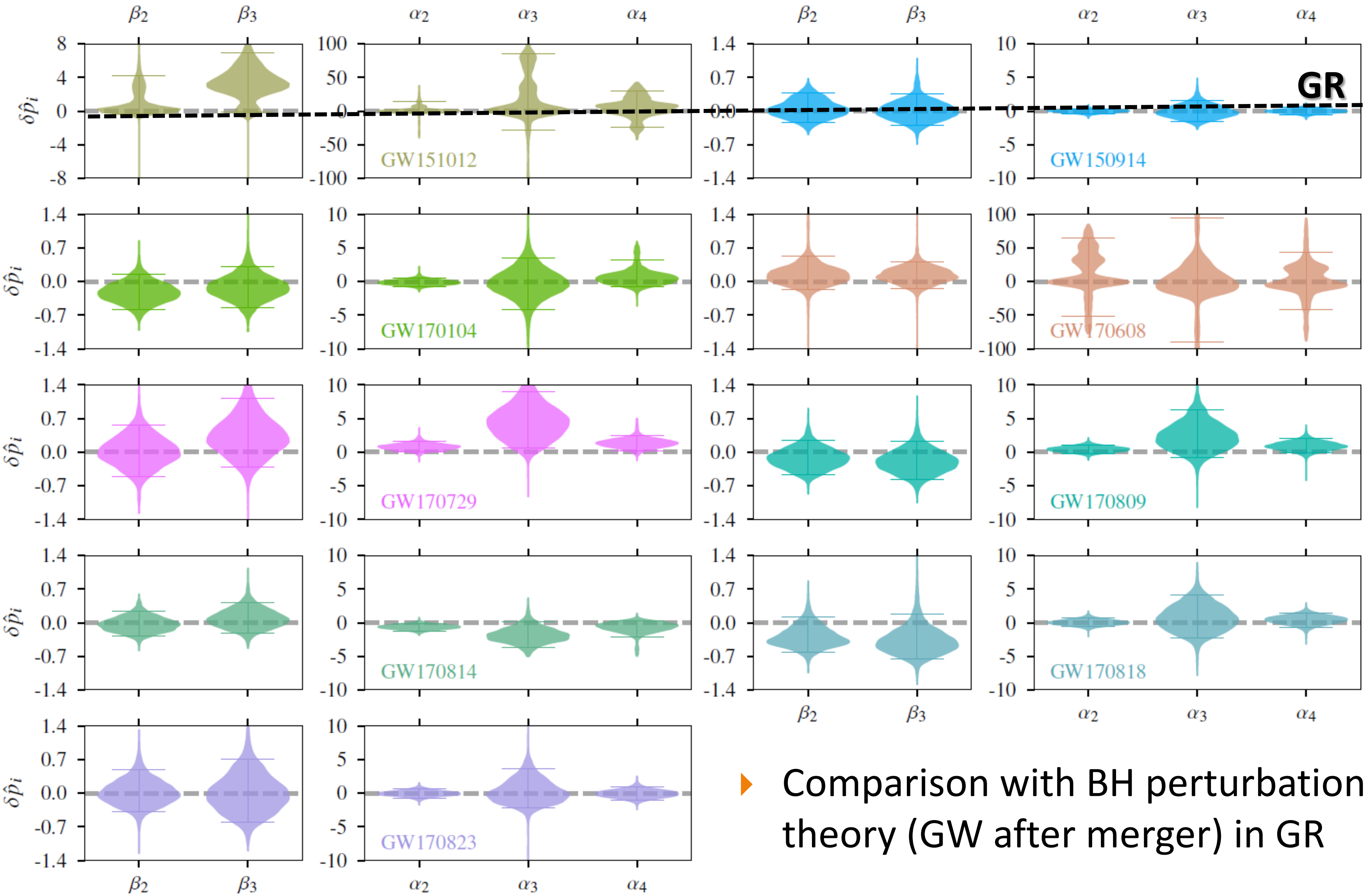
Tests of general relativity by BH-BH (2)

- Comparison of Post-Newtonian waveform (perturbation expansion in terms of G, v^2/c^2)
- Upper limit of GW phase-parameter error in each PN order
- -1PN order corresponds to dipolar radiation
  - In GR, GW can be generated only from quadrupolar radiation
  - In some alternative theories, dipolar radiation is predicted



LIGO Virgo Collaboration GWTC-1 paper (2018)

Tests of general relativity by BH-BH (3)



- Comparison with BH perturbation theory (GW after merger) in GR

Introduction

- Era of Gravitational-wave astronomy has come !

- Qualitatively new information provided by GW enabled us
  - Test of Einstein's theory of general relativity as the theory of gravity especially in strong field regimes
  - Test of cosmology via independent estimates of Hubble constant
  - Exploring the physics of dense nuclear matter using GW from binary neutron star (BNS) mergers

- Also, together with observations of electromagnetic signals,
  - The origin of short-hard gamma-ray bursts
  - The origin of heavy elements, like gold, lanthanides and actinoides
  - Constraining the maximum mass of neutron star by inferring the remnant of BNS merger

Test of Cosmology

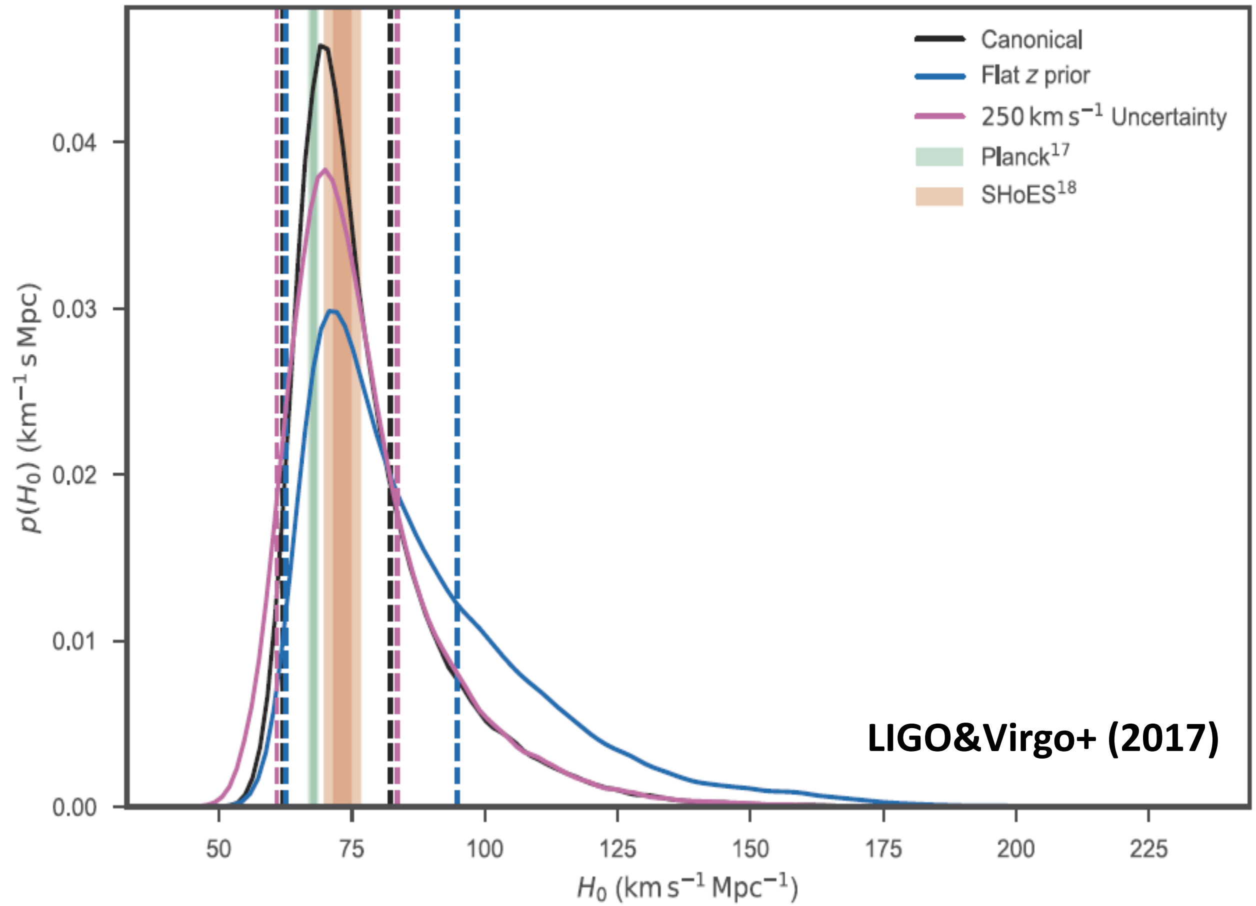
- Hubble's law

  $$cz = H_0 D_L$$

  z : cosmological redshift

  D_L : luminosity distance

  H_0 : Hubble constant



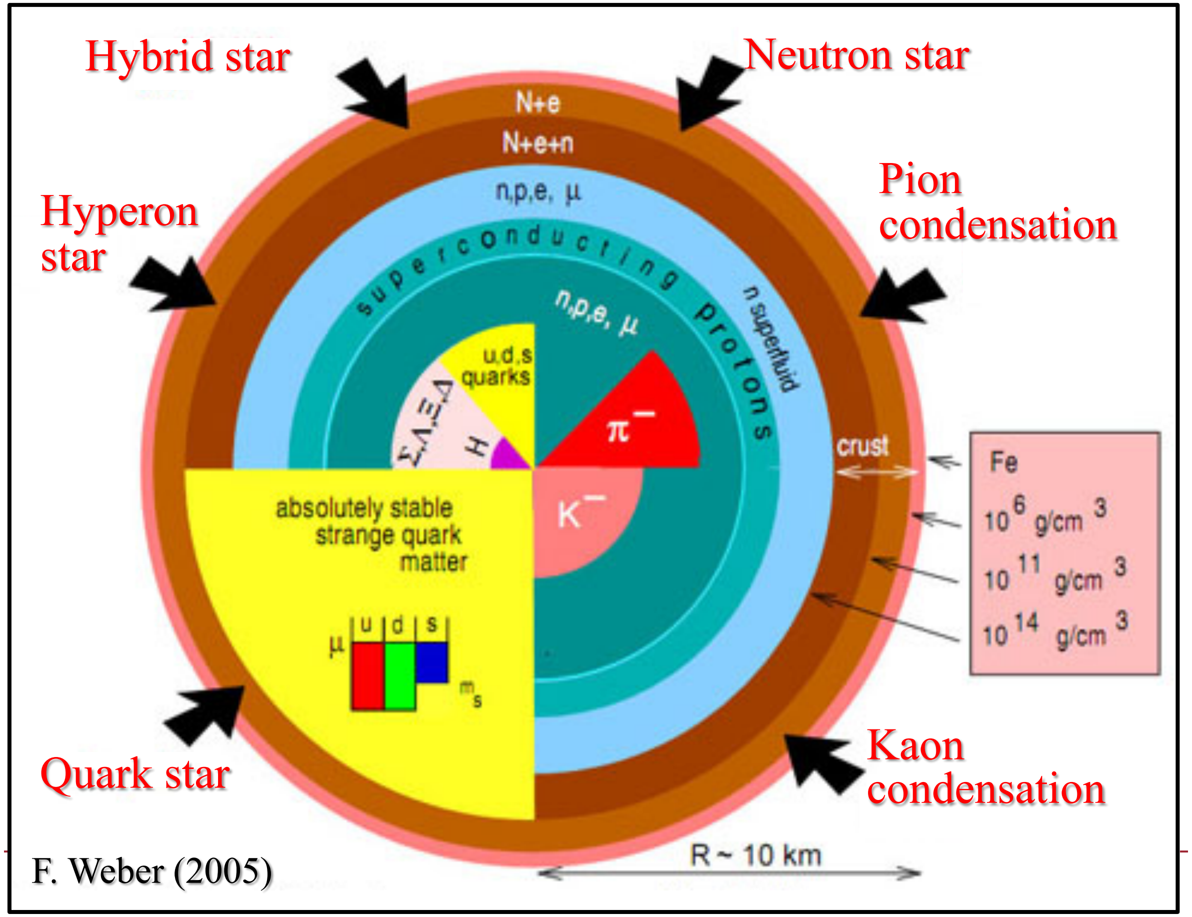
- We have two D_L : of EM and GW luminosities
- They can differ, for example, in theories with extra dimensions: only GW (gravity) can propagate in extra dimensions
- Measurement of H_0 could be used to test Cosmology

Introduction

- Era of Gravitational-wave astronomy has come !
- Qualitatively new information provided by GW enabled us
  - Test of Einstein's theory of general relativity as the theory of gravity especially in strong field regimes
  - Test of cosmology via independent estimates of Hubble constant
  - **Exploring the physics of dense nuclear matter using GW from binary neutron star (BNS) mergers**
- Also, together with observations of electromagnetic signals,
  - The origin of short-hard gamma-ray bursts
  - The origin of heavy elements, like gold, lanthanides and actinoides
  - Constraining the maximum mass of neutron star by inferring the remnant of BNS merger

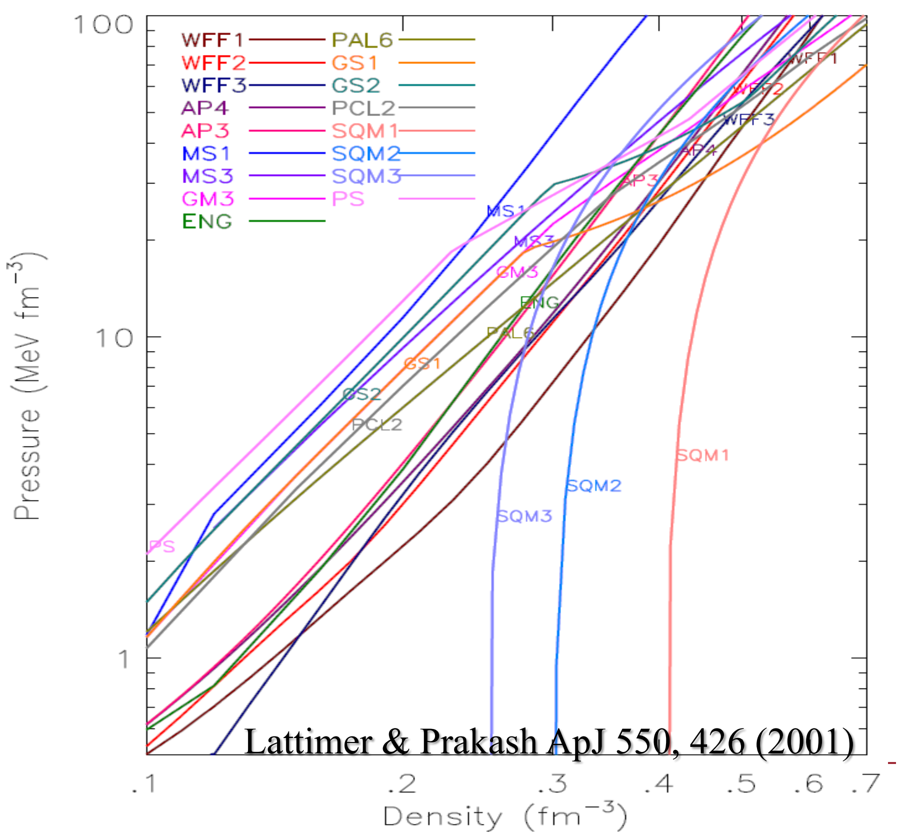
Neutron star structure and EOS

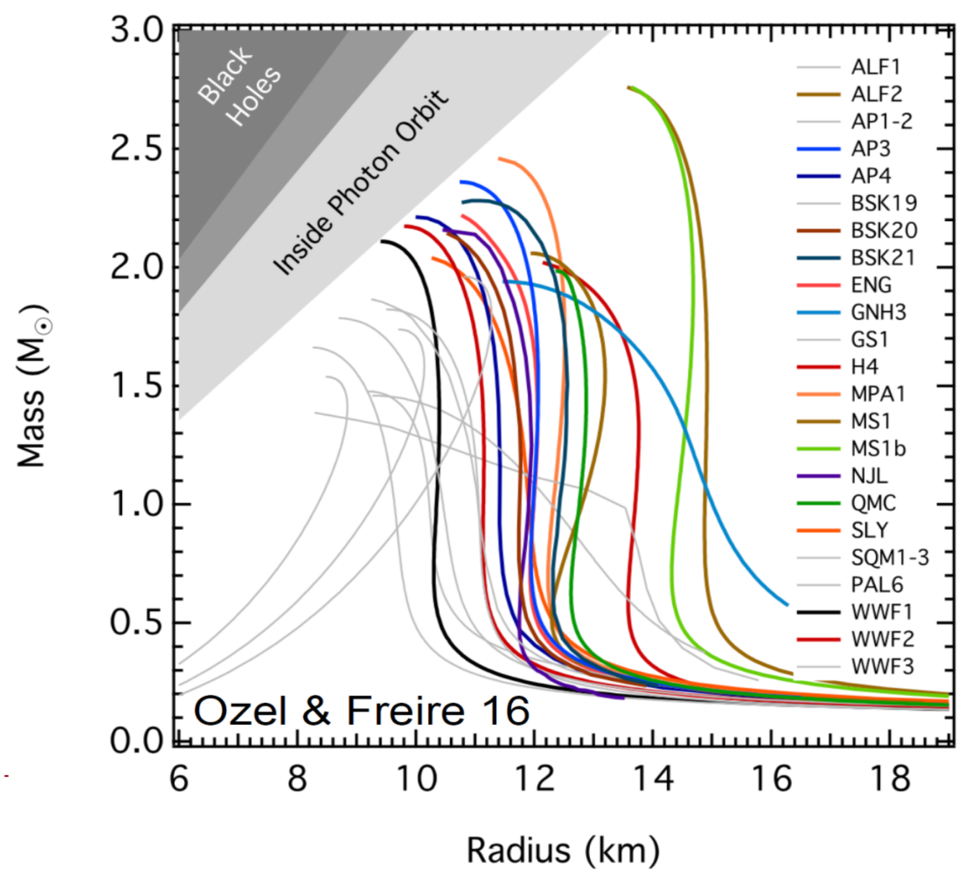
- Deep interiors of NS is still poorly known : many theoretical models
  - Each model predicts its own EOS : NS structure is uniquely determined
  - Model (EOS) ⇒ NS structure



Neutron star structure and EOS

- Deep interiors of NS is still poorly known : many theoretical models
  - Each model predicts its own EOS : NS structure is uniquely determined
  - Model (EOS) ⇒ NS structure
- Inverse problem : NS structure ⇒ constraining the models/EOS

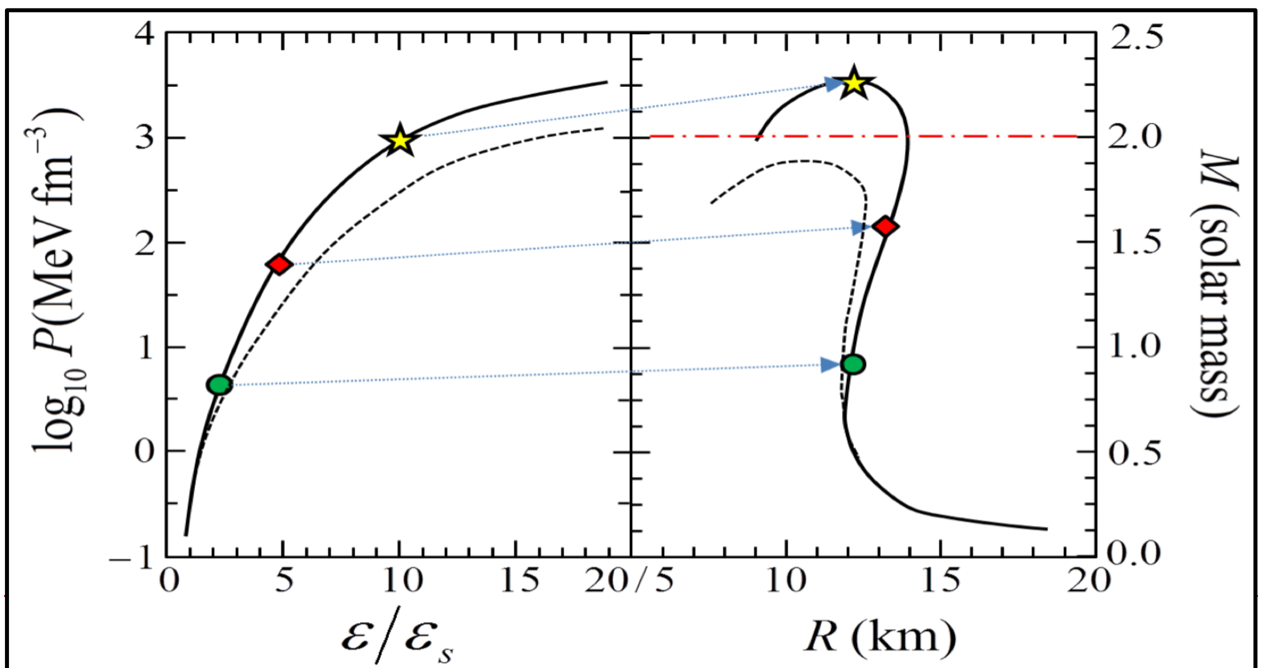




TOV (Tolman-Oppenheimer-Volkov) equations

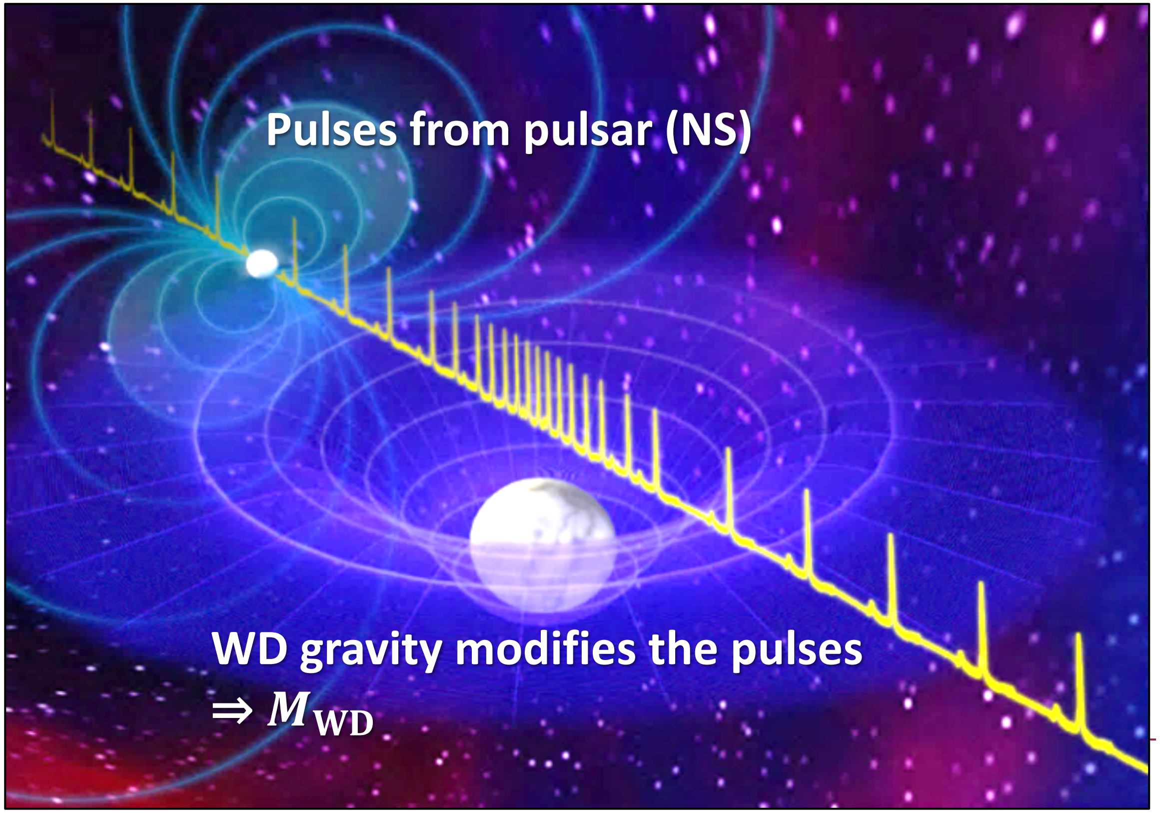
$$\frac{dP}{dr} = -\frac{Gm}{r^2}\left(1+\frac{P}{\varepsilon}\right)\left(1+\frac{4\pi r^3 P}{Mc^2}\right)\left(1-\frac{2GM}{r}\right)^{-1}, \qquad \frac{dM}{dr} = 4\pi r^2 \frac{\varepsilon}{c^2}$$

- put one-to-one correspondence between EOS ⇔ NS M-R relation
  - Lindblom ApJ 398, 569 (1992)
- set maximum mass $M_{\mathrm{EOS,max}}$ of NS associated with EOS
  - models with $M_{\mathrm{EOS,max}}$ not compatible with $M_{\mathrm{obs,max}}$ should be discarded



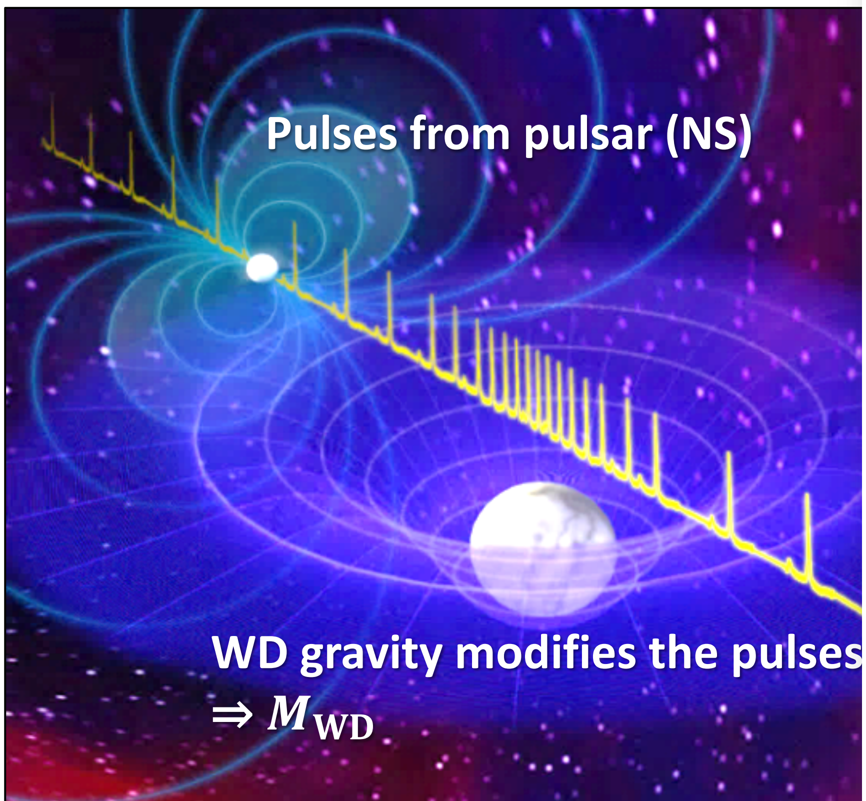
The most massive NS so far

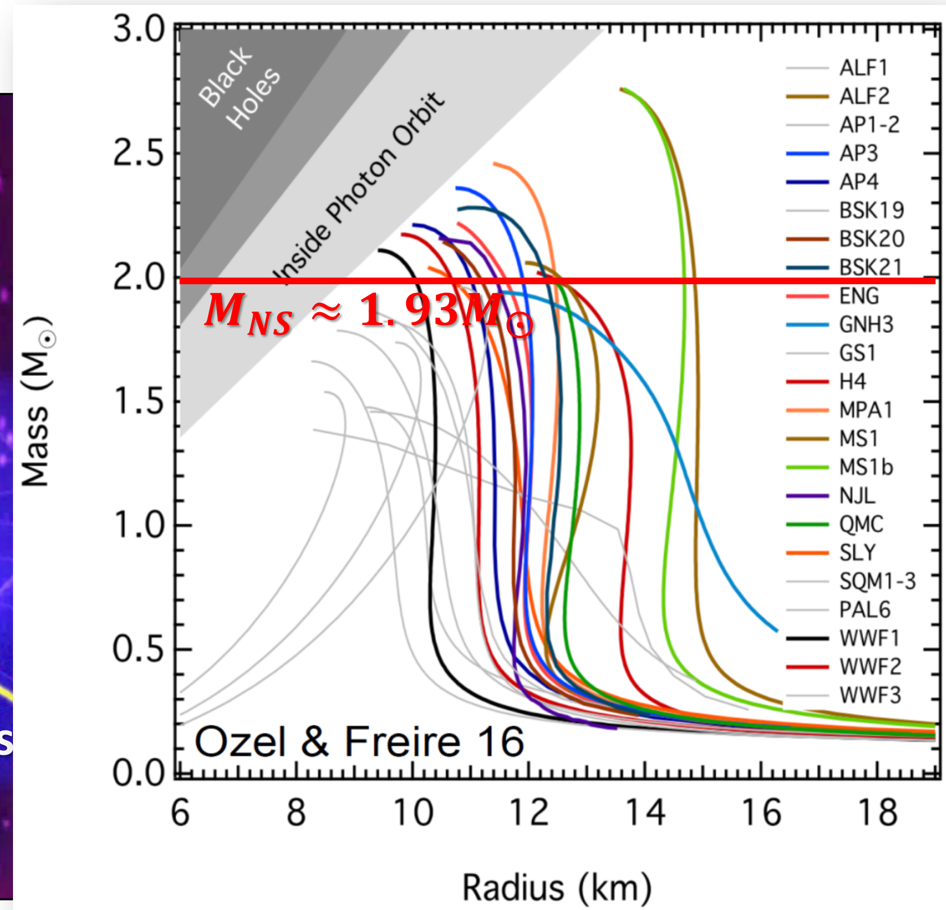
- A pulsar (PSR J1614-2230) – White dwarf (WD) binary
  - pulse profile is modified by the gravity of the WD
  - Mass of WD is determined ⇒ NS mass is determined (since total mass is known)
  - $M_{NS} \approx 1.93 M_{\odot}$



The most massive NS so far

- A pulsar (PSR J1614-2230) – White dwarf (WD) binary
  - pulse profile is modified by the gravity of the WD
  - Mass of WD is determined ⇒ NS mass is determined (total mass is known)
  - $M_{NS} \approx 1.93 M_{\odot}$





Constraining NS EOS with GW

- We will consider to extract information of both mass and radius (infact, mass and tidal deformability, a quantity which represent finite-size effect) from GW emitted in NS-NS merger
- Then, we can challenge the inverse problem : NS structure ⇒ constraining the models/EOS

- We will also consider to put a constraint on the maximum mass of NS using GW from NS-NS merger

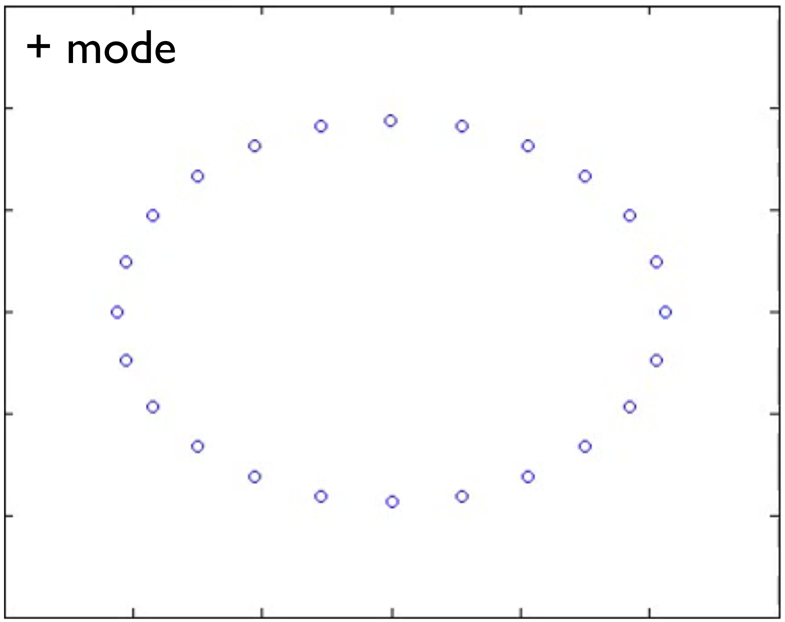
Contents

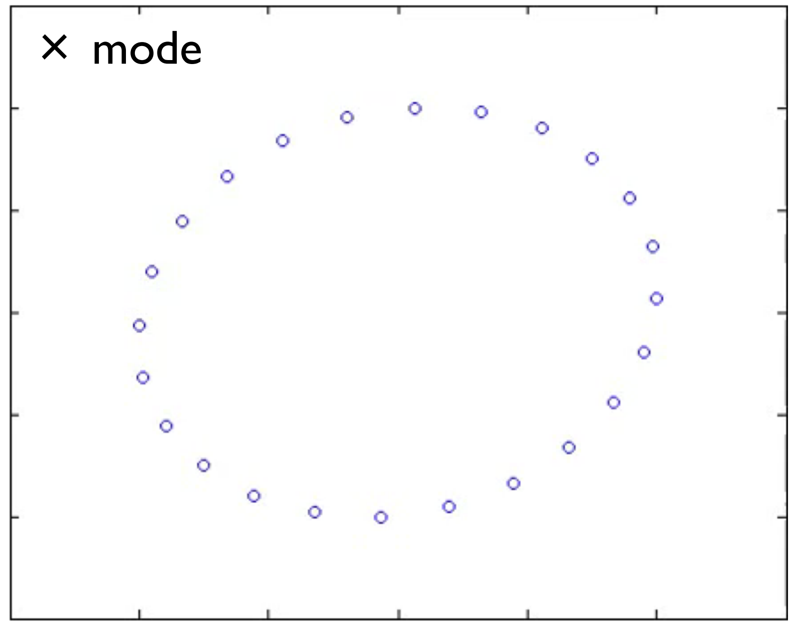
- Aim : Introduce physics of GW from NS-NS in a fundamental viewpoint
- Lecture 1: Linearized theory
  - GW propagation, TT gauge, polarization of GW (+, and × modes)
  - GW production, quadrupole formula
- Lecture 2: GW from binary system in circular orbit
  - the (point-particle) chirp signal, tidal deformability
  - Post-Newtonian GW and Numerical Relativity
- Lecture 3: Achievement in GW170817
  - Extraction of tidal deformability and its interpretation
  - Current constraint on EOS (combining with EM signals)
- Lecture 4: Future prospects
  - higher density regions, proving hadron-quark transition
  - Importance of numerical relativity

Lecture 1: Linearized Theory

The goals of lecture 1

- Understanding two polarization modes of GW propagation from fundamental viewpoints (using linearized Einstein's equaion)





- Deriving the energy flux formula $$\frac{dP_{\mathrm{gw}}}{d\Omega} = \frac{r^2 c^3}{32\pi G}\left\langle \dot{h}_+^2 + \dot{h}_\times^2 \right\rangle$$
- Deriving the quadrupole formula $$h_{ij}^{TT,\,\mathrm{quad}} = \frac{1}{r}\frac{2G}{c^4}\ddot{Q}^{ij}(t - r/c)$$

Lecture 1: Linearized Theory

the linearized Einstein's equations

Derivation of the linearized equation (1)

- In general relativity, the spacetime metric g_{ab} is determined by Einstein's equations :

$$\boxed{\boxed{G_{ab}} = \frac{8\pi G}{c^4} T_{ab}}$$

Einstain's equation

$$\boxed{\boxed{G_{ab}} = \boxed{R_{ab}} - \frac{1}{2} g_{ab} \sum_c \sum_d g^{cd} \boxed{R_{cd}}}$$

Definition of Einstain tensor G_{ab}

$$\boxed{\boxed{R_{ab}} = \sum_c \frac{\partial}{\partial x^c} \boxed{\Gamma^c{}_{ab}} - \frac{\partial}{\partial x^a}\left[\sum_c \boxed{\Gamma^c{}_{cb}}\right] + \sum_c \sum_e \left(\boxed{\Gamma^e{}_{ab}\Gamma^c{}_{ec}} - \boxed{\Gamma^e{}_{cb}\Gamma^c{}_{ea}}\right)}$$

Coordinate component of Ricci tensor R_{ab}

$$\boxed{\boxed{\Gamma^c{}_{ab}} = \frac{1}{2} \sum_d g^{cd} \left[\frac{\partial g_{bd}}{\partial x^a} + \frac{\partial g_{ad}}{\partial x^b} - \frac{\partial g_{ab}}{\partial x^d}\right]}$$

Christoffel symbol

- Now we expand g_{ab} in the background Minkowski metric η_{ab} as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad |h_{\mu\nu}| \ll 1$$

and linearize Einstein's equation

Derivation of the linearized equation (2)

- With a straightforward calculation, the linearized Einstein tensor (the left-hand of Einstein's eqs) is

$$2G_{\mu\nu} = -\Box h_{\mu\nu} + \partial^\rho \partial_\mu h_{\nu\rho} + \partial^\rho \partial_\mu h_{\nu\rho} - \partial_\mu \partial_\nu h - \eta_{\mu\nu}\left(\partial^\rho \partial^\sigma h_{\rho\sigma} - \Box h\right)$$

which can be simplified (by using the so-called trace-reversed tensor)

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$$

  - So as to satisfy $g^{\mu\lambda} g_{\lambda\nu} = \delta^\mu_\nu$, the inverse metric is given by $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$.
  - Note that the indices of tensors can be raised or lowered in linearized theory by the Minkowski metric η^{ab}, η_{ab} : e.g., $h = h^\mu_\mu = g^{\mu\nu} h_{\mu\nu} = \eta^{\mu\nu} h_{\mu\nu}$.

- In terms of $\bar{h}_{\mu\nu}$, the linearized Einstein's equations become

$$\Box \bar{h}_{\mu\nu} + \eta_{\mu\nu} \partial^\rho \partial^\sigma \bar{h}_{\rho\sigma} - \partial^\rho \partial_\nu \bar{h}_{\mu\rho} - \partial^\rho \partial_\mu \bar{h}_{\nu\rho} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

- We can further simplify it by using the gauge degree of freedom of general relativity

Derivation of the linearized equation (3)

- The transformation law of components of the 2nd lank tensor via $x^\mu \longrightarrow x'^\mu(x)$

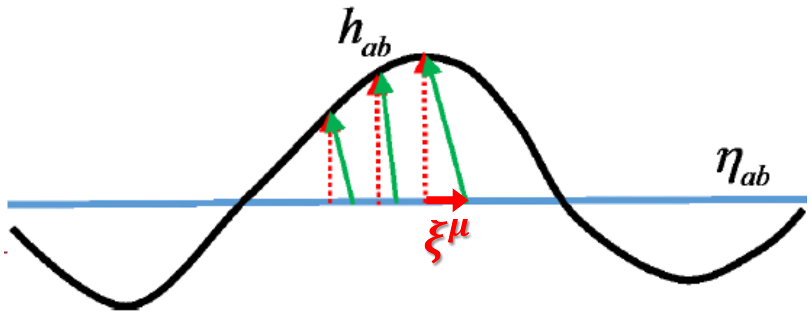
$$g_{\mu\nu}(x) \longrightarrow g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}$$

gives the linear transformation law of $h_{\mu\nu}$ under the infinitesimal coordinate transformation $x'^\mu = x^\mu - \xi^\mu$, as

$$h_{\mu\nu} \longrightarrow h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu = h_{\mu\nu} + \mathcal{L}_\xi \eta_{\mu\nu}$$

- Because all observers (all coordinate system) are equivalent in general relativity, $g_{\mu\nu}$ and $g'_{\mu\nu}$ (or $h_{\mu\nu}$ and $h'_{\mu\nu}$) must be equivalent as descriptions of gravitational fields
- This is the gauge degree of freedom of general relativity

- geometrically, this reflects the degree of freedom in description of the perturbed field $h_{\mu\nu}$: from which point $h_{\mu\nu}$ deviates



Derivation of the linearized equation (4)

- We can use this gauge degree of freedom to impose de Donder gauge condition
$$\partial^\nu \bar{h}_{\mu\nu} = 0$$
  - Because $\partial^\nu \bar{h}_{\mu\nu} \longrightarrow \partial^\nu \bar{h}_{\mu\nu} + \Box \xi_\mu$, de Donder gauge can be chosen by setting ξ_μ to be the solution of $\Box \xi_\mu = -\partial^\nu \bar{h}_{\mu\nu}$
- In de Donder gauge, the linearized Einstein's equations become simple wave equations :
$$\Box \bar{h}_{\mu\nu} + \eta_{\mu\nu} \partial^\rho \partial^\sigma \bar{h}_{\rho\sigma} - \partial^\rho \partial_\nu \bar{h}_{\mu\rho} - \partial^\rho \partial_\mu \bar{h}_{\nu\rho} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$
$$\Longrightarrow \boxed{\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}}$$
  - Note that the de Donder gauge condition gives 4 conditions, so that $h_{\mu\nu}$ have 6 independent components, not 10
- The above linearized Einstein's equation are the start point to study GW

Lecture 1: Linearized Theory

Propagation of Gravitational Waves

The transverse-traceless gauge (1)

- Outside the source (propagation of GW), we can greatly simplify the form of the metric
  - The following discussion closely parallels the situation in electrodynamics
- First, note that de Donder condition does not fix the gauge completely
  - de Donder gauge condition can be imposed by solving $\Box\xi^{\mu} = -\partial_{\nu}\bar{h}^{\mu\nu}$, but we still have degrees of freedom to add a homogeneous solution $\Box\xi^{\mu} = 0$
- We use these 'extra' degrees of freedom as follows
  - We set $\bar{h} = 0$ (traceless) using the time component ξ^0
  - We set $h_{0i} = 0$ using spatial components ξ^i
- Then $h_{\mu\nu} = \bar{h}_{\mu\nu}$, and de Donder condition gives

$$\partial^{\mu} h_{\mu 0} = \partial^0 h_{00} + \partial^i h_{i0} = \partial^0 h_{00} = 0$$

which indicates that h_{00} describes a static 'Newtonian' potential outside the source ⇒ for GW, we will set $h_{00} = 0$

The transverse-traceless gauge (2)

- Thus, we have $h_{0\mu} = 0$ $(h_{00} = 0, h_{0i} = 0)$
- Traceless condition is now $h_i^i = 0$
- de Donder gauge condition is now $0 = \partial^\nu h_{\mu\nu} = \partial^j h_{ij}$
- To summarize, we have imposed the so-called TT (Transverse-Traceless) gauge condition for GW outside the source :

$$h^{0\mu} = 0 \qquad h_i^i = 0 \qquad \partial^j h_{ij} = 0$$

- Note that in the TT gauge, only 2 degrees of freedom remain $(6 - 4 = 2)$

Propagation of GW in TT gauge (1)

- For a plane wave propagating along z direction, in TT gauge, we have

$$h_{ij}^{\mathrm{TT}}(t,z) = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \cos\omega\left(t - z/c\right)$$

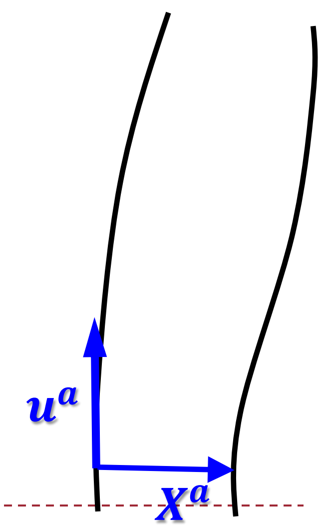
- What will happen to two nearby test bodies (in geodesics) when GW propagate through them ?
- The separation vector $X^i = \left(x_0 + \delta x(t),\, y_0 + \delta y(t)\right)$ obeys the geodesic deviation equation

$$\frac{d^2 X^a}{d\tau^2} = R_{bcd}{}^{a} X^b u^c u^d \qquad \frac{d}{d\tau} = u^a \nabla_a$$

  - Here $R^{\mu}_{\ \nu\rho\sigma} = \partial_\rho \Gamma^{\mu}_{\ \nu\sigma} - \partial_\sigma \Gamma^{\mu}_{\ \nu\rho} + \Gamma^{\mu}_{\ \alpha\rho}\Gamma^{\alpha}_{\ \nu\sigma} - \Gamma^{\mu}_{\ \alpha\sigma}\Gamma^{\alpha}_{\ \nu\rho}$ is Riemann curvature tensor

- For the linearized theory in TT gauge

$$\ddot{X}^i = \frac{1}{2}\ddot{h}_{ij}^{TT} X^j \quad \text{(dot denotes time-derivative)}$$



$$
\begin{aligned}
\frac{\partial^2 X^a}{\partial \tau^2} &= T^c \nabla_c (T^b \nabla_b X^a) \overset{(3.8)}{=} T^c \nabla_c (X^b \nabla_b T^a) \\
&= (T^c \nabla_c X^b)(\nabla_b T^a) + X^b T^c \nabla_c \nabla_b T^a \\
&= (X^c \nabla_c T^b)(\nabla_b T^a) + X^b T^c \nabla_b \nabla_c T^a - X^b T^c \left[\nabla_b \nabla_c T^a - \nabla_c \nabla_b T^a \right] \\
&= X^c \nabla_c (T^b \nabla_b T^a) + X^b T^c R_{bcd}{}^a T^d \\
&= R_{bcd}{}^a X^b T^c T^d
\end{aligned}
$$

$$
T^a \nabla_a X^b = X^a \nabla_a T^b \tag{3.8}
$$

Propagation of GW in TT gauge (2)

- It is easy to solve the geodesic deviation equation and we have two solutions

$$\ddot{X}^i = \frac{1}{2}\ddot{h}_{ij}^{TT} X^j \qquad h_{ij}^{\mathrm{TT}}(t,z) = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \cos\omega(t - z/c)$$

- **For h_+ (+ polarization)**

$$\delta\ddot{x} = -\frac{h_+}{2}(x_0 + \delta x)\omega^2 \cos\omega(t - z/c) \approx -\frac{h_+}{2} x_0 \omega^2 \cos\omega(t - z/c)$$

$$\delta\ddot{y} = +\frac{h_+}{2}(y_0 + \delta y)\omega^2 \cos\omega(t - z/c) \approx +\frac{h_+}{2} y_0 \omega^2 \cos\omega(t - z/c)$$

  - The solutions are

$$\boxed{\begin{aligned} \delta x &= +\frac{h_+}{2} x_0 \cos\omega(t - z/c) \\ \delta y &= -\frac{h_+}{2} y_0 \cos\omega(t - z/c) \end{aligned}}$$

- **For $h_\times$ ($\times$ polarization)**

$$\boxed{\begin{aligned} \delta x &= \frac{h_\times}{2} y_0 \cos\omega(t - z/c) \\ \delta y &= \frac{h_\times}{2} x_0 \cos\omega(t - z/c) \end{aligned}}$$

Propagation of GW in TT gauge (3)

- **For h_+(+ polarization)**

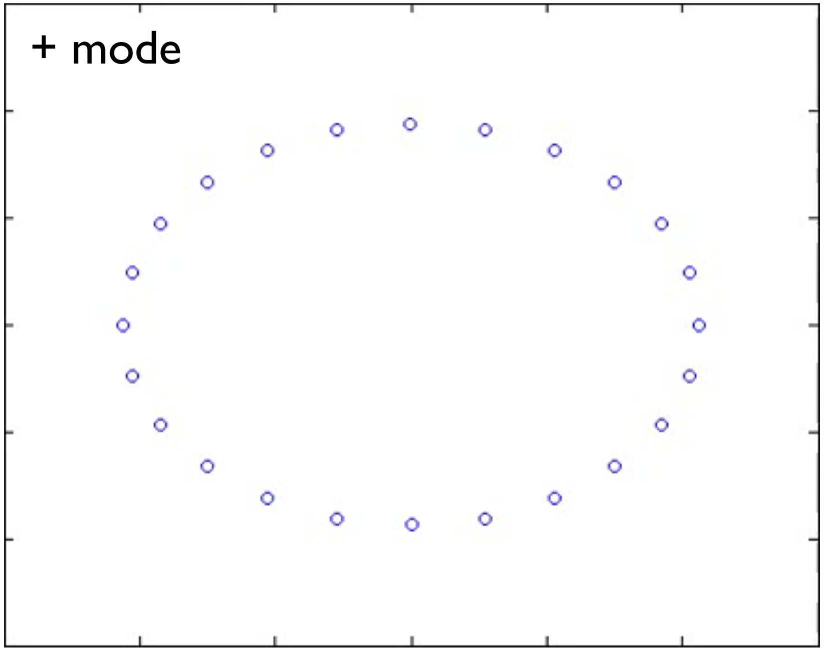
$$\delta x = +\frac{h_+}{2} x_0 \cos\omega(t - z/c)$$

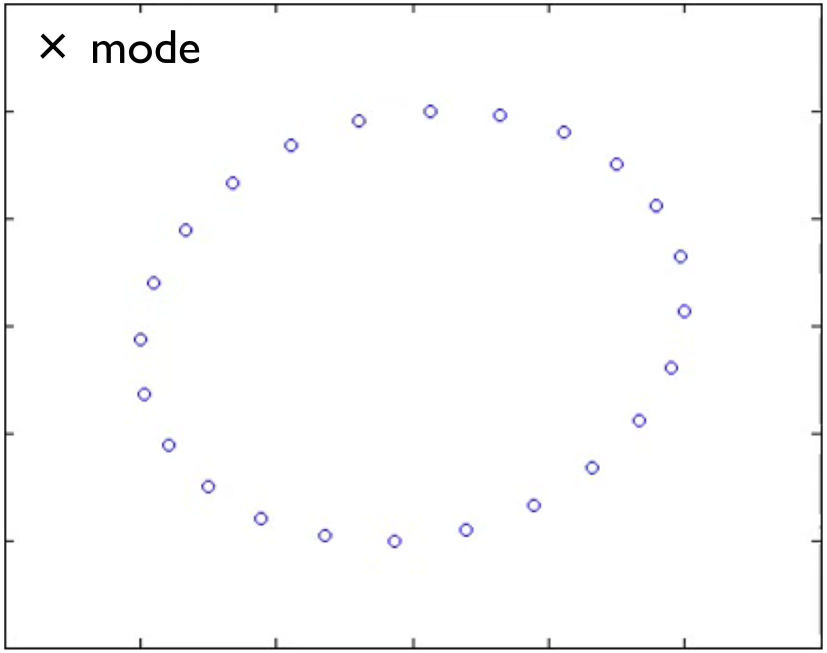
$$\delta y = -\frac{h_+}{2} y_0 \cos\omega(t - z/c)$$

- **For $h_\times$(× polarization)**

$$\delta x = \frac{h_\times}{2} y_0 \cos\omega(t - z/c)$$

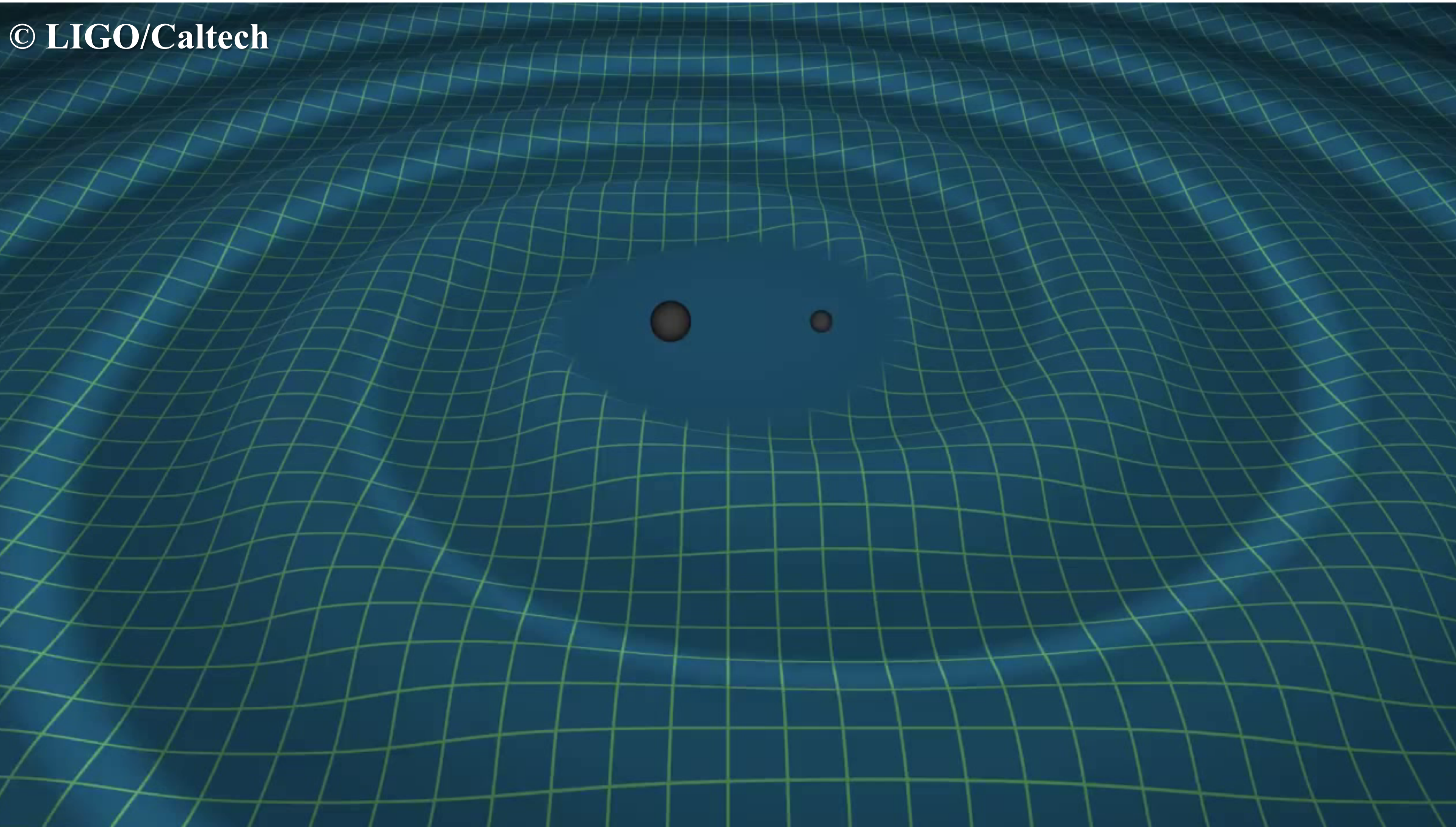
$$\delta y = \frac{h_\times}{2} x_0 \cos\omega(t - z/c)$$





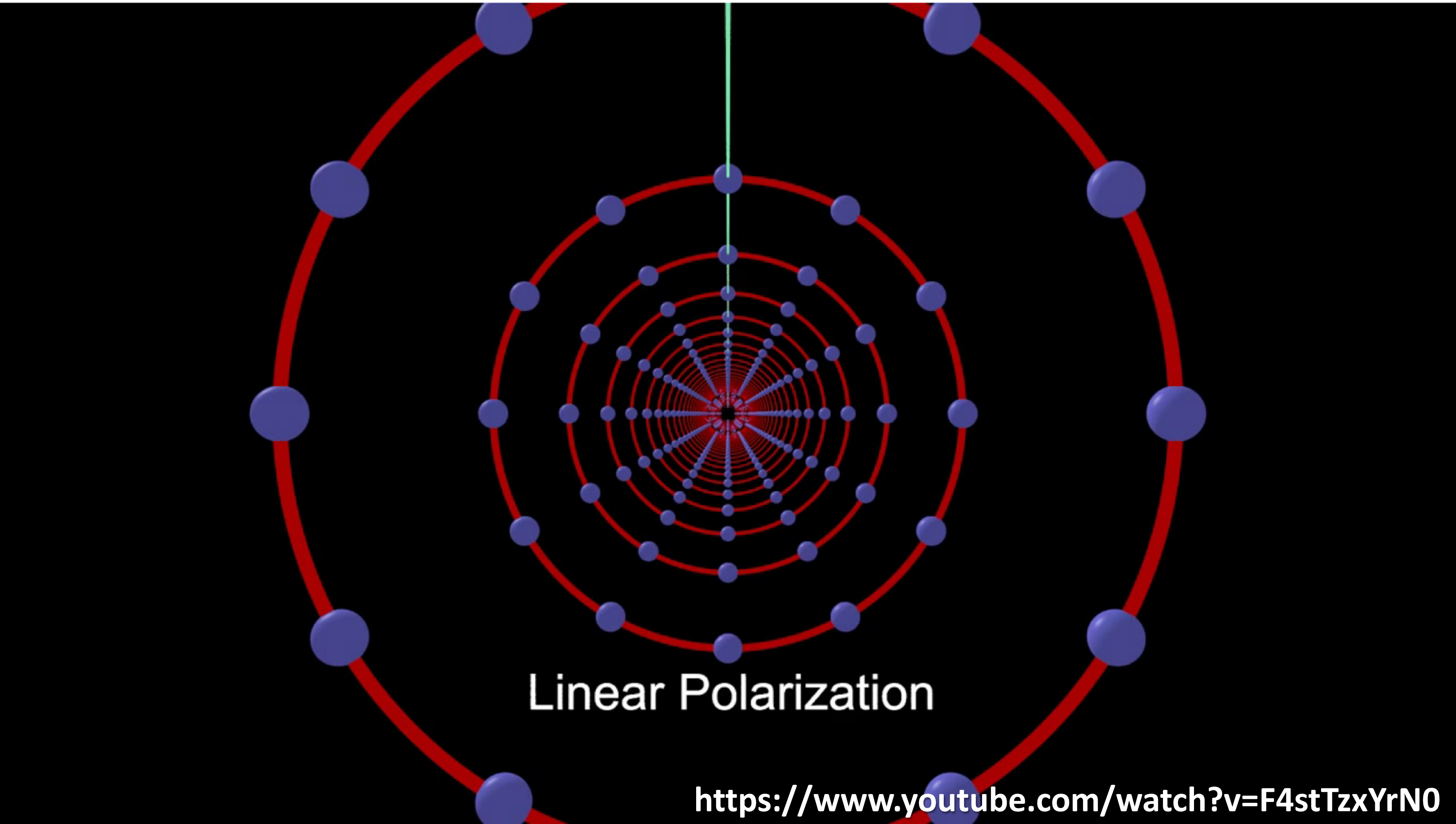
Propagation of GW in TT gauge (4)

- Somewhat different from naïve notion of ripples in spacetime



Propagation of GW in TT gauge (4)

- Somewhat different from naïve notion of ripples in spacetime



Generalization

- Given a plane wave GW solution $h_{\mu\nu}$ propagating in the direction $\vec{n}$, we can find solution in the TT gauge as follows
- First we introduce a projection tensor : $P_{ij}(\boldsymbol{n}) = \delta_{ij} - n_i n_j$
  - Note that this tensor is symmetric and transverse ($n^i P_{ij} = 0$)
  - Projection means $P_{ik} P_{kj} = P_{ij}$
- Then we construct a projection tensor

$$\Lambda_{ij,kl}(\boldsymbol{n}) = P_{ik} P_{jl} - \frac{1}{2} P_{ij} P_{kl}$$

  - Which is transverse on all indices : $n^i \Lambda_{ij,kl} = n^j \Lambda_{ij,kl} = \cdots = 0$
  - Traceless with respect to (i,j) and (k,l)
  - Explicitly $\Lambda_{ij,kl} = \delta_{ik}\delta_{jl} - \frac{1}{2}\delta_{ij}\delta_{kl} - n_j n_l \delta_{ik} - n_i n_k \delta_{jl} + \frac{1}{2} n_k n_l \delta_{ij} + \frac{1}{2} n_i n_j \delta_{kl} + \frac{1}{2} n_i n_j n_k n_l$
- The GW in the TT gauge is given by $h_{ij}^{\mathrm{TT}} = \Lambda_{ij,kl} h_{kl}$

Energy (& momentum) carried by GW (1)

- Let us regard the linearized gravity h_{ab} as a classical field on Minkowski spacetime
- Then, by Noether's theorem, we can derive the canonical energy momentum tensor for GW from the Einstein-Hilbert action

$$S_G = \frac{c^3}{16\pi G}\int d^4x\sqrt{-g}R$$

$-g = \det(-g_{\mu\nu})$: determinant of the metric

$R = g^{ab}R_{ab}$: Ricci tensor

- A messy calculation gives

$$S_G = -\frac{c^3}{64\pi G}\int d^4x\left[\partial_\mu h_{\alpha\beta}\partial^\mu h^{\alpha\beta} - \partial_\mu h\partial^\mu h + 2\partial_\mu h^{\mu\nu}\partial_\nu h - 2\partial_\mu h^{\mu\nu}\partial_\rho h^\rho_\nu\right]$$

- Then, we obtain (canonical) energy-momentum tensor for $h_{\mu\nu}$, as

$$t^{\mu\nu} = -\frac{\partial\mathcal{L}}{\partial(\partial_\mu h_{\alpha\beta})}\partial^\nu h_{\alpha\beta} + \eta^{\alpha\beta}\mathcal{L} = \frac{c^4}{32\pi G}\langle\partial^\mu h^{\alpha\beta}\partial^\nu h_{\alpha\beta}\rangle$$

  - Where we used Einstein's equation $\Box h_{\mu\nu} = 0$ and de Donder condition
  - $\langle\quad\rangle$ denotes an average over GW wavelength (for completeness)

Energy (& momentum) carried by GW (2)

- Let us compute the energy flux of GWs, t^{0r}
- The conservation of the energy-momentum tensor $\partial_\mu t^{\mu\nu} = 0$ implies

$$\int d^3x \left(\partial_0 t^{00} + \partial_i t^{0i}\right) = 0$$

  - Integration is taken over a volume V, bounded by a surface S

- In terms of the energy inside the volume V in TT gauge, $E = \int d^3x\, t^{00}$ the conservation of energy becomes

$$\frac{1}{c}\frac{dE}{dt} = -\int d^3x\, \partial_i t^{0i} = -\int dS\, n_i t^{0i} = -\int dS\, t^{0r}$$

  - where

$$t^{0r} = \frac{c^4}{32\pi G}\left\langle \partial^0 h^{TT,ij} \partial^r h_{ij}^{TT} \right\rangle = \frac{c^3}{32\pi G}\left\langle \partial^t h^{TT,ij} \partial^r h_{ij}^{TT} \right\rangle \qquad \frac{\partial}{\partial x^0} = \frac{\partial}{\partial (ct)}$$

- We are interested in the energy flux in a far distance from the source for which a more useful expression can be derived

Energy (& momentum) carried by GW (3)

- For GWs propagating radially outward, $h^{\mathrm{TT}}_{ij}(t,r) = \dfrac{1}{r} f_{ij}(t - r/c)$, then

$$\partial^r h^{TT}_{ij} = \partial_r h^{TT}_{ij} = \frac{1}{r}\frac{\partial f_{ij}}{\partial r} - \frac{f_{ij}}{r^2} = -\frac{1}{r}\frac{\partial f_{ij}}{\partial (ct)} - \frac{f_{ij}}{r^2}$$

$$= -\partial_0 h^{TT}_{ij} + O(r^{-1}) = +\partial^0 h^{TT}_{ij} + O(r^{-1})$$

- So that at large distances,

$$t^{0r} = \frac{c^4}{32\pi G}\left\langle \partial^0 h^{TT,ij} \partial^r h^{TT}_{ij} \right\rangle = \frac{c^2}{32\pi G}\left\langle \dot{h}^{TT,ij} \dot{h}^{TT}_{ij} \right\rangle$$

- Thus the energy flux of GW is

$$\frac{1}{c}\frac{dE}{dt} = -\int dS\, t^{0r} \quad \Rightarrow \quad \boxed{\begin{aligned}\frac{dP_{\mathrm{gw}}}{d\Omega} &\equiv -\frac{r^2 dE}{dSdt} = \frac{r^2 c^3}{32\pi G}\left\langle \dot{h}^{TT,ij} \dot{h}^{TT}_{ij} \right\rangle \\ &= \frac{r^2 c^3}{32\pi G}\left\langle \dot{h}_+^2 + \dot{h}_\times^2 \right\rangle\end{aligned}}$$

- [This formula will be used in deriving GW from a binary system in lecture 2]

Lecture 1: Linearized Theory

Generation of Gravitational Waves

Multipole and low velocity expansion (1)

- Let us start from the following qualitative consideration

- Let ω_s : the typical frequency of the motion inside the source

  d : the source size,

  then the typical velocity will be $v \sim \omega_s d$

- The frequency of GW will be $\omega_{\rm gw} \sim \omega_s \sim v/d$, so that GW wavelength is

$$\lambda_{\rm gw} = c/\omega_{\rm gw} \sim d(c/v)$$

  which is much larger than the source size ($\lambda_{\rm gw} \gg d$) for low velocity

- This implies that generation of GW will not depends on fine features, but determined by coarse/bulk features of the source

  $\Rightarrow$ lower-order multipoles will contribute to GW in lower velocity sources (we will explicitly derive in below)

Multipole and low velocity expansion (2)

- In the linearized theory, generation of GWs is governed by the linearized eq.

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

- The solution is given using the retarded Green's function as

$$\bar{h}_{\mu\nu}(t,\boldsymbol{x}) = \frac{4G}{c^4}\int d^3x' \frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} T_{\mu\nu}(t_{\mathrm{ret}},\boldsymbol{x}')$$

  - where $t_{\mathrm{ret}} = t - |\boldsymbol{x}-\boldsymbol{x}'|/c$ is the retardation time

- We are interested in GWs far from the source, so that with $n^i = x^i/r$,

$$h_{ij}(t,\boldsymbol{x}) = \frac{1}{r}\frac{4G}{c^4}\int d^3x' T_{ij}\left(t - \frac{r}{c} + \frac{\boldsymbol{x}'\cdot\boldsymbol{n}}{c}, \boldsymbol{x}'\right) \qquad |\boldsymbol{x}-\boldsymbol{x}'| = r - x'^i n_i + O(1/r)$$

- Now we we write T_{ij} in terms of its Fourier transform

$$T_{ij}\left(t - \frac{r}{c} + \frac{\boldsymbol{x}'\cdot\boldsymbol{n}}{c}, \boldsymbol{x}'\right) = \int \frac{d^4k}{(2\pi)^4}\tilde{T}(\omega,\boldsymbol{k})\exp\left[-i\omega\left(t - \frac{r}{c} + \frac{\boldsymbol{x}'\cdot\boldsymbol{n}}{c}\right) + i\boldsymbol{k}\cdot\boldsymbol{x}'\right]$$

Multipole and low velocity expansion (3)

$$T_{ij}\left(t-\frac{r}{c}+\frac{\boldsymbol{x}'\cdot\boldsymbol{n}}{c},\boldsymbol{x}'\right)=\int\frac{d^4k}{(2\pi)^4}\tilde{T}(\omega,\boldsymbol{k})\exp\left[-i\omega\left(t-\frac{r}{c}+\frac{\boldsymbol{x}'\cdot\boldsymbol{n}}{c}\right)+i\boldsymbol{k}\cdot\boldsymbol{x}'\right]$$

- Here we note that
  - $\tilde{T}_{ij}$ will be peaked around the typical frequency ω_s
  - Integration is restricted within the source size : $|x'| < d$

then, we have for low velocity sources $\quad \dfrac{\omega}{c}\boldsymbol{x}'\cdot\boldsymbol{n}\lesssim\dfrac{\omega_s}{c}d\sim\dfrac{v}{c}\ll 1$

- Then, in the right-hand-side, we may expand

$$\exp\left[-i\omega\left(t-\frac{r}{c}+\frac{\boldsymbol{x}'\cdot\boldsymbol{n}}{c}\right)\right]=e^{-i\omega(t-r/c)}\left[1-\frac{i\omega}{c}x'_in^i+\frac{1}{2}\left(-\frac{i\omega}{c}\right)^2(x'_in^i)(x'_jn^j)+\cdots\right]$$

- Substituting to the Fourier transform, we obtain

$$T_{ij}\left(t-\frac{r}{c}+\frac{\boldsymbol{x}'\cdot\boldsymbol{n}}{c},\boldsymbol{x}'\right)\approx T_{ij}\left(t-\frac{r}{c},\boldsymbol{x}'\right)+\frac{1}{c}x'_kn^k\partial_0T_{ij}+\frac{1}{2c^2}x'_kx'_ln^kn^l\partial_0^2T_{ij}+\cdots$$

  - Note that there is no mention to the low velocity nature if we regard the above as a result of the direct expansion for small $\boldsymbol{x}'\cdot\boldsymbol{n}/r$

Multipole and low velocity expansion (4)

$$h_{ij}(t,\boldsymbol{x}) = \frac{1}{r}\frac{4G}{c^4}\int d^3x' \, T_{ij}\left(t-\frac{r}{c}+\frac{\boldsymbol{x}'\cdot\boldsymbol{n}}{c},\boldsymbol{x}'\right)$$

$$T_{ij}\left(t-\frac{r}{c}+\frac{\boldsymbol{x}'\cdot\boldsymbol{n}}{c},\boldsymbol{x}'\right) \approx T_{ij}\left(t-\frac{r}{c},\boldsymbol{x}'\right)+\frac{1}{c}x'_k n^k \partial_0 T_{ij}+\frac{1}{2c^2}x'_k x'_l n^k n^l \partial_0^2 T_{ij}+\cdots$$

- Let us define moments

$$S^{ij} = \int d^3x \, T^{ij}(t-\tfrac{r}{c},\boldsymbol{x}), \quad S^{ij,k} = \int d^3x \, T^{ij}(t-\tfrac{r}{c},\boldsymbol{x})x^k, \quad S^{ij,kl} = \int d^3x \, T^{ij}(t-\tfrac{r}{c},\boldsymbol{x})x^k x^l$$

then, we get

$$\boxed{\bar{h}_{ij}^{\mathrm{TT}}(t,\boldsymbol{x}) = \frac{4G}{c^4}\frac{1}{r}\Lambda_{ij,kl}\left[S^{kl}+\frac{1}{c}n_m \dot{S}^{kl,m}+\frac{1}{2c^2}n_m n_p \ddot{S}^{kl,mp}+\cdots\right]}$$

- This is multipole formula for GW generation in linearized theory
- Note again that higher multipoles corresponds to higher order in v/c

Quadrupole formula (1)

$$\bar{h}_{ij}^{\mathrm{TT}}(t,\boldsymbol{x}) = \frac{4G}{c^4}\frac{1}{r}\Lambda_{ij,kl}\left[S^{kl} + \frac{1}{c}n_m\dot{S}^{kl,m} + \frac{1}{2c^2}n_m n_p\ddot{S}^{kl,mp} + \cdots\right]$$

- The multipole formula is not very useful because it is given in terms of moments of matter stress tensor T_{ij}
- Using the conservation law $\partial_\mu T^{\mu\nu} = 0$, in the leading order, we can rewrite it using mass (energy) moment M, M^i, M^{ij}, …

$$c^2 M(t) = \int d^3x\, T^{00}(t,\boldsymbol{x}), \quad c^2 M^i(t) = \int d^3x\, T^{00}(t,\boldsymbol{x})x^i, \quad c^2 M^{ij}(t) = \int d^3x\, T^{00}(t,\boldsymbol{x})x^i x^j$$

$$cP^i(t) = \int d^3x\, T^{0i}(t,\boldsymbol{x}), \quad c\; P^{i,j}(t) = \int d^3x\, T^{0i}(t,\boldsymbol{x})x^j, \quad c\; P^{i,jk}(t) = \int d^3x\, T^{0i}(t,\boldsymbol{x})x^j x^k$$

- Because $\partial_0 T^{i0} = -\partial_k T^{ik}$ and $\partial_0 T^{00} = -\partial_k T^{0k}$,

$$c\partial_0 P^{i,j} = \int d^3x\, x^j \partial_0 T^{i0} = -\int d^3x\, x^j \partial_k T^{ik} = \int d^3x\, (\partial_k x^j) T^{ik} = \int d^3x\, \delta_k^j T^{ik} = S^{ij}$$

$$c^2\partial_0 M^{ij} = -\int d^3x\, x^i x^j \partial_k T^{0k} = \int d^3x\, T^{0k}\partial_k(x^i x^j) = c(P^{i,j} + P^{j,i})$$

so we obtain

$$2S^{ij} = c^2\partial_0^2 M^{ij} = \ddot{M}^{ij}$$

Quadrupole formula (2)

- Thus, for the leading order, GW are generated by the 2nd time derivative of mass quadrupole moment

$$\boxed{h_{ij}^{\mathrm{TT,\,quad}} = \frac{1}{r}\frac{2G}{c^4}[\ddot{M}^{ij}]^{TT} = \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij,kl}\ddot{Q}^{kl}(t-r/c)}$$

where

$$Q^{ij} \equiv M^{ij} - \frac{1}{3}\delta^{ij}M_k^k = \int d^3x\rho(t,\boldsymbol{x})\left(x^i x^j - \frac{1}{3}r^2\delta^{ij}\right)$$

- This is [the quadrupole formula]

- Order of magnitude estimate gives

$$h_{ij} \sim \epsilon\frac{1}{r}\frac{GMR^2}{c^4}\omega_s^2 \sim \epsilon\frac{1}{r}\frac{GMR^2}{c^4}\left(\frac{v}{R}\right)^2 \sim \epsilon\frac{R}{r}\frac{GM}{c^2R}\left(\frac{v}{c}\right)^2$$

  - (nearby $R\sim r$), Non-spherical $(\epsilon\sim 1)$, high-velocity $(v\sim c)$, strong-gravity $(GM/c^2R \sim 1)$ phenomena are promising sources of GW