The 8th Huada school on QCD @ CCNU, Wuhan, China

Foundations of GW from BNS merger

with application to nuclear/hadron physics

Yuichiro Sekiguchi (Toho University)

Contents

- Aim : Introduce physics of GW from NS-NS in a pedagogical viewpoint
- Lecture 1: Linearized theory
  - GW propagation, TT gauge, polarization of GW (+, and × modes)
  - GW production, quadrupole formula
- Lecture 2: GW from binary system in circular orbit
  - the (point-particle) chirp signal, tidal deformability
  - Post-Newtonian GW and Numerical Relativity
- Lecture 3: Achievement in GW170817
  - Extraction of tidal deformability and its interpretation
  - Current constraint on EOS (combining with EM signals)
- Lecture 4: Future prospects
  - higher density regions, proving hadron-quark transition
  - Numerical relativity

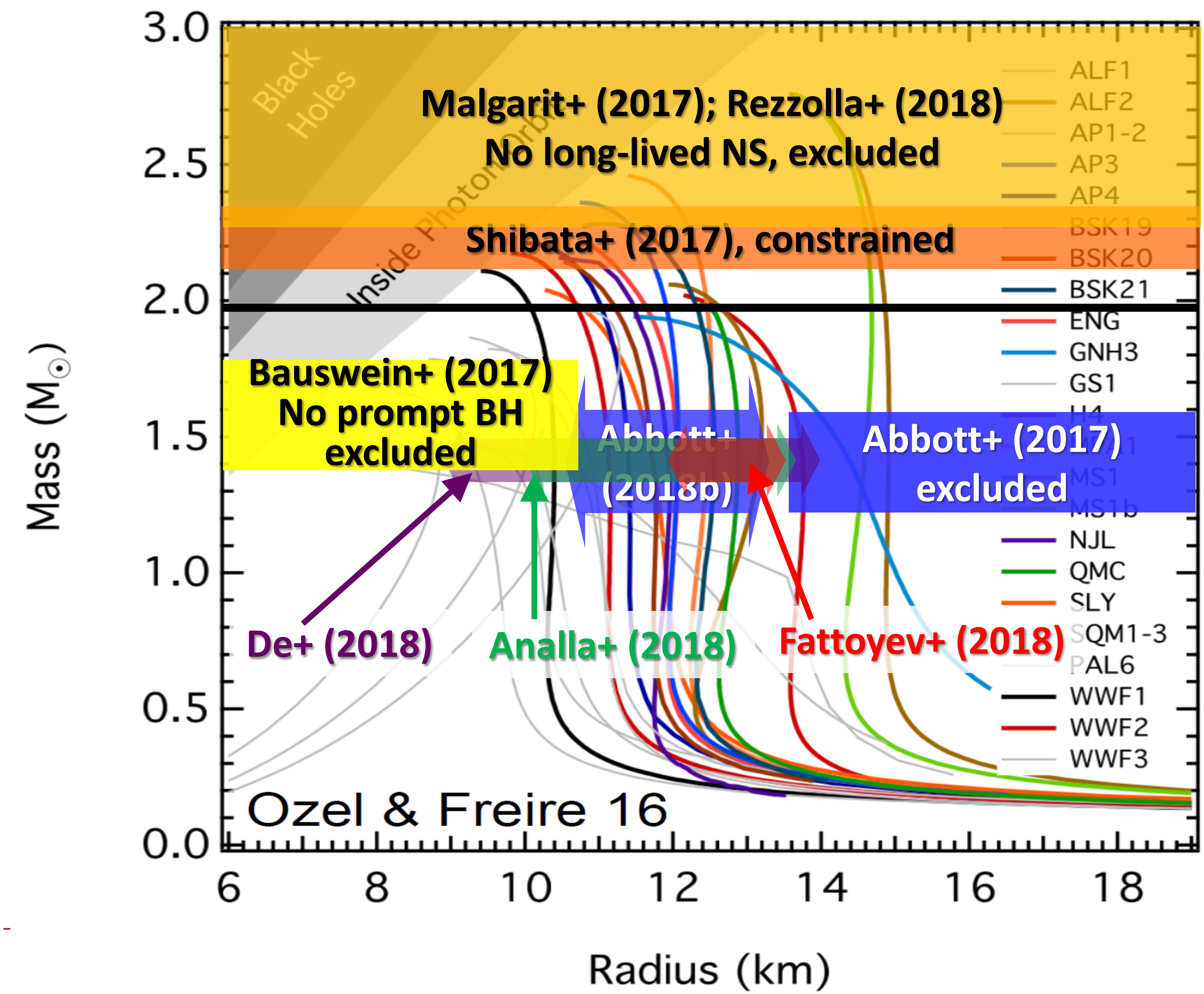
- Significance of detection

$$\text{Significance} \sim \mathbf{S/N} \sim \int \boldsymbol{df}\, \frac{\langle \mathbf{data} | \mathbf{template} \rangle}{\boldsymbol{S_n(f)}}$$

- Significance of parameter determination

$$\text{Significance} \sim \mathbf{S/N} \sim \int \boldsymbol{df}\, \frac{\langle \mathbf{data} | \Delta\mathbf{template} \rangle}{\boldsymbol{S_n(f)}}$$

Summary of lecture 3



Hotokezaka et al. (2016)

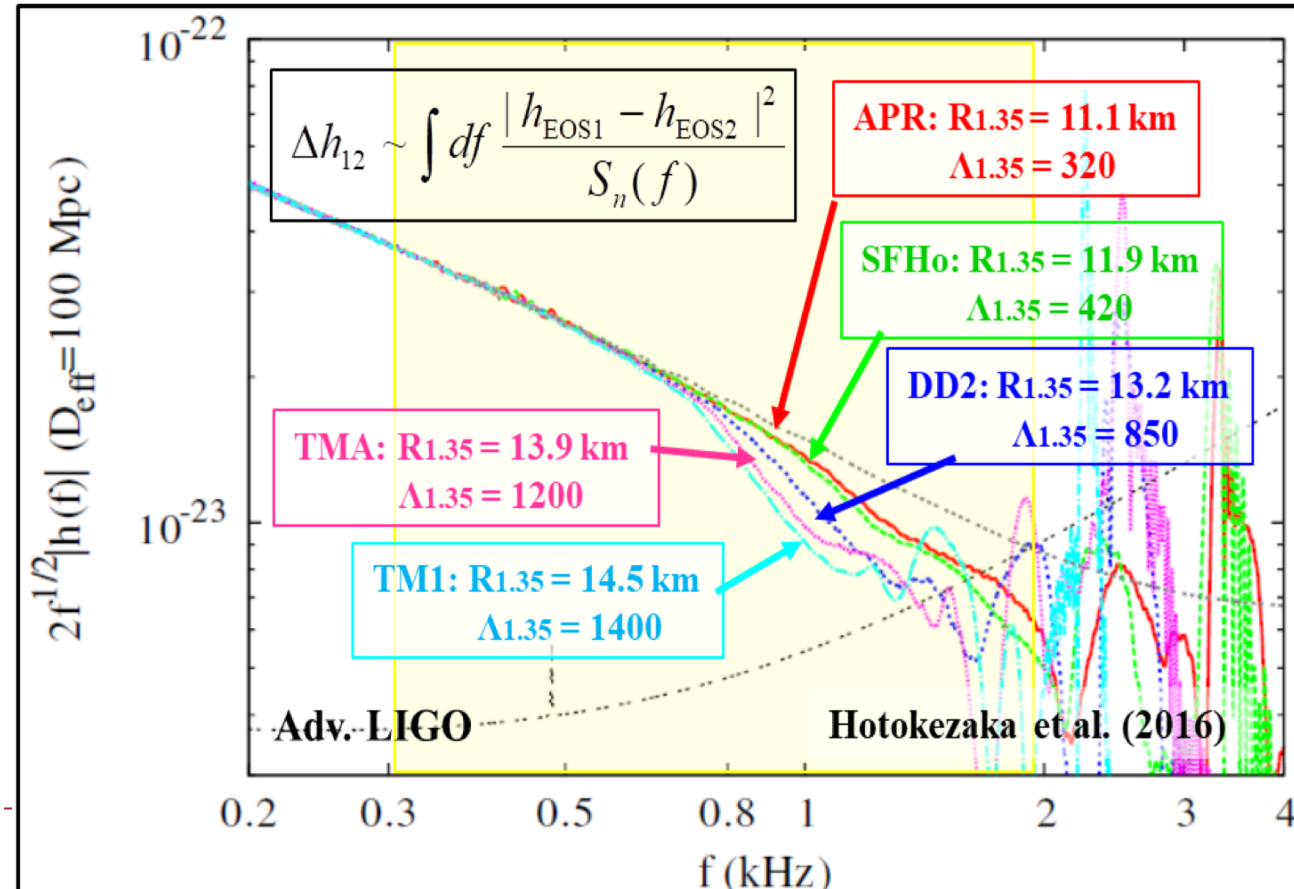
Future prospects: Constraining EOS by Λ

- Define distinguishability Δh_{12}
  - $\Delta h_{12} = 1$: marginally distinguishable
  - E.g. APR and TM1 are distinguishable (~3-σ level) for Deff = 200 Mpc

	APR	SFHo	DD2	TMA	TM1
APR	—	σ: 0.7	2.3	3.0	3.5
SFHo	0.8 km	—	1.9	2.7	3.3
DD2	2.1 km	1.3 km	—	1.3	2.5
TMA	2.8 km	2.0 km	0.7 km	—	1.7
TM1	3.4 km	2.6 km	1.3 km	0.6 km	—

ΔR :

- ΔR < 1 km @ 200Mpc
  - for $R_{1.35}$ > 14 km (2-σ)
  - ~ 10 event / yr
- ΔR < 1 km @ 100Mpc
  - for $R_{1.35}$ > 12 km (2-σ)
  - ~ 1 event / yr
- ΔR < 1 km @ 50Mpc
  - for $R_{1.35}$ > 11 km (2-σ)
  - ~ 0.1 event / yr



an optimal estimate

Future prospects
Stiffening vs. Softening of EOS

- Accurate determinations of NS radius via Λ for different masses will tell us weather EOS get stiffened or not as density increases

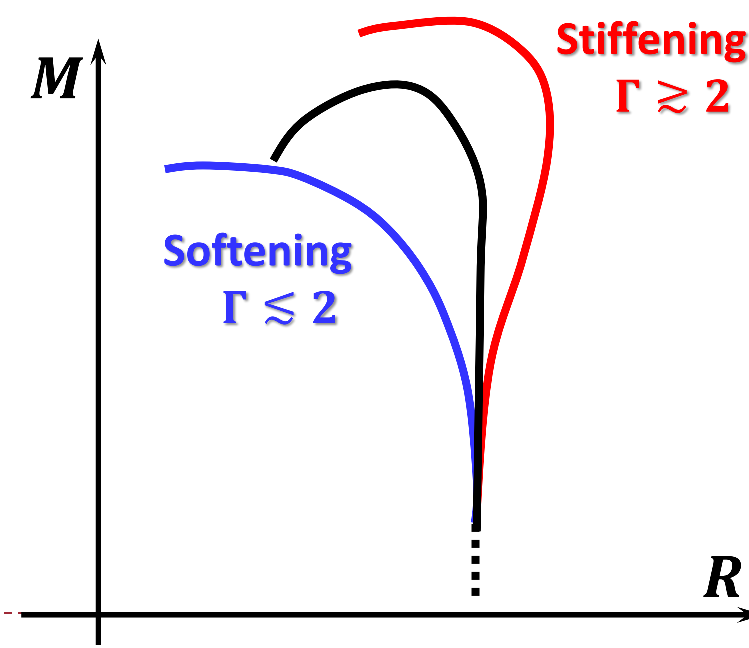
- Newtonian configuration with polytropic EOS (Lane-Emden equation) gives

$$R \propto M^{(n-1)/(n-3)} K^{n/(3-n)} \qquad P = K\rho^{\Gamma} = K\rho^{1+1/n}$$

- General relativistic correction :

$$\Gamma \Rightarrow \Gamma - O(1)\frac{2GM}{c^2 R}$$

- Accurate Radius determination provides EOS information
  - $dR/dM < 0$: Softening of EOS
  - $dR/dM > 0$: Stiffening of EOS



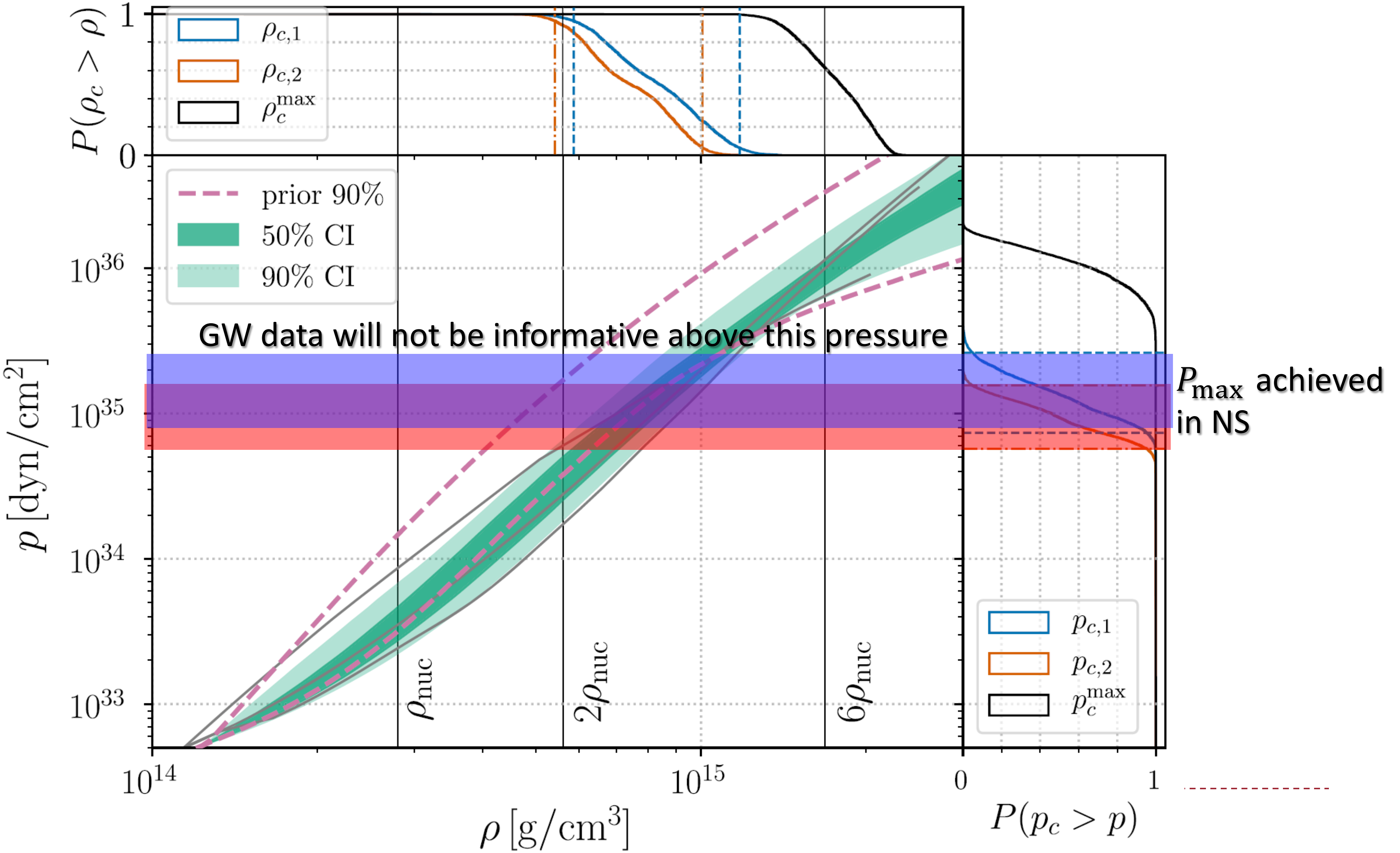
Accurate/precise GW template required

- Numerical relativity (NR) is a good tool for it, however computational cost is very high
- We need sufficiently dense (ideally continuous) template in the parameter space $(m_1, m_2, \vec{s}_1, \vec{s}_2, \Lambda_1, \Lambda_2, \cdots)$

  $\Rightarrow$ phenomenological template calibrated by NR will be necessary
- Current status : systematic error $\Delta\Lambda \sim 100$, which corresponds to $\Delta R \sim 1$ km
- We will need more systematic study in preparing GW template

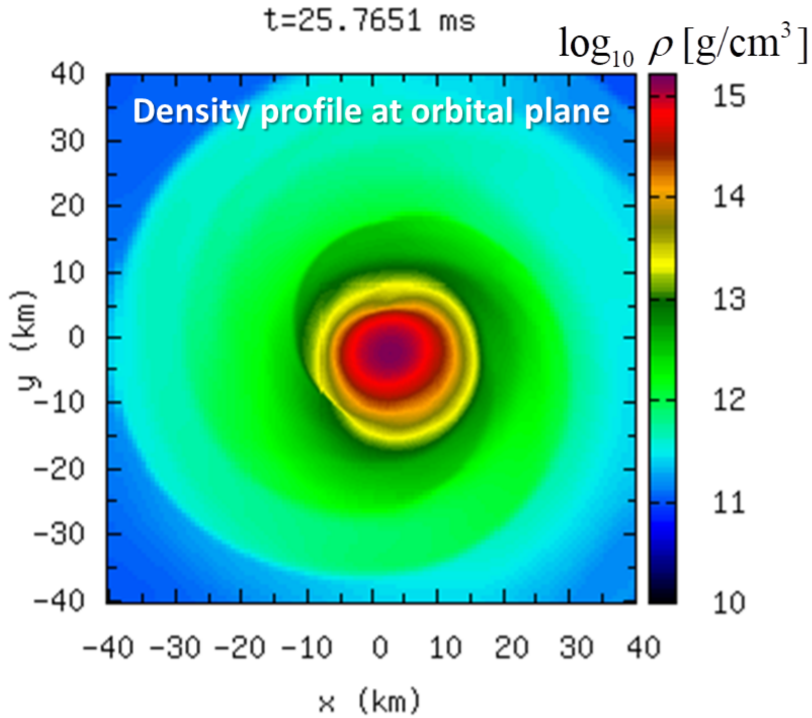
Abbott et al. PRL 121, 161101 (2018)

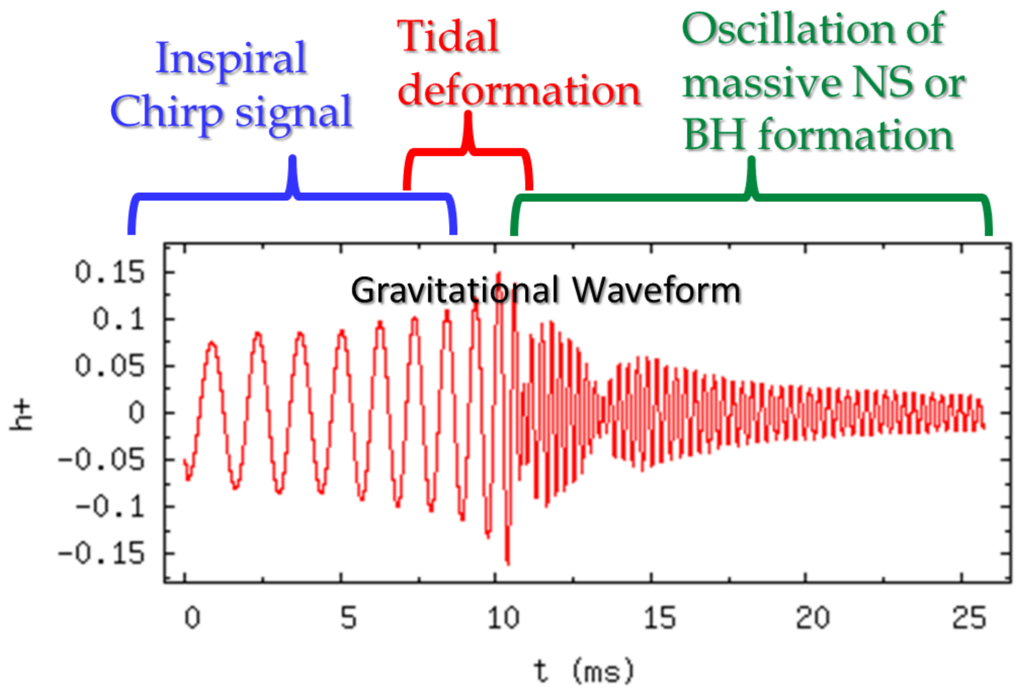
Summary of lecture 3



GW from object formed after the merger

Numerical relativity simulation modelling GW170817





- point particle approx.
- information of binary parameter (**NS mass**, etc)

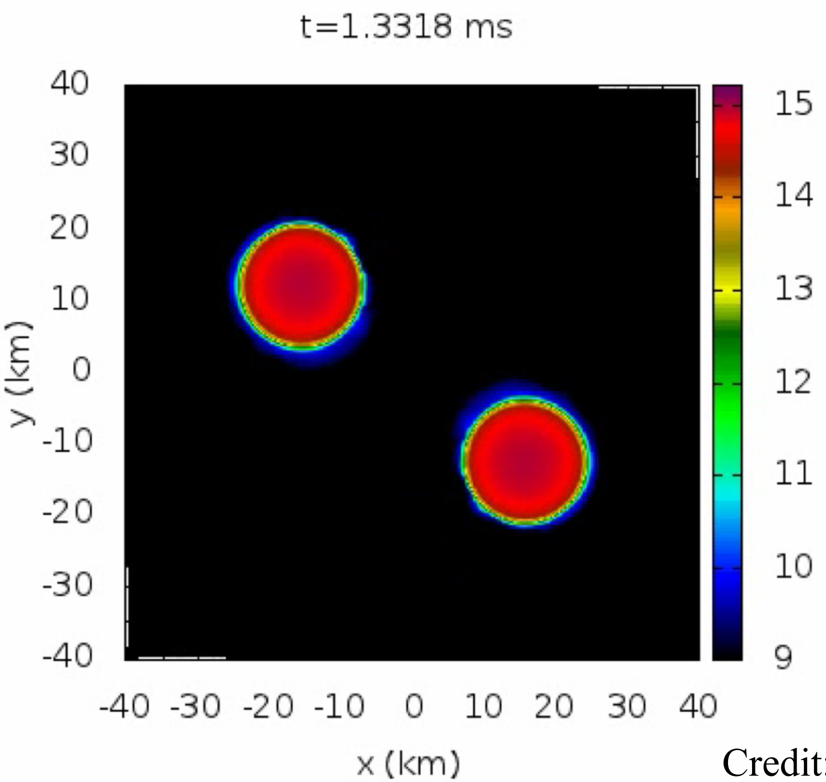
- finite size effect
- **NS tidal deformability**
- ⇒ **NS radius**

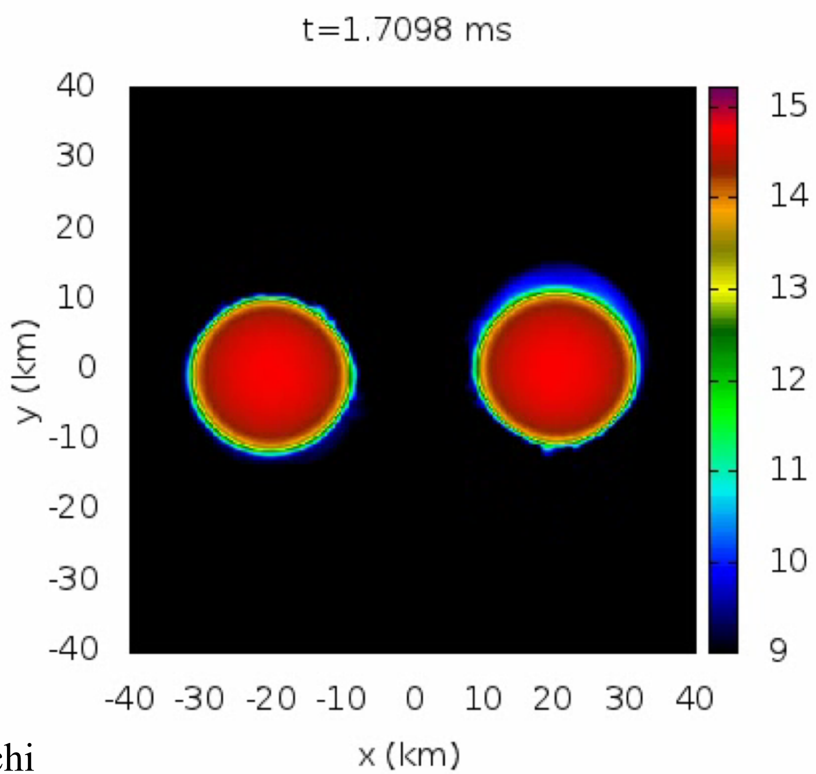
- BH or NS ⇒ **maximum mass**
- GWs from massive NS
  ⇒ **NS radius of *massive* NS**

Future prospects:
Listening GW from merger remnant NS

Soft EOS
⇒ more compact and higher density
⇒ higher frequency GW

Stiff EOS
⇒ less compact and lower density
⇒ lower frequency GW

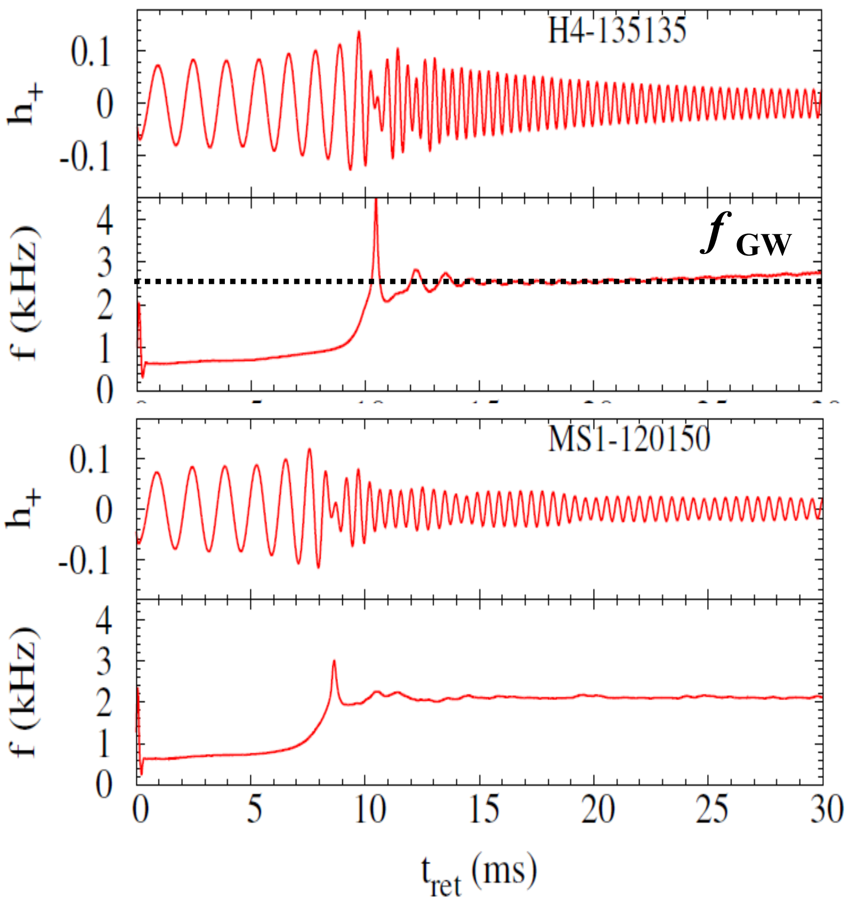


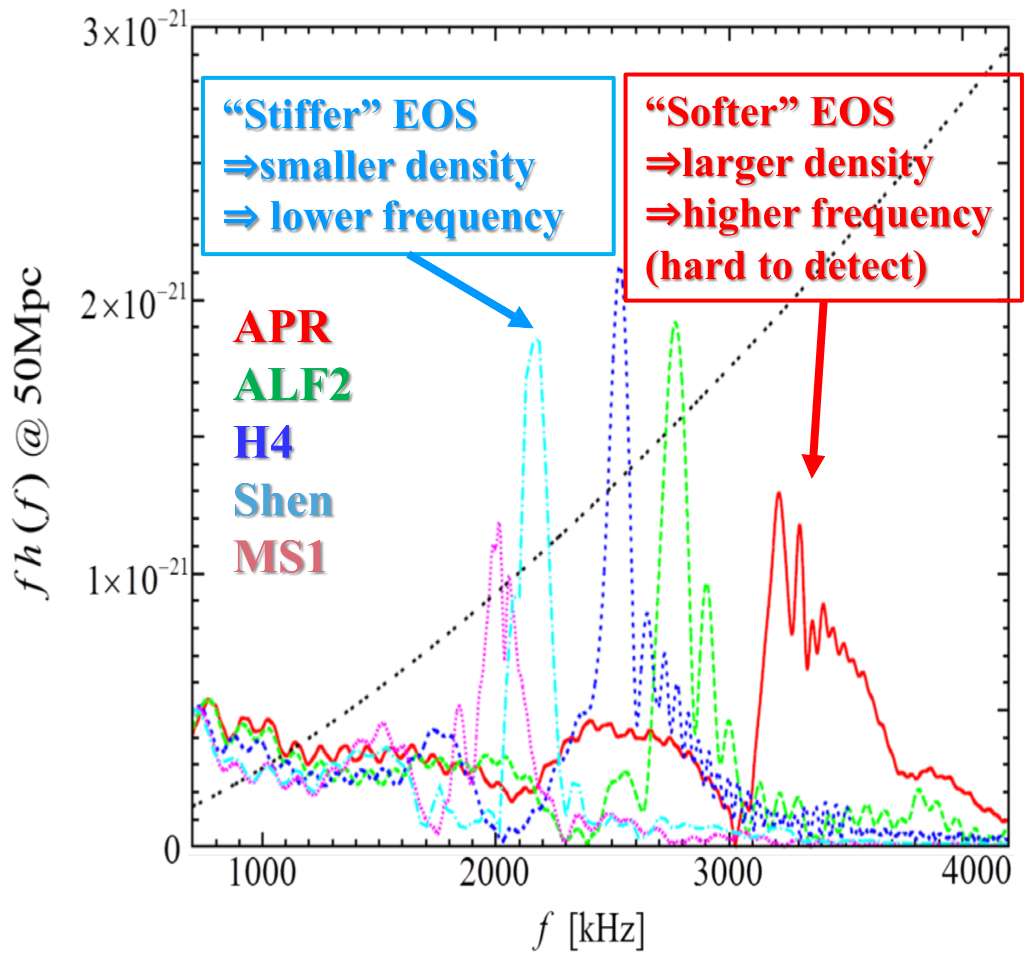


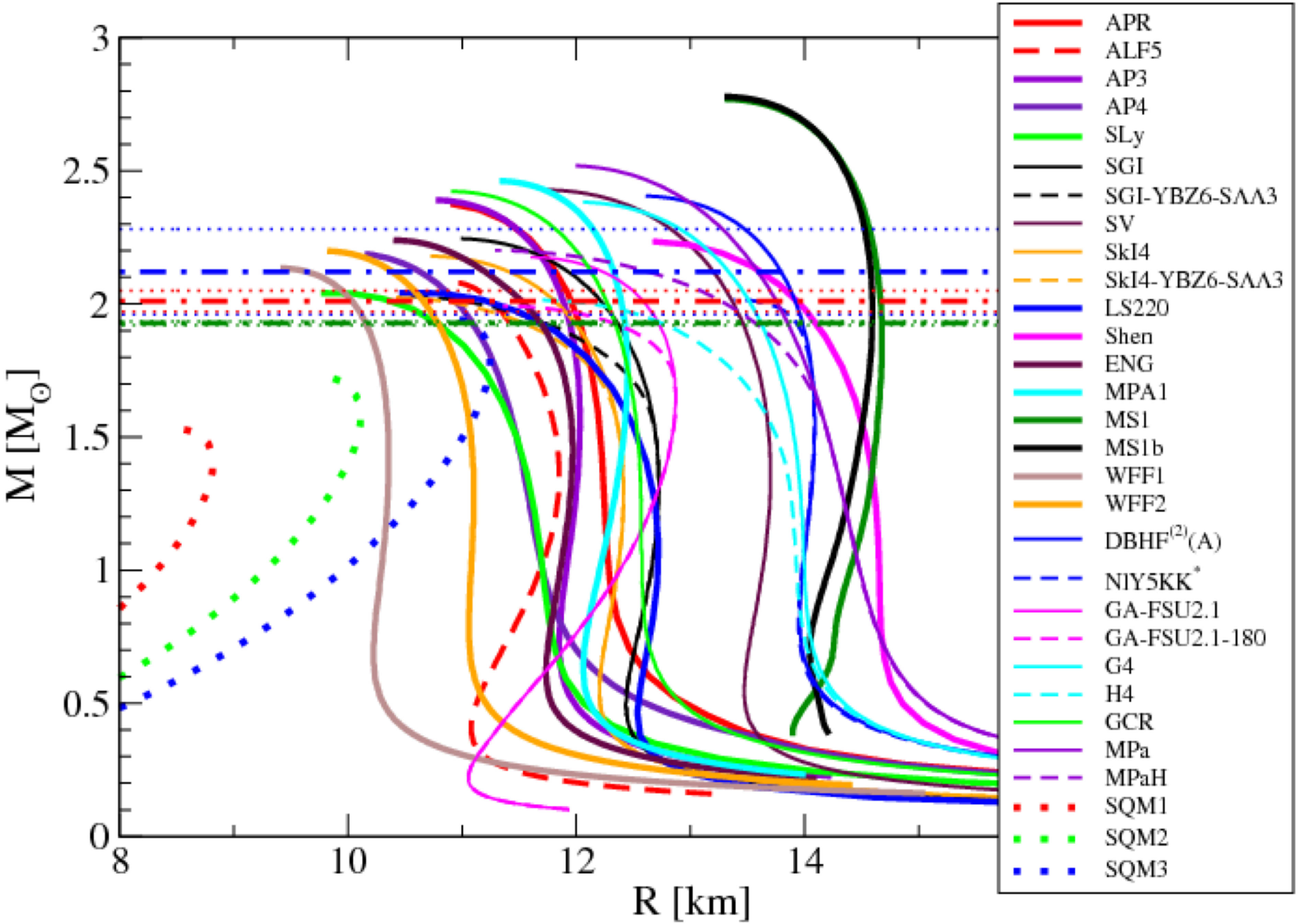
Credit: K. Kawaguchi

Future prospects: Listening GW from merger remnant NS

- GWs have characteristic frequency ('line') depending on EOS : f_{GW}



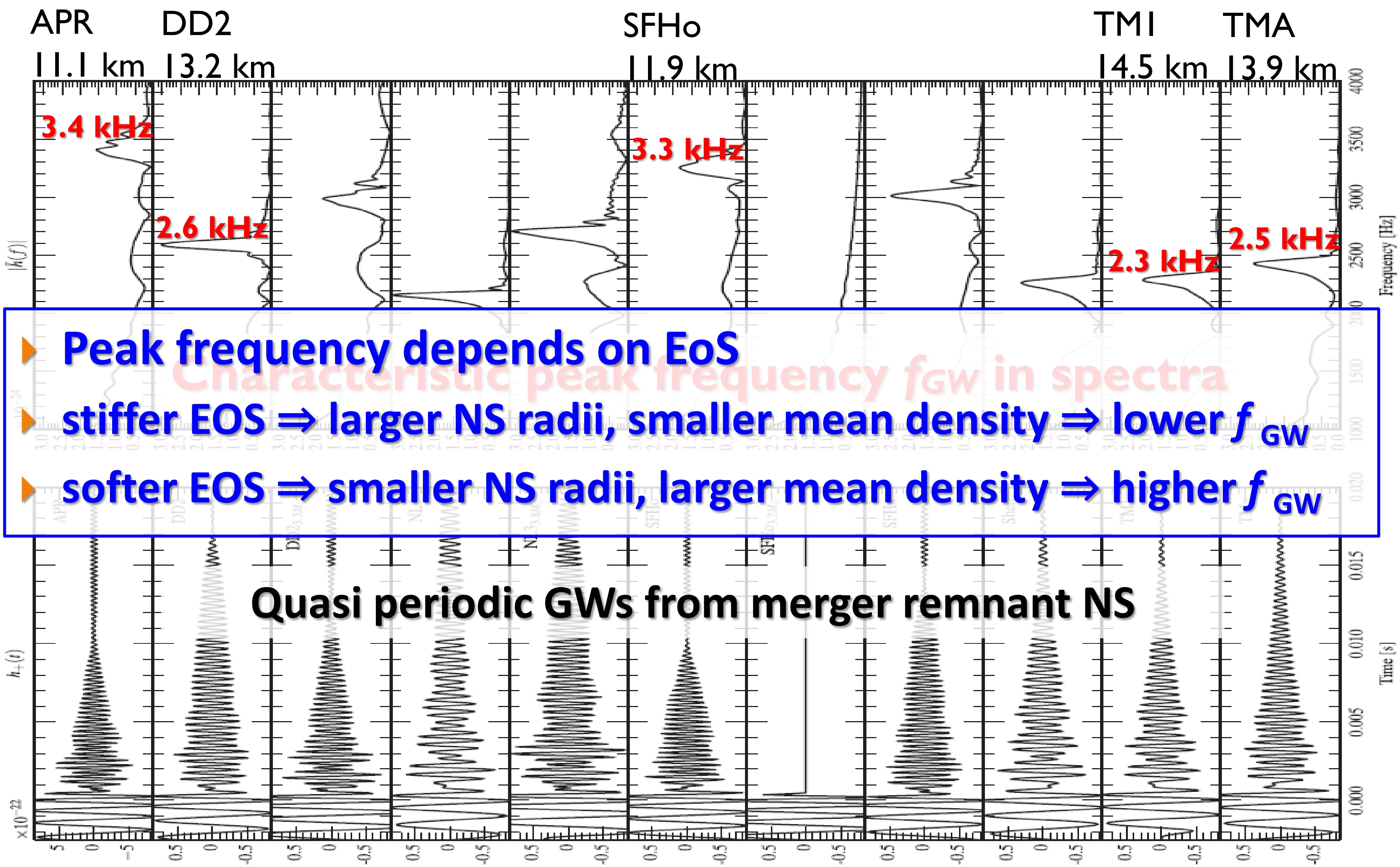




APR
ALF5
AP3
AP4
SLy
SGI
SGI-YBZ6-SΛΛ3
SV
SkI4
SkI4-YBZ6-SΛΛ3
LS220
Shen
ENG
MPA1
MS1
MS1b
WFF1
WFF2
DBHF(2)(A)
NlY5KK*
GA-FSU2.1
GA-FSU2.1-180
G4
H4
GCR
MPa
MPaH
SQM1
SQM2
SQM3
M [M⊙]
R [km]
0
0.5
1
1.5
2
2.5
3
8
10
12
14

Clark et al. (2014)

Future prospects: Listening GW from merger remnant NS

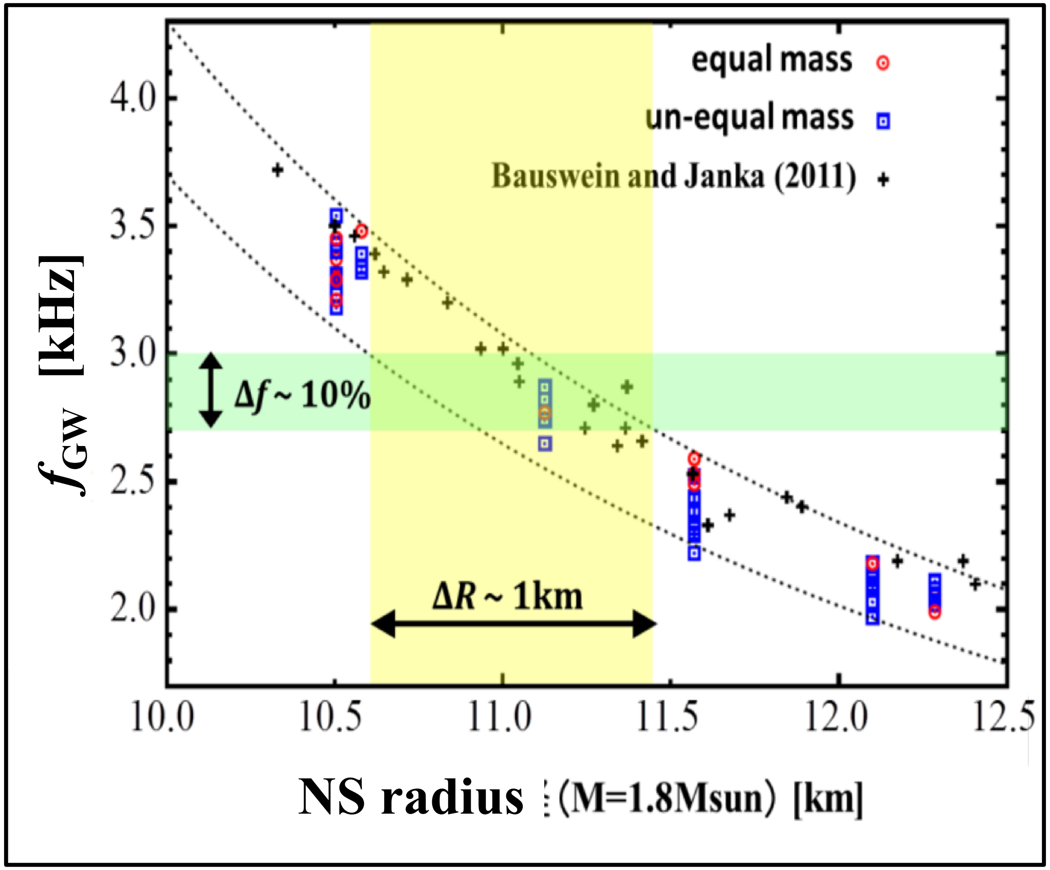


- **Peak frequency depends on EoS**
- **stiffer EOS ⇒ larger NS radii, smaller mean density ⇒ lower f_{GW}**
- **softer EOS ⇒ smaller NS radii, larger mean density ⇒ higher f_{GW}**

Hotokezaka et al. 2013; Bauswein et al. 2013

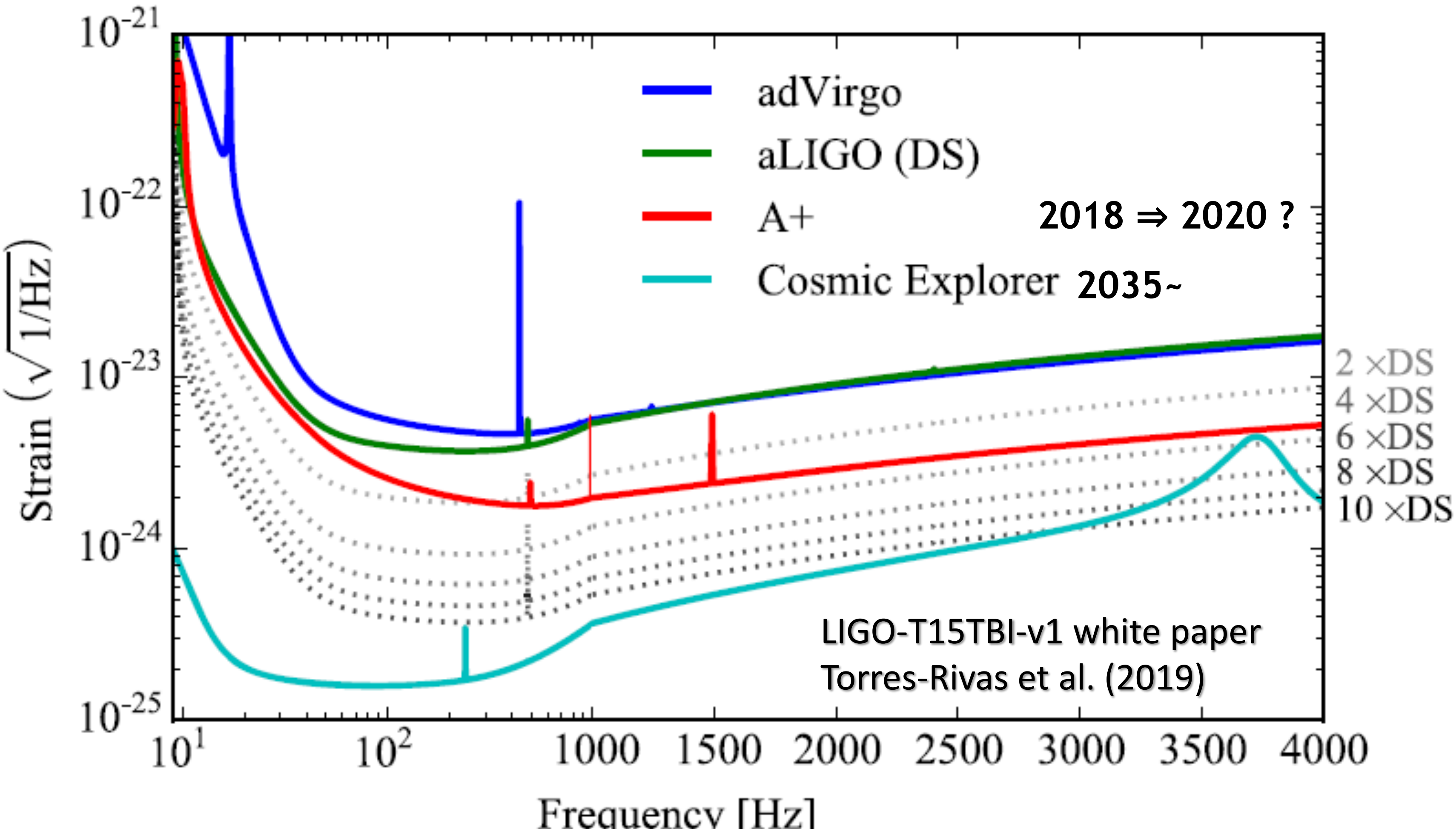
Future prospects: Listening GW from merger remnant NS

- stiff EOS ⇒ larger NS radii, smaller mean density ⇒ low f_{GW}
- soft EOS ⇒ smaller NS radii, larger mean density ⇒ high f_{GW}
- **Empirical relation for f_{GW}**
  - Good correlation with
  - **radius of 1.6Msolar NS**
    - **Bauswein et al. (2012)**
    - **Approx. GR study**
  - **radius of 1.8Msolar NS**
    - **Hotokezaka et al. (2013)**
    - **Full GR study**
- tight correlation : ΔRmodel ~ 1 km
- Further developments
  - Takami + (2015)
  - Bauswein + (2014, 2015, 2016)
  - Rezzolla + (2016)



Sensitivities of future detectors

- 5-8 times more sensitive in kHz band than adv. LIGO design sensitivity for an event @ 100Mpc (Torres-Rivas et al. (2019) PRD 98 084061)



Future detectors in LIGO white paper

LIGO A+ [74, 75] a set of upgrades to the existing LIGO facilities, including frequency-dependent squeezed light, improved mirror coatings and potentially increased laser beam sizes. Noise amplitude spectral sensitivity would be improved by a factor of $\sim$ 2.5-3 over 1–4 kHz. A+ could begin operation as early as 2017–18.

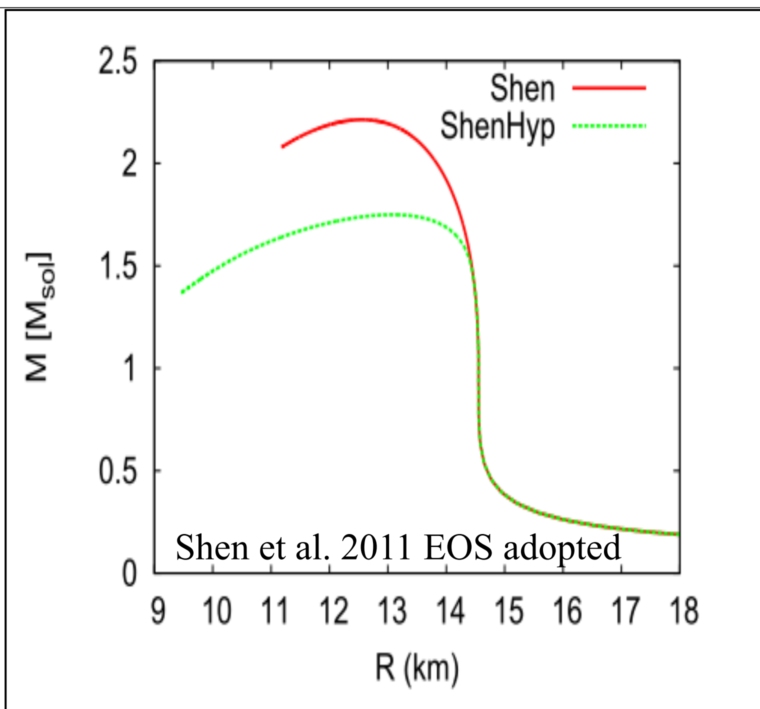
LIGO Voyager (LV) [75] a major upgrade to the existing LIGO facilities, including higher laser power, changes to materials used for suspensions and mirror substrates and, possibly, low temperature operation. LV would become operational around 2027–28 and offer noise amplitude spectral sensitivity improvements of $\sim$ 4.5-5 over 1–4 kHz.

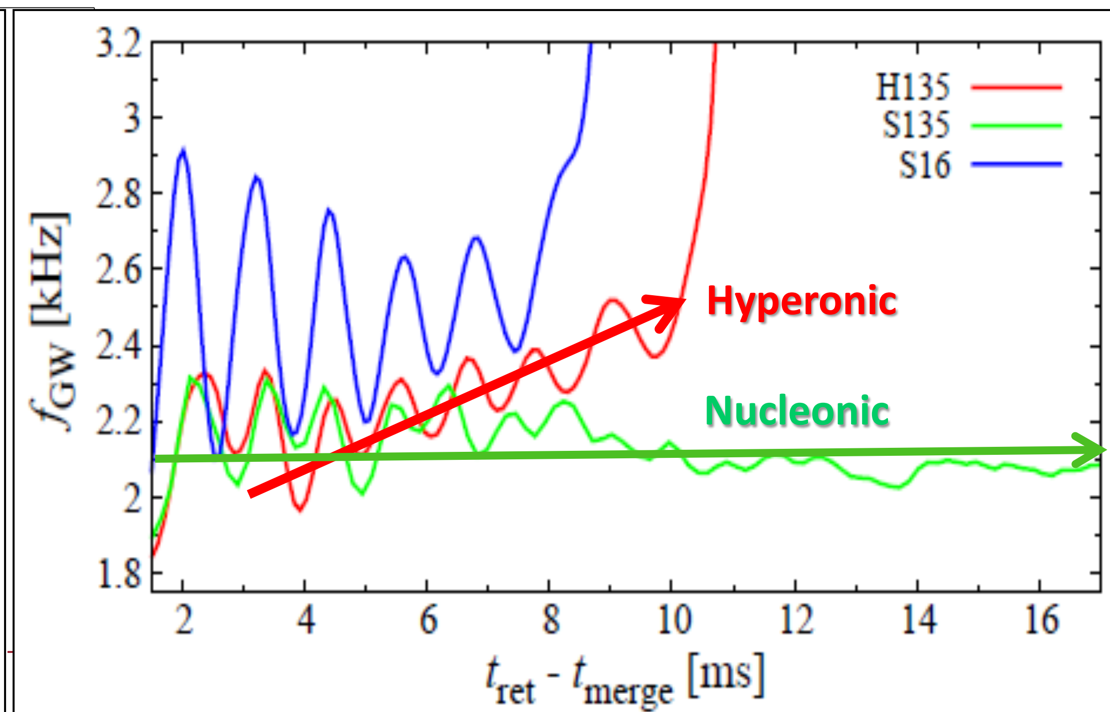
LIGO Cosmic Explorer (CE) [75] a new LIGO facility rather than an upgrade, with operation envisioned to commence after 2035, probably as part of a network with LIGO Voyager. In its simplest incarnation, Cosmic Explorer would be a straightforward extrapolation of A+ technology to a much longer arm length of 40 km, referred to as CE1 which would be $\sim 14\times$ more sensitive than aLIGO over 1–4 kHz. An alternative extrapolation is that of Voyager technology to the 40 km

Sekiguchi et al. PRL (2011b)

Future prospects: Proving emergence of hyperons by GW

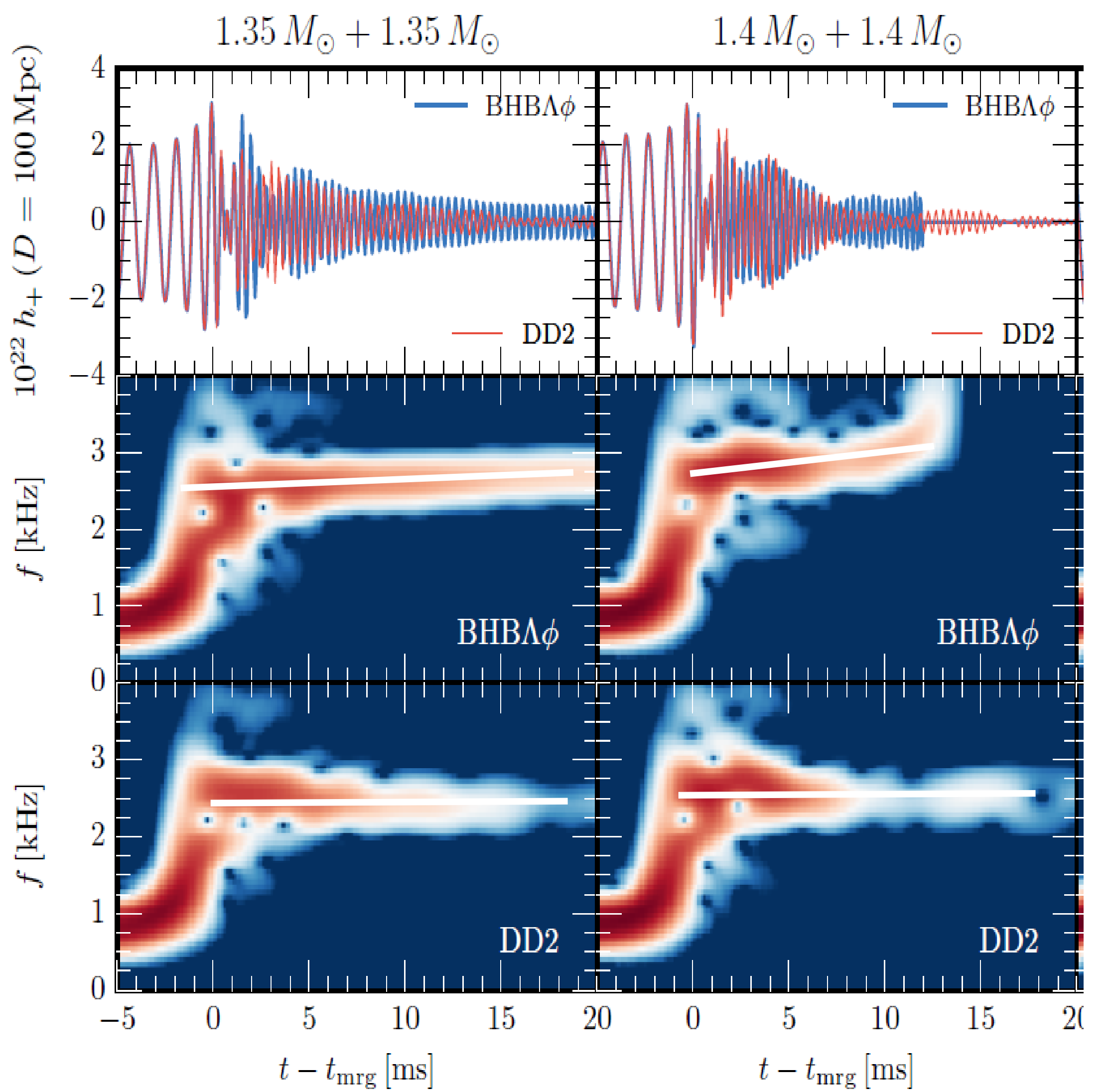
- **Nucleonic** : NS shrinks by angular momentum loss in a **long** GW timescale
- **Hyperonic** : GW emission ⇒ NS shrinks ⇒ More Hyperons appear ⇒
  EOS becomes softer ⇒ NS shrinks more ⇒ ….
- **⇒ the characteristic frequency of GW for hyperonic EOS increases with time**
  - ***Could*** provide potential way to tell existence of hyperons (exotic particles)





Future prospects:
Proving emergence of hyperons by GW

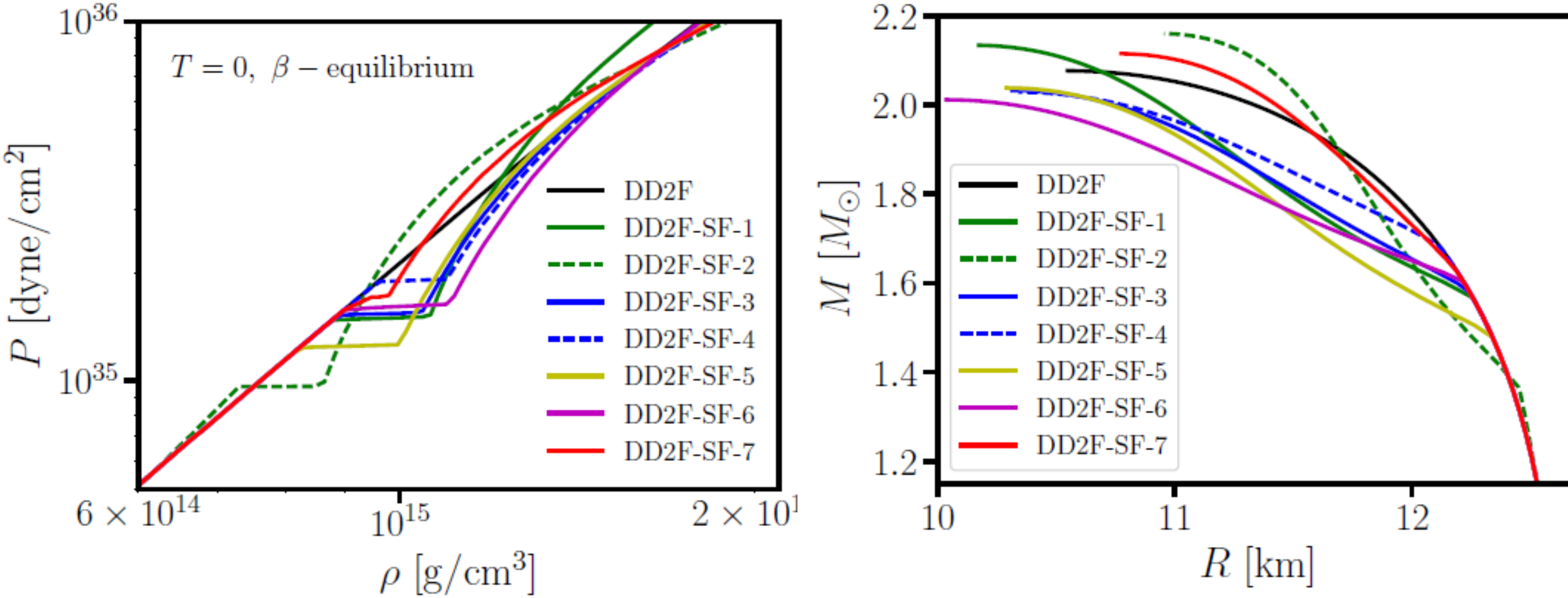
- Simulation with a hyperon EOS compatible with 2Msun NS
- Softening due to hyperon appearance is reduced
- Increase of f_{GW} will not be prominent as previously expected



Radice et al. (2017)

Bauswein et al. (2019) PRL 122 061102

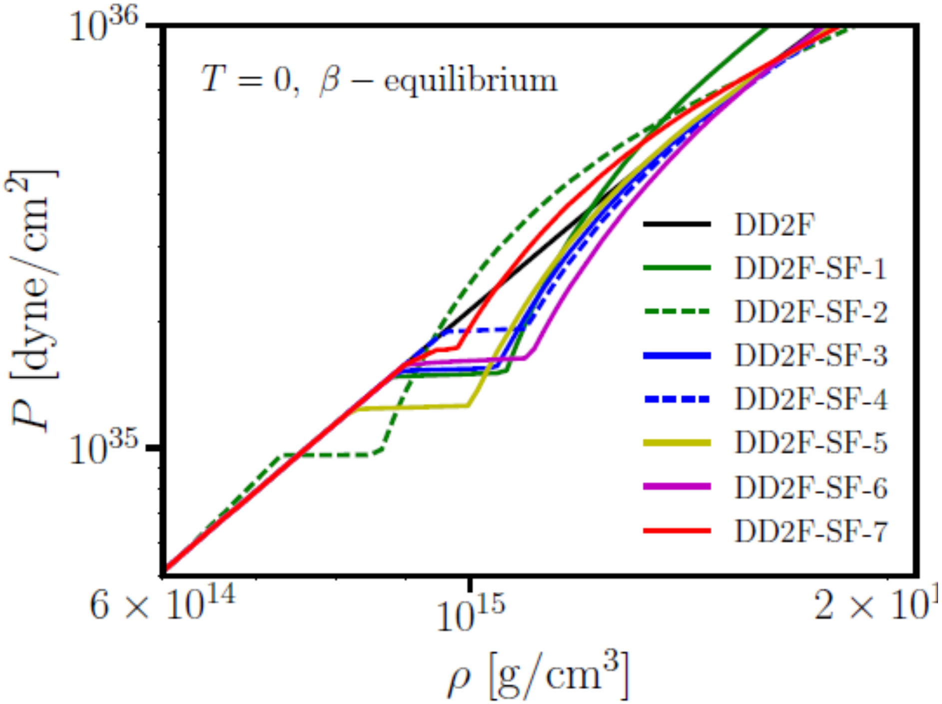
Future prospects: Proving 1st order hadron-quark transition

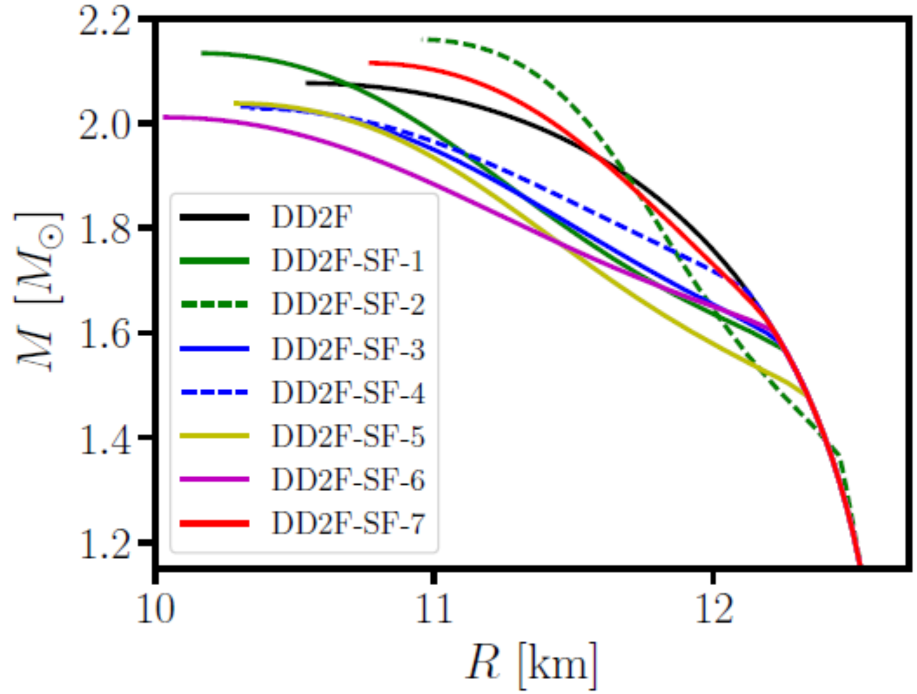


- Except for DD2F-SF-2 model, hadron-quark phase transition occurs at higher densities, so that the structure of $< 1.4M_\odot$ NS is same as that of DD2F (without transition)

  $\Rightarrow$ tidal deformability will be almost same

Future prospects:
Proving 1st order hadron-quark transition



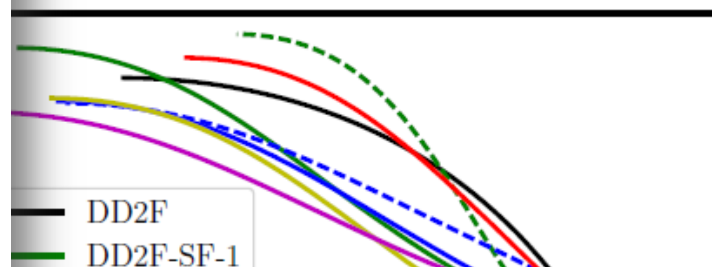


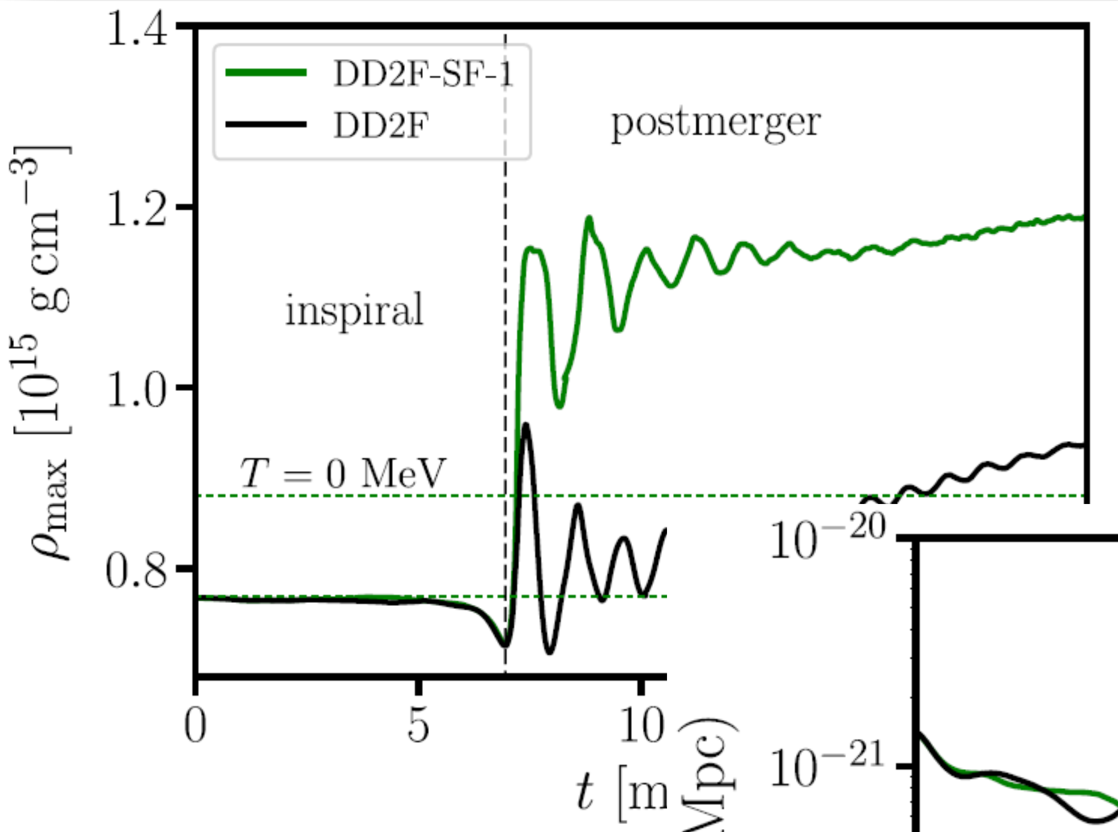
- On the other hand, structure of more massive NS is different

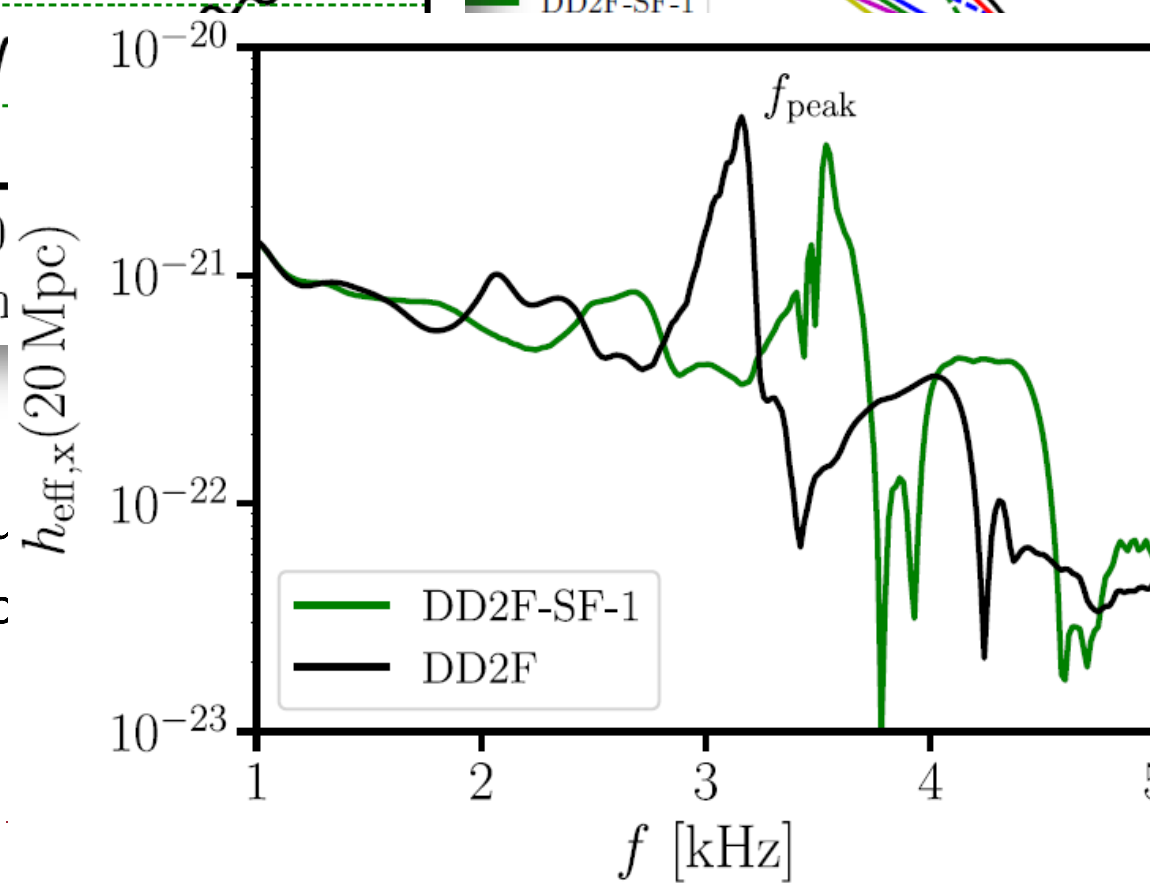
  ⇒ the peak frequency of GW from post-merger system will be different

Bauswein et al. (2019) PRL 122 061102

ark transition





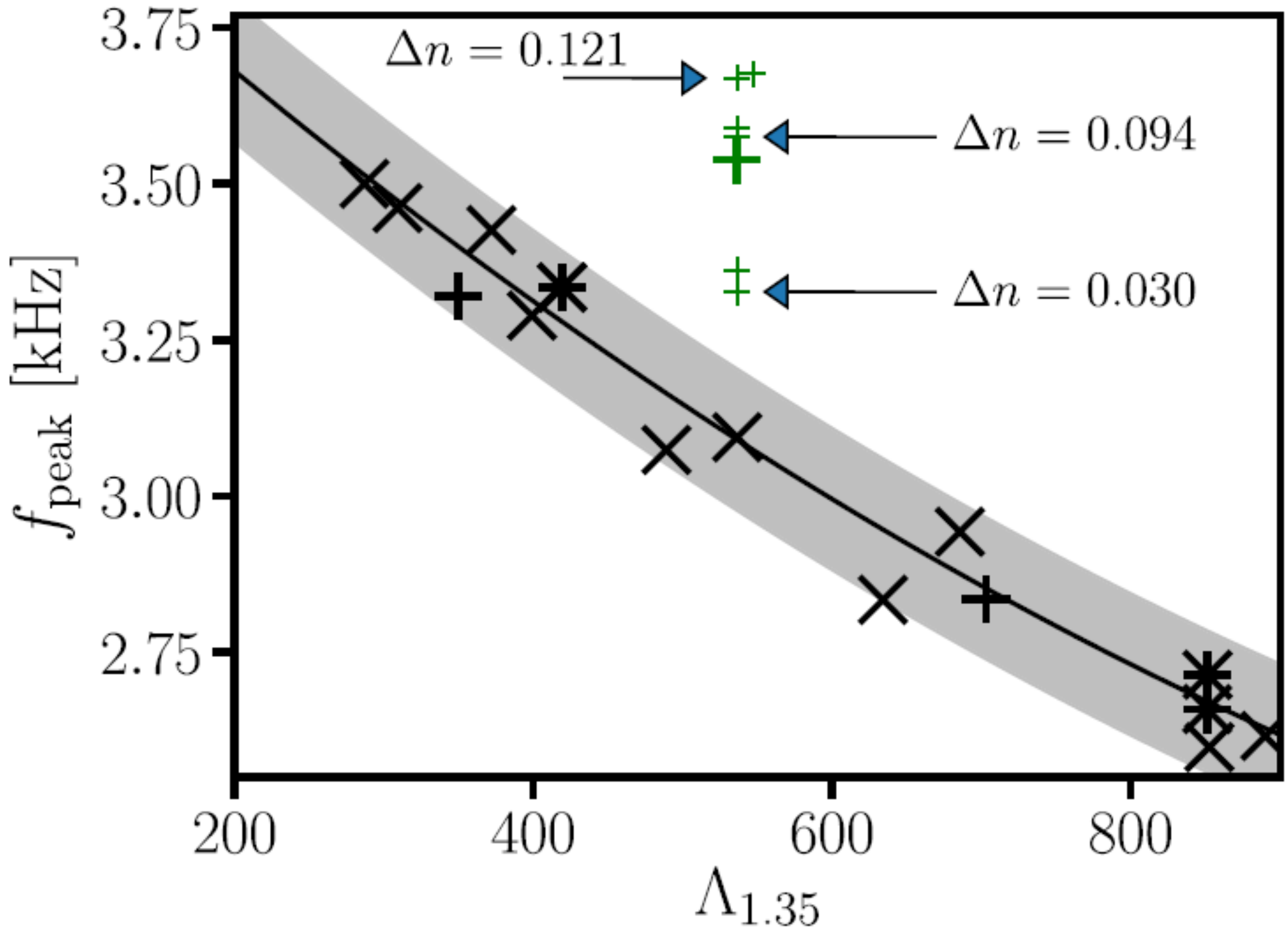


- On the other hand, stru

  ⇒ the peak frequency c

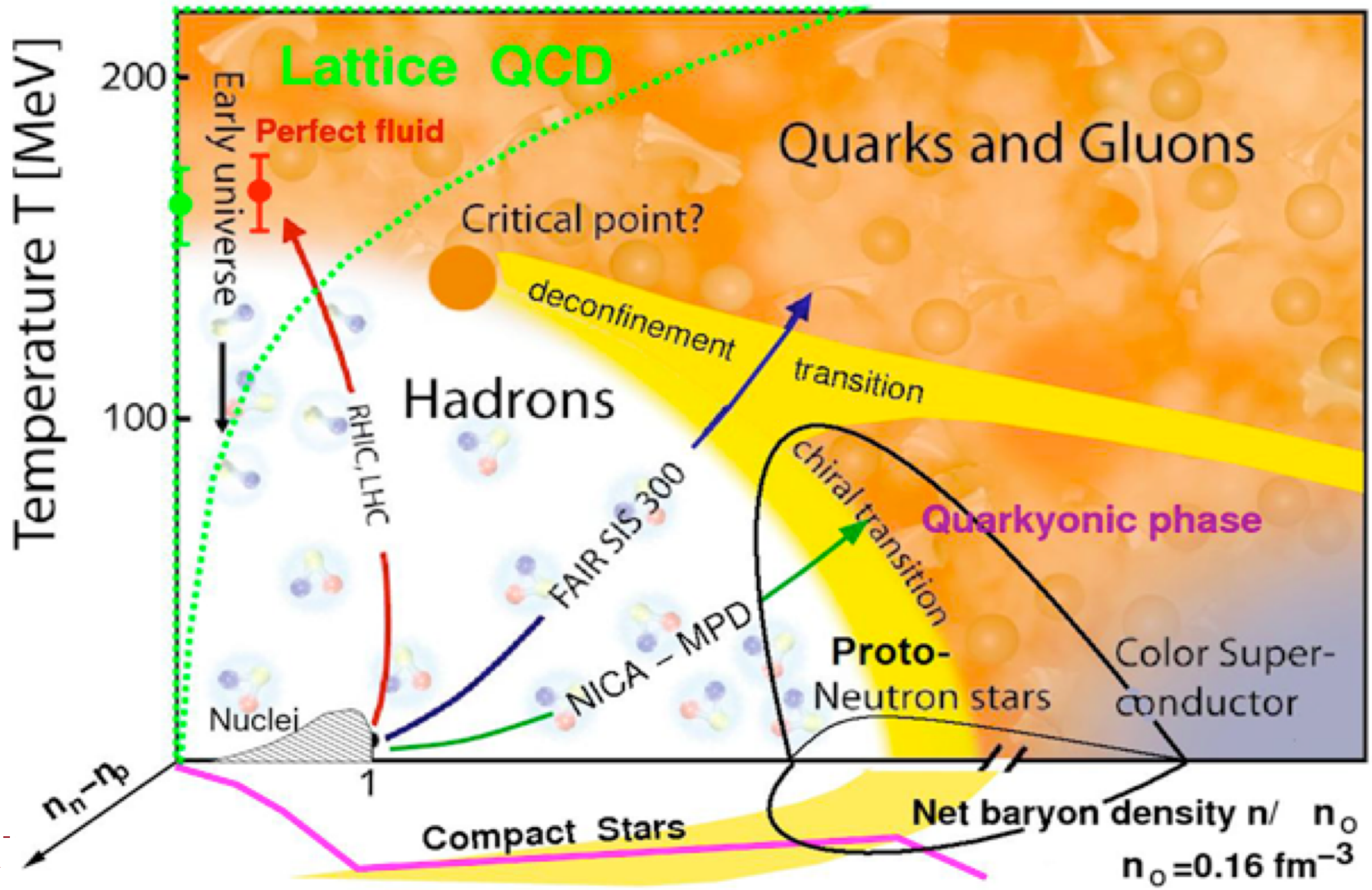
Bauswein et al. (2019) PRL 122 061102

Future prospects: Proving 1st order hadron-quark transition



QCD Phase diagram and NS

McLerran, Nucl.Phys.Proc.Suppl. 195, 275 (2009)



No phenomenological GW for post-merger

- Numerical relativity (NR) is a unique tool for it, however computational cost is very high

- We need sufficiently dense (ideally continuous) template in the parameter space $(m_1, m_2, \vec{s}_1, \vec{s}_2, \mathbf{EOS}, \cdots)$
- It would be difficult to construct phenomenological EOS

$$(m_1, m_2, \vec{s}_1, \vec{s}_2, \mathbf{EOS}, \cdots) \Rightarrow (m_1, m_2, \vec{s}_1, \vec{s}_2,\ f_1^{\mathrm{GW}}, f_2^{\mathrm{GW}} \cdots) \ \ ???$$

- It may be better to try to construct a physically motivated, useful empirical method to efficiently extract information of EOS from GW

Numerical relativity for pedestrians

The Einstein's equation

$$\boxed{G_{ab}} = \frac{8\pi G}{c^4} T_{ab}$$

$$\boxed{G_{ab}} = \boxed{R_{ab}} - \frac{1}{2} g_{ab} \sum_c \sum_d g^{cd} \boxed{R_{cd}}$$

$$\boxed{R_{ab}} = \sum_c \frac{\partial}{\partial x^c} \boxed{\Gamma^c{}_{ab}} - \frac{\partial}{\partial x^a} \left[\sum_c \boxed{\Gamma^c{}_{cb}} \right] + \sum_c \sum_e \left(\boxed{\Gamma^e{}_{ab} \Gamma^c{}_{ec}} - \boxed{\Gamma^e{}_{cb} \Gamma^c{}_{ea}} \right)$$

$$\boxed{\Gamma^c{}_{ab}} = \frac{1}{2} \sum_d g^{cd} \left[\frac{\partial g_{bd}}{\partial x^a} + \frac{\partial g_{ad}}{\partial x^b} - \frac{\partial g_{ab}}{\partial x^d} \right]$$

2nd order partial differential equations for g_{ab}

Contain complicated mixed derivatives in terms of space and time

The 'type' of partial differential equation is not clear (elliptic ? Hyperbolic ?)

⇒ We need a formulation in which the 'type' of Einstein equation becomes clear

Analogy to Maxwell equation

- Decomposition of covariant Maxwell equation

$$\boxed{\sum_{\mu=0}^{3}\frac{\partial F^{\mu\nu}}{\partial x^{\mu}}=4\pi J^{\nu}} \qquad \boxed{\sum_{\mu=0}^{3}\frac{\partial F_{*}^{\mu\nu}}{\partial x^{\mu}}=0}$$

$$\boldsymbol{F}^{\mu\nu}=\begin{pmatrix} \mathbf{0} & \boldsymbol{E}^x & \boldsymbol{E}^y & \boldsymbol{E}^z \\ -\boldsymbol{E}^x & \mathbf{0} & \boldsymbol{B}^z & -\boldsymbol{B}^y \\ -\boldsymbol{E}^y & -\boldsymbol{B}^z & \mathbf{0} & \boldsymbol{B}^x \\ -\boldsymbol{E}^z & \boldsymbol{B}^y & -\boldsymbol{B}^x & \mathbf{0} \end{pmatrix}$$

$$\boldsymbol{F}_{*}^{\mu\nu}=\begin{pmatrix} \mathbf{0} & \boldsymbol{B}^x & \boldsymbol{B}^y & \boldsymbol{B}^z \\ -\boldsymbol{B}^x & \mathbf{0} & -\boldsymbol{E}^z & \boldsymbol{E}^y \\ -\boldsymbol{B}^y & \boldsymbol{E}^z & \mathbf{0} & -\boldsymbol{E}^x \\ -\boldsymbol{B}^z & -\boldsymbol{E}^y & \boldsymbol{E}^x & \mathbf{0} \end{pmatrix}$$

- Vector equations (need to specify a direction)
  - The time components (seen from $\vec{u}=\vec{e}_t$)

$$\boxed{\vec{e}_t\cdot\sum_{\mu=0}^{3}\frac{\partial F^{\mu\nu}}{\partial x^{\mu}}=\vec{e}_t\cdot 4\pi J^{\nu} \;\Rightarrow\; \mathrm{div}\,\boldsymbol{E}=4\pi\rho} \qquad \boxed{\vec{e}_t\cdot\sum_{\mu=0}^{3}\frac{\partial F_{*}^{\mu\nu}}{\partial x^{\mu}}=0 \;\Rightarrow\; \mathrm{div}\,\boldsymbol{B}=0}$$

  - The space components ($\vec{e}_i:\ i=x,y,z$)

$$\boxed{\vec{e}_i\cdot\sum_{\mu=0}^{3}\frac{\partial F^{\mu\nu}}{\partial x^{\mu}}=\vec{e}_i\cdot 4\pi J^{\nu} \;\Rightarrow\; \mathrm{rot}\,\boldsymbol{B}=4\pi\boldsymbol{j}+\frac{\partial\boldsymbol{E}}{\partial t}} \qquad \boxed{\vec{e}_i\cdot\sum_{\mu=0}^{3}\frac{\partial F_{*}^{\mu\nu}}{\partial x^{\mu}}=0 \;\Rightarrow\; \mathrm{rot}\,\boldsymbol{E}=-\frac{\partial\boldsymbol{B}}{\partial t}}$$

Elliptic equations from time components (Gauss's law, no monopole condition)

Hyperbolic equations form space components (Ampere, Faraday law)

There are no absolute space and time

- What is 'time/space' in the 'time/space component' ?
  - Time-like and space-like coordinates can be exchanged in extreme spacetime like black hole (e.g., Schwarzschild BH in the standard coordinate)

- Setting of time and space coordinates was first studied in the canonical formulation of gravitation

The theory of gravitation in Hamiltonian form

By P. A. M. Dirac, F.R.S.
St John's College, Cambridge

(*Received* 13 *March* 1958—*Revised* 21 *April* 1958)

PHYSICAL REVIEW VOLUME 114, NUMBER 3 MAY 1, 1959

Fixation of Coordinates in the Hamiltonian Theory of Gravitation

P. A. M. Dirac*
Institute for Advanced Study, Princeton, New Jersey

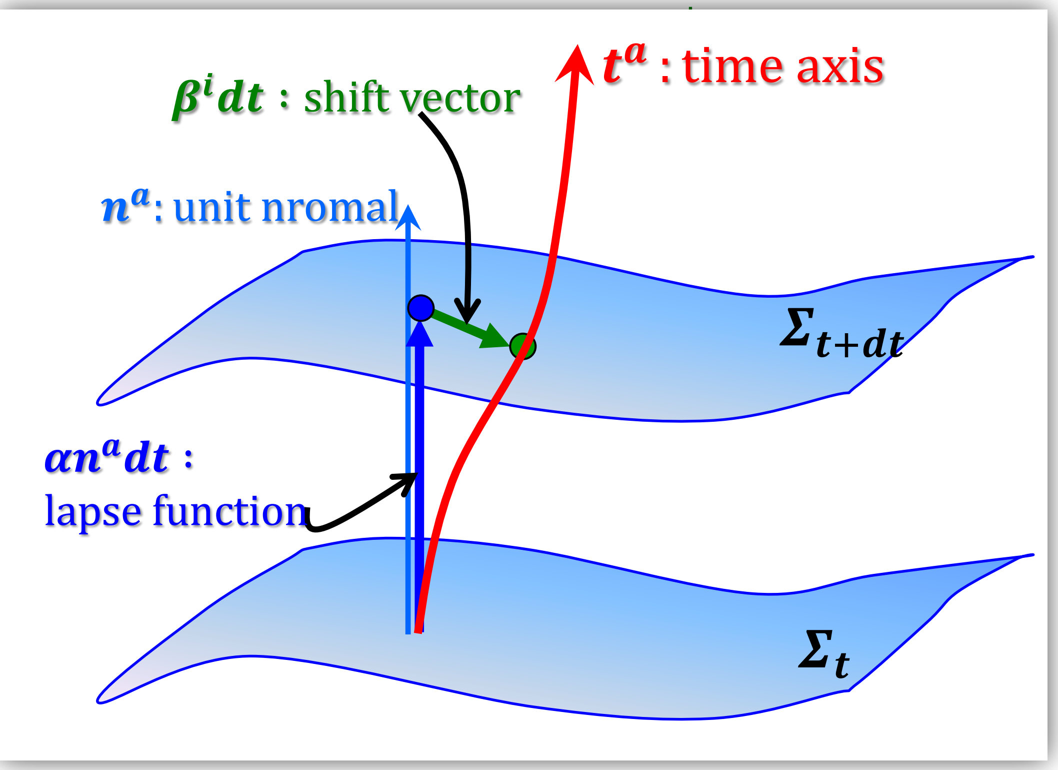
(Received December 10, 1958)

The theory of gravitation is usually expressed in terms of an arbitrary system of coordinates. This results in the appearance of weak equations connecting the Hamiltonian dynamical variables that describe a state at a certain time, leading to supplementary conditions on the wave function after quantization. It is then difficult to specify the initial state in any practical problem.

To remove the difficulty one must eliminate the weak equations by fixing the coordinate system. The general procedure for this elimination is here described. A particular way of fixing the coordinate system is then proposed and its effect on the Poisson bracket relations is worked out.

Introducing time and space coordinates

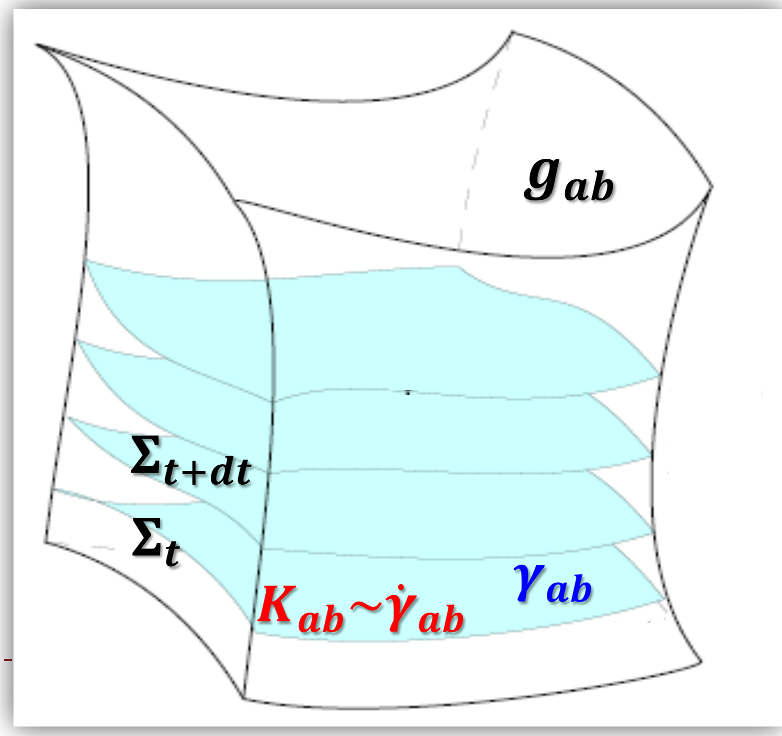
- In NR, we must specify the time and space coordinates
  - Prepare the initial slice Σ_t $\Rightarrow$ **unit normal vector n^a** is specified
  - Degree of freedom in advancing the time : α (**lapse function**)
    - In Newtonian framework, the time pass uniformly
  - Degree of freedom in specifying direction of time axis (**shift vector**)
  - Time axis : $t^a = \alpha n^a + \beta^a$
  - The next slice Σ_{t+dt} can be given (sequentially)
  - It is important problem how determine α, β^a



Generating spacetime

- （if α, β^a are given） we can construct a future slice Σ_{t+dt}
  - **spatial metric γ_{ab}** on Σ_t is also given as initial data
  - **'velocity' of γ_{ab} : $K_{ab} \sim \dot{\gamma}_{ab}$** （**extrinsic curvature** ） is also given as initial data
    - Einstein equation is 2nd order equations of metric

- Everything of spacetime is contained in the **metric g_{ab}**
  - From given initial data $\Sigma_t, \gamma_{ab}, K_{ab}$
  - **Spacetime is generated if we can construct the spacetime metric g_{ab}**
    - α, β^a are also necessary
  - The equation to determine the evolution of γ_{ab}, K_{ab} is the space-space component of Einstein equation



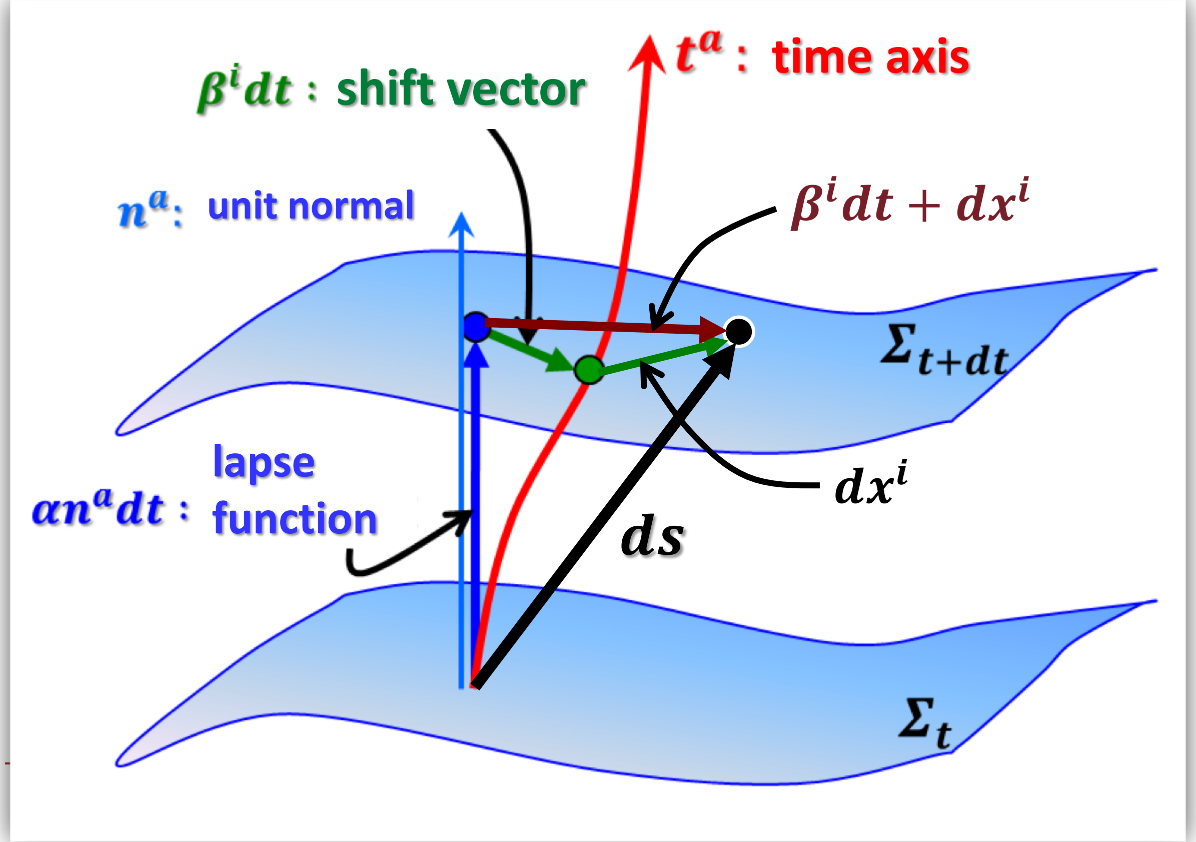
Construction of the metric g_{ab}

- **The metric defines (spacetime) Pythagorean theorem**
  - Consider a point at distance dx^i from the time axis
  - Displacement from n^a is $\beta^i dt + dx^i$

$$\boxed{ds^2 = -(\alpha dt)^2 + \gamma_{ij}(\beta^i dt + dx^i)(\beta^j dt + dx^j)}$$

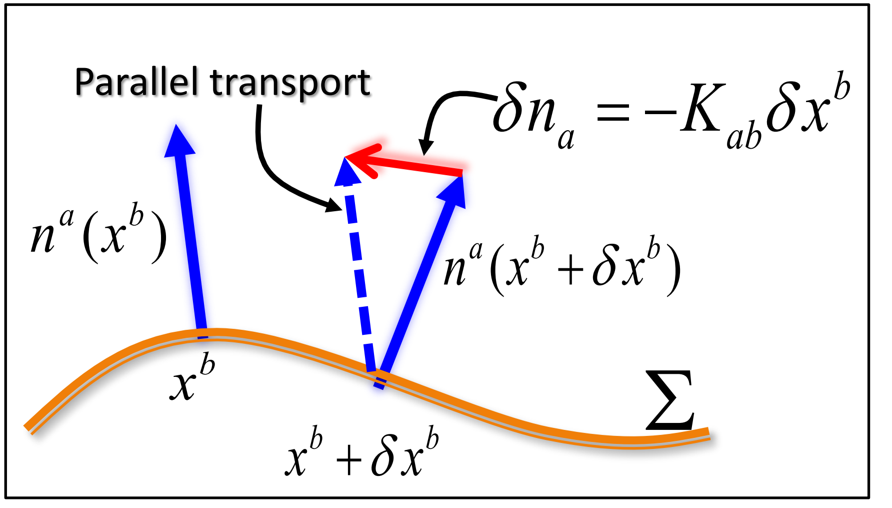
- **Calculate ds**
  - We can construct the metric g_{ab} from $\alpha, \beta^a, \gamma_{ab}$

- Issues remained
  - How to choose α, β^a



Geometrical meaning of K_{ab}

- K_{ab} is associated with the difference between the original and parallel-transported unit normal vector n^a on Σ_t
  - For a slice Σ_t embedded in the spacetime in a 'flat' manner, $K_{ab} = 0$
  - How Σ_t is embedded in spacetime (curvature seen from outside) ⇒ extrinsic curvature



3+1 (ADM) decomposition of Einstein's eq

- Einstein equation : (0,2) tensor equation (need to specify 2 directions)
  - Time-time component ⇒ energy conservation including gravity（elliptic）

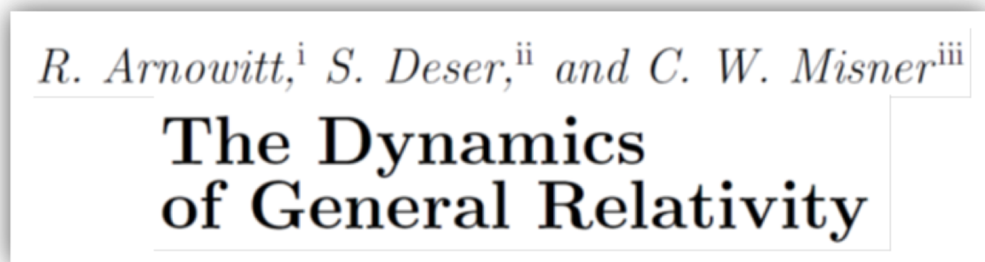
$$\boxed{R + K^2 - K_{ab}K^{ab} = 16\pi E}$$

  - Time-space ⇒ momentum conservation（vector elliptic）

$$\boxed{D_b K^b_a - D_a K = 8\pi P_a}$$

  - Space-space component ⇒ evolution equation（hyperbolic）

$$\boxed{(\boldsymbol{L}_t - \boldsymbol{L}_\beta)K_{ab} = -D_a D_b \alpha + \alpha[R_{ab} + KK_{ab} - 2K_{ac}K^c_b] - 4\pi\alpha\left(2S_{ab} - \gamma_{ab}(S-E)\right)}$$


R. Arnowitt,[i] S. Deser,[ii] and C. W. Misner[iii]

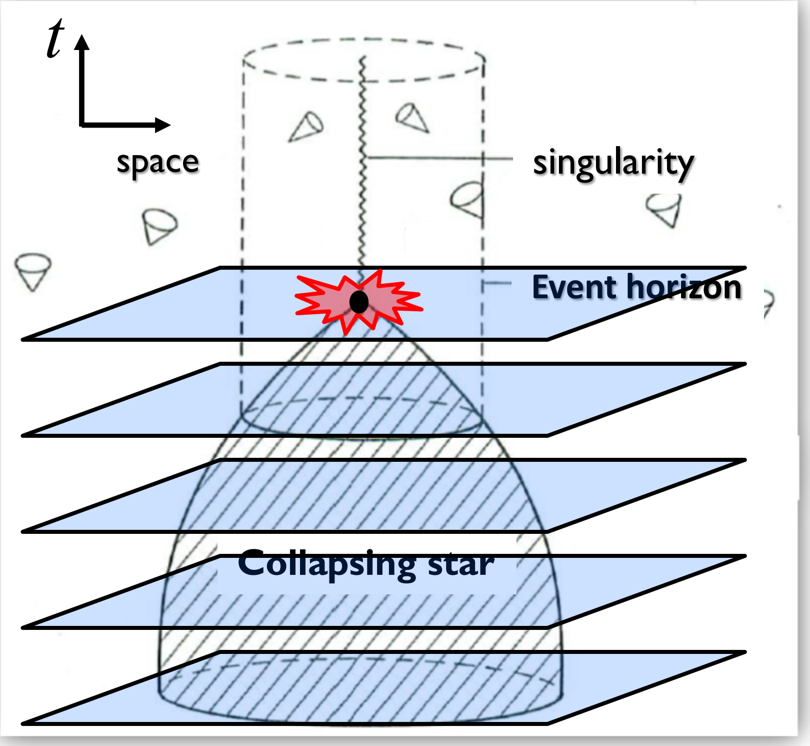
The Dynamics of General Relativity

In 1962 !!

Choice of the lapse function α

- In numerical relativity, we aggressively utilize these gauge degrees of freedom instead of regarding as troublesome issues
  - To increase the stability of simulation
  - Avoiding appearance of the coordinate singularities

- There is singularities inside black hole
    - Singularity theorem
    - (Hawking & Penrose 1970)
  - For the lapse $\alpha = 1$
    - the slice hits the singularity just after its formation
    - The simulation crash there



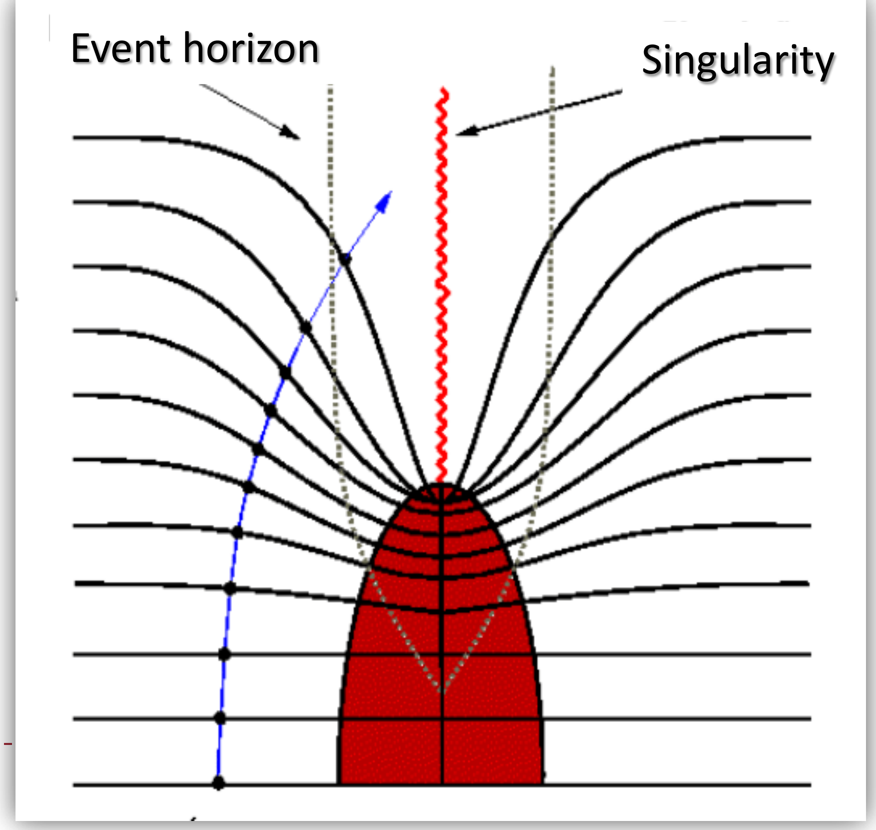
Choice of the lapse function α

- Maximal slicing (Smarr & York 1978)
  - Because the decrease in time of the volume element $\sqrt{\gamma}$ results in a coordinate singularity, let us maximize the volume element
  - We take the volume element of a 3D-domain S and consider a variation along the time vector $V[S] = \int_S \sqrt{\gamma}\, d^3x$

$$\boxed{\mathcal{L}_t V[S] = \int_S d^3x \left[-\alpha K \sqrt{\gamma} + \partial_i (\sqrt{\gamma} \beta^i) \right] = -\int_S \alpha K \sqrt{\gamma}\, d^3x}$$

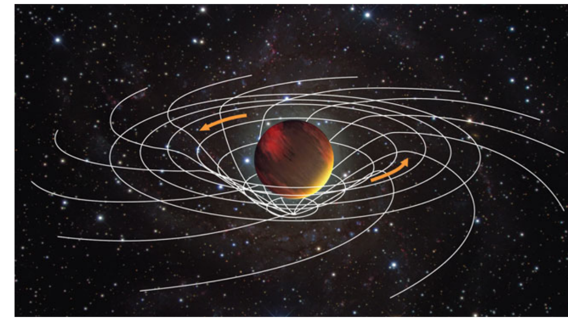
  - If $K = 0$ on the slice, the volume is maximal
    - Time evolution is delayed in strong gravity
    - has strong singularity avoidance property
    - But the normal vector gets focused
    - $\Rightarrow$ eventually simulation crash
      - Necessary to use β^a
  - Maximal slicing condition is elliptic equation for α

$$\boxed{0 = (\mathcal{L}_t - \mathcal{L}_\beta) K = -D_i D^i \alpha + \alpha [K_{ij} K^{ij} + 4\pi (E + S)]}$$



- Hyperbolic lapse has been developed

Utilizing the shift vector $\boldsymbol{\beta}^a$



- Distortion of the time vector is problematic
  - Distortion of n^a due to the black hole formation
  - The dragging of the frame around a rotating object
- We can use β^a to minimize these distortions
- Minimal distortion shift (Smarr & York 1978)
  - The covariant derivative of any timelike unit vector can be decomposed as (Helmholtz's theorem)

$$\boxed{\nabla_a z_b = \omega_{ab} + \sigma_{ab} + \frac{1}{3} h_{ab}\theta - z_a \zeta_b}$$

$h_{ab} \equiv g_{ab} + z_a z_b$, (induced metric)

$\omega_{ab} \equiv \perp \nabla_{[a} z_{b]}$, (twist) $\qquad \theta \equiv \nabla_c z^c$, (expansion)

$\sigma_{ab} \equiv \perp \nabla_{(a} z_{b)}{}^{\mathrm{TF}}$, (shear) $\qquad \zeta^a \equiv z^c \nabla_c z^a$, (acceleration)

  - Minimize distortion functional

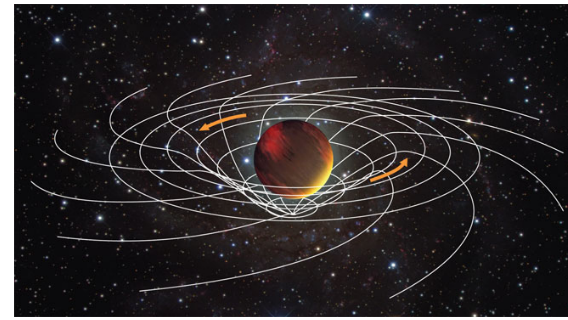
$$\boxed{I \equiv \int \Sigma_{ab} \Sigma^{ab} \sqrt{\gamma}\, dx^3} \qquad \boxed{\Sigma_{ab} \equiv \frac{1}{2} \perp \mathbf{L}_t \gamma_{ab}{}^{\mathrm{TF}} \sim -K_{ab}{}^{\mathrm{TF}} = \perp \nabla_{(a} n_{b)}{}^{\mathrm{TF}}}$$

  - Gives a condition for shift
    - Vector elliptic equation
    - Hyperbolic shift has been developed

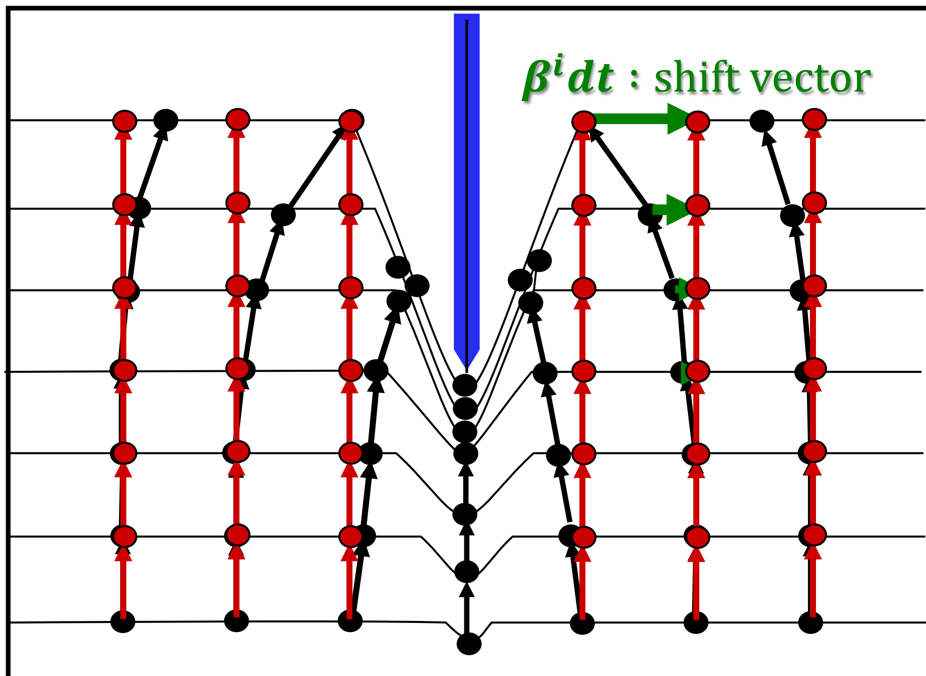
$$\boxed{D_c D^c \beta^a + D_a D_c \beta^c + R_{ab}\beta^b = D^b\left[2\alpha A_{ab}\right] = 2A^{ab} D_b \alpha + \alpha\left(\frac{4}{3}\gamma^{ab} D_b K + 16\pi P_a\right)}$$

Utilizing the shift vector $\boldsymbol{\beta^a}$

- Distortion of the time vector is problematic
  - Distortion of n^a due to the black hole formation
  - The dragging of the frame around a rotating object
- We can use β^a to minimize these distortions



- Minimal distortion shift (Smarr & York 1978)
  - vector
  - n)



$_a z_b$, (induced metric)

$_{b]}$, (twist) $\theta \equiv \nabla_c z^c$, (expansion)

$_{b)}^{\mathrm{TF}}$, (shear) $\zeta^a \equiv z^c \nabla_c z^a$, (acceleration)

$_{ab}^{\mathrm{TF}} = \perp \nabla_{(a} n_{b)}^{\mathrm{TF}}$

$$_{ab}\beta^b = D^b\left[2\alpha A_{ab}\right] = 2A^{ab}D_b\alpha + \alpha\left(\frac{4}{3}\gamma^{ab}D_b K + 16\pi P_a\right)$$

ADM formulation is unstable !

- Numerical relativity simulations based on ADM formulation is unstable
- This crucial limitation may be captured in terms of hyperbolicity
  - Consider a first-order system : $\partial_t u_i + (A_{ij})^c \partial_c u^j = 0.$ This system is called
    - Strongly hyperbolic : if a matrix (representation) of A has real eigenvalues and a complete set of eigenvectors
    - Weakly hyperbolic : if A has real eigenvalues but not a complete set of eigenvectors
  - Hyperbolicity is a key property for the stability
    - Strongly hyperbolic system is well-posed and only characteristic fields corresponding to negative eigenvalues need boundary conditions
    - Weakly hyperbolic system is not well-posed and the solution can be unbounded faster than exponential
  - Note that Einstein's equation is nonlinear (2^{nd} order quasi-linear) so that the above arguments may not be adopted directly
- (a first order formulation version of) the ADM system is weakly hyperbolic
- seeking (at least) strongly hyperbolic reformulation is a central issue in NR

Three ways to achieve better hyperbolicity

- We need formulations for the Einstein's equation which is (at least) strongly hyperbolic (in a linearized regime) (as 'wave-like' as possible)
  - Caution ! : better hyperbolicity is necessary condition, not sufficient
  - Let us consider **Maxwell's equation** in flat spacetime to capture what we should do to obtain a more stable system

$$\boxed{-\partial_t^2 A_i + \nabla^k \nabla_k A_i \boxed{- \nabla_i \nabla_k A^k} = \nabla_i \partial_t \phi}$$

Divergence terms prevents the system from achieving better hyperbolicity

Adopting better gauge

Lorenz gauge : $\partial_\mu A^\mu = 0$

$$\partial^\mu \partial_\mu A^i = 0$$

Coulomb gauge : $\nabla_k A^k = 0$

$$\partial^\mu \partial_\mu A^i = \nabla_i \partial_t \phi$$

Introduce new variables

$$F = \nabla_k A^k$$

$$\partial^\mu \partial_\mu A_i = \nabla_i \partial_t \phi \boxed{+ \nabla_i F}$$

Evolution eq. for F

$$\partial_t F = -\nabla_k E^k - \nabla^k \nabla_k \phi$$

Utilize constraints

$$\nabla_k E^k = 4\pi \rho_e$$
$$\partial_t F = \boxed{-\nabla_k E^k} - \nabla^k \nabla_k \phi$$

$$\nabla_k E^k = 4\pi \rho_e$$
$$\partial_t F = \boxed{-4\pi \rho_e} - \nabla^k \nabla_k \phi$$

Reformulating Einstein's equation

- Strongly/Symmetric hyperbolic reformulations of Einstein's equation
  - **Choosing a better gauge**
    - Better hyperbolicity vs. Singularity avoidance/frame dragging
      - Generalized harmonic gauge (Pretorius, CQG **22**, 425 (2005))
      - Z4 formalism (Bona et al. PRD **67**, 104005 (2003))
  - **Introducing new, independent variables**
    - BSSN (Shibata & Nakamura PRD 52, 5428 (1995); Baumgarte & Shapiro PRD 59, 024007 (1999))
    - Kidder-Scheel-Teukolsky (Kidder et al. PRD 64, 064017 (2001)) symmetric hyp.
      - Bona-Masso (Bona et al. PRD **56**, 3405 (1997))
      - Nagy-Ortiz-Reula (Nagy et al. PRD **70**, 044012 (2004))
  - **Using the constraint equations to improve the hyperbolicity**
    - adjusted ADM/BSSN (Shinkai & Yoneda, gr-qc/0209111)

BSSN formulation (Shibata & Nakamura 1995; Baumgarte & Shapiro 1998)

- Strategy: as wave-like as possible
- Introduce new variables $\Gamma^a = \partial_b \gamma^{ab}$
- Extract the 'true' degrees of freedom of gravity (GW)
  - Conformal decomposition by York (PRL 26, 1656 (1971); PRL 28, 1082 (1972))
    - the two degrees of freedom of the gravitational field are carried by the conformal equivalence classes of 3-metric, which are related each other by the conformal transformation :

$$\gamma_{ab} = \psi^4 \tilde{\gamma}_{ab}$$

  - Extrinsic curvature is also conformally decomposed
    - Trace of K is associated with the lapse function (c.f. maximal slicing) ⇒ split

$$K_{ab} = \psi^4 \tilde{A}_{ab} + \frac{1}{3}\gamma_{ab} K$$

- Reformulation based on new variables :

$$\psi, \tilde{\gamma}_{ab}, \tilde{A}_{ab}, K = \mathrm{tr}(K), \Gamma^a = \partial_b \gamma^{ab}$$

BSSN reformulation

$$\tilde{D}_i \tilde{D}^i \psi - \frac{1}{8}\tilde{R}\psi + \left(\frac{1}{8}\tilde{A}_{ij}\tilde{A}^{ij} - \frac{1}{12}K^2 + 2\pi E\right)\psi^5 = 0$$

$$\tilde{D}_j \tilde{A}^{ij} + 6\tilde{A}^{ij}\tilde{D}_j \ln\psi - \frac{2}{3}\tilde{D}^i K = 8\pi\psi^4 P^i$$

$$(\partial_t - \beta^k \partial_k)\ln\psi = -\frac{1}{6}\alpha K + \frac{1}{6}\partial_k \beta^k$$

$$(\partial_t - \beta^k \partial_k)\tilde{\gamma}_{ij} = -2\alpha\tilde{A}_{ij} + \tilde{\gamma}_{ik}\partial_j \beta^k + \tilde{\gamma}_{jk}\partial_i \beta^k - \frac{2}{3}\tilde{\gamma}_{ij}\partial_k \beta^k$$

$$(\partial_t - \beta^k \partial_k)K = -D_i D^i \alpha + \alpha[K_{ij}K^{ij} + 4\pi(E+S)]$$

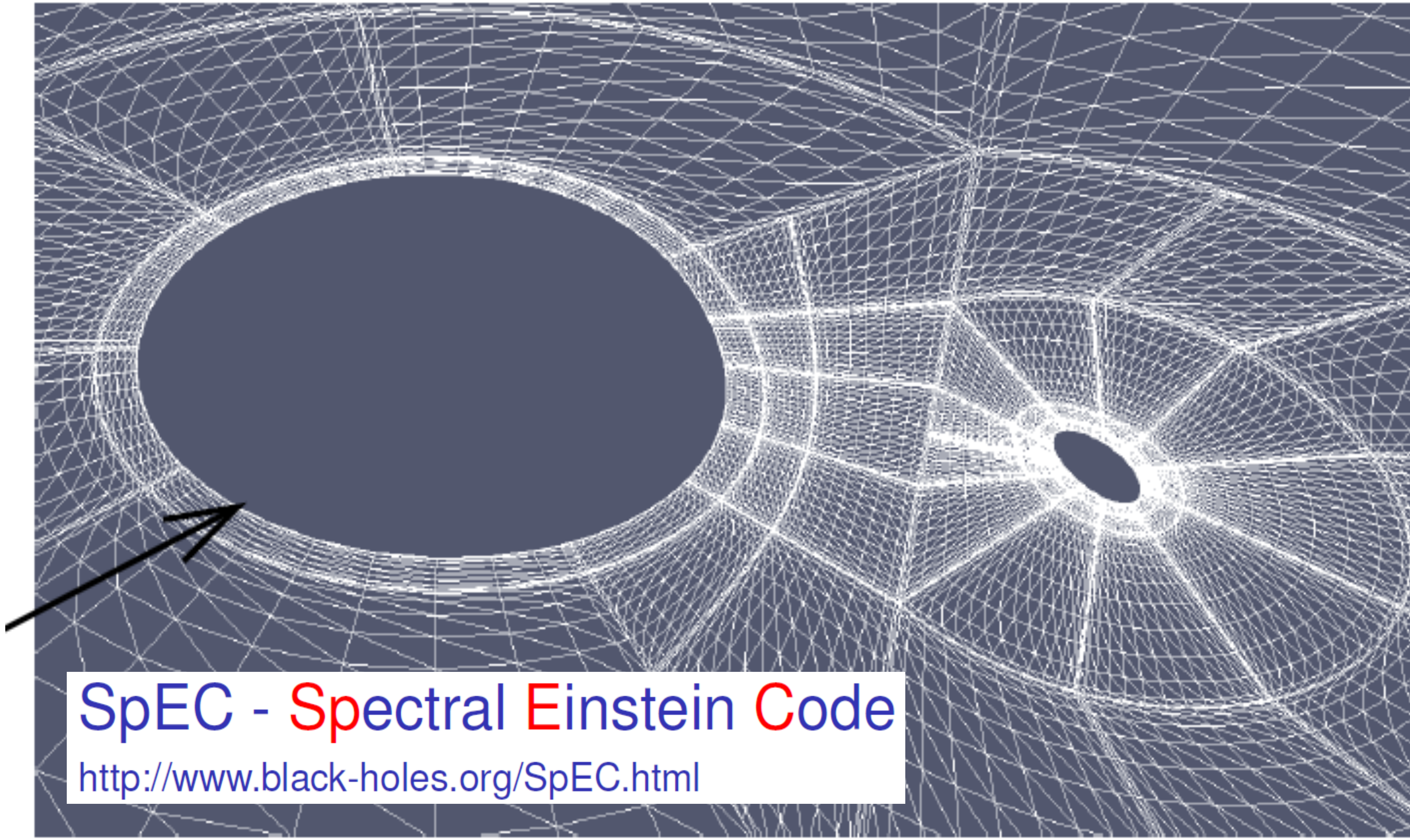
$$(\partial_t - \beta^k \partial_k)\tilde{A}_{ij} = \psi^{-4}\left[-(D_i D_j \alpha)^{\mathrm{TF}} + \alpha(R_{ij}{}^{\mathrm{TF}} - 8\pi S_{ij}{}^{TF})\right] + \alpha\left[K\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k_j\right]$$
$$+ \tilde{A}_{ik}\partial_j \beta^k + \tilde{A}_{jk}\partial_i \beta^k - \frac{2}{3}\tilde{A}_{ij}\partial_k \beta^k$$

Hamiltonian constraint is used

$$(\partial_t - \beta^k \partial_k)\Gamma^i = -16\pi\alpha P^i + 2\alpha\left[\tilde{\Gamma}^i_{jk}\tilde{A}^{jk} + 6\tilde{A}^{ij}\partial_j \ln\psi - \frac{2}{3}\tilde{\gamma}^{ij}\partial_j K\right] - 2\tilde{A}^{ij}\partial_j \alpha$$
$$+ \beta^j \partial_j \Gamma^i - \Gamma^j \partial_j \beta^i + \frac{2}{3}\Gamma^i \partial_j \beta^j + \frac{1}{3}\tilde{\gamma}^{ij}\partial_j \partial_k \beta^k + \tilde{\gamma}^{jk}\partial_j \partial_k \beta^i$$

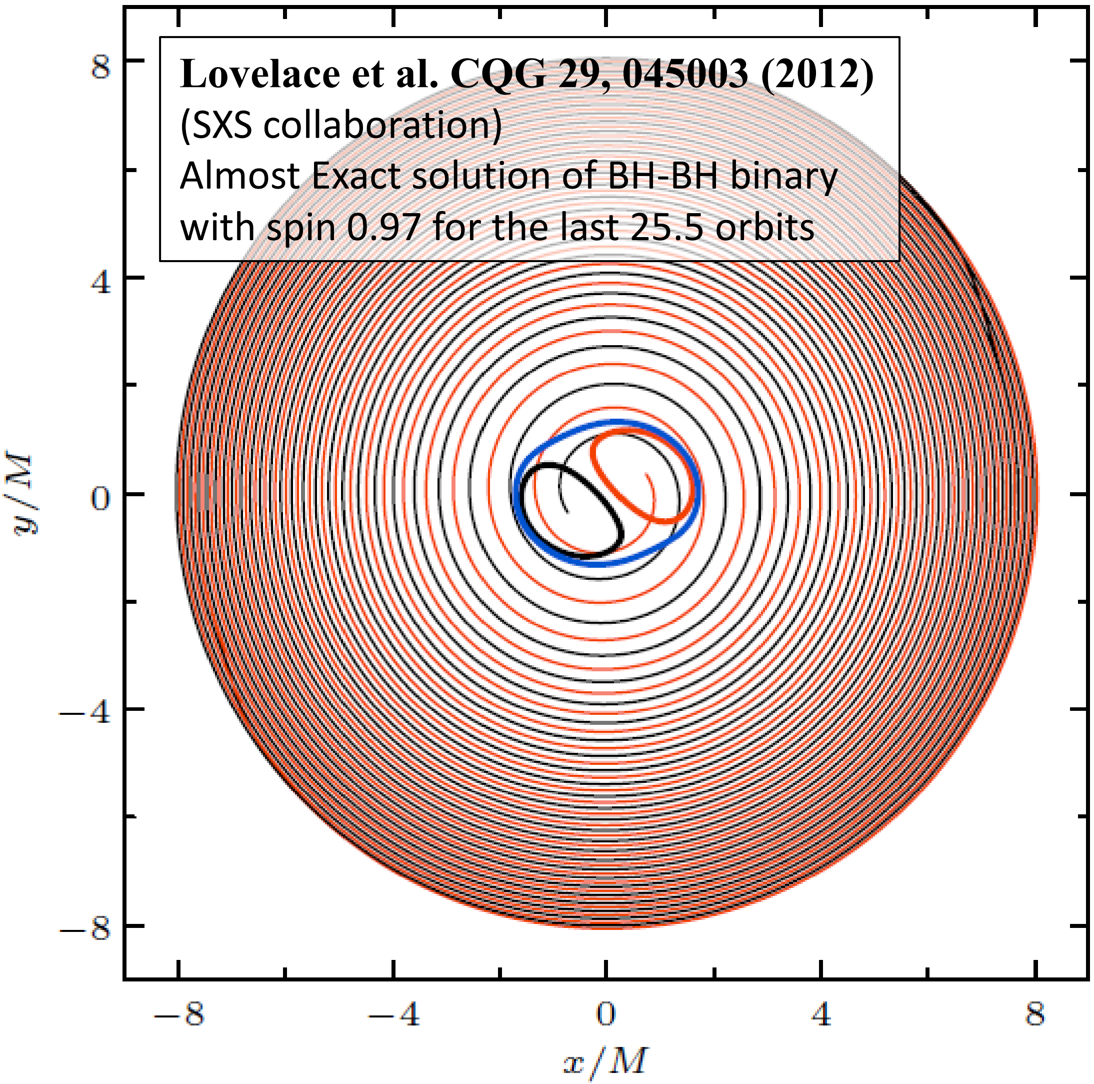
Momentum constraint is used

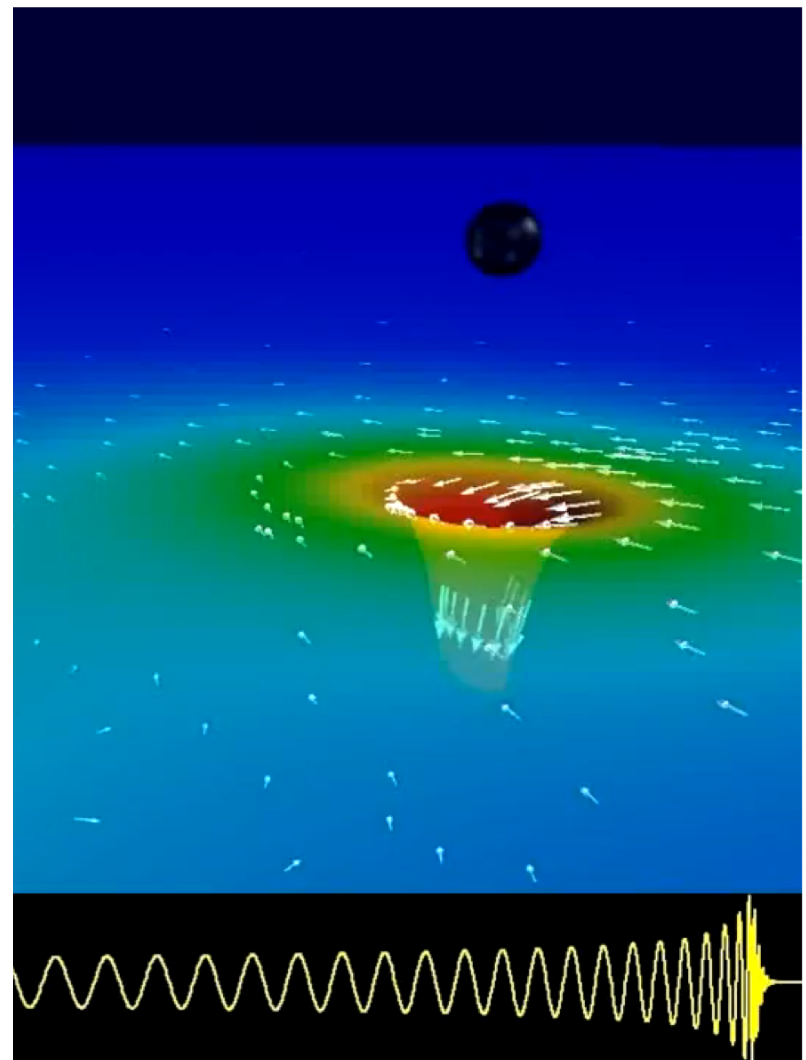
A milestone simulation by SXS collaboration: Long-term simulation of BH-BH merger



A milestone simulation by SXS collaboration: Long-term simulation of BH-BH merger

Scheel et. al., Phys. Rev. D 79, 024003 (2009); Cohen et. al., Class. Quantum Grav. 26 035005 (2009)



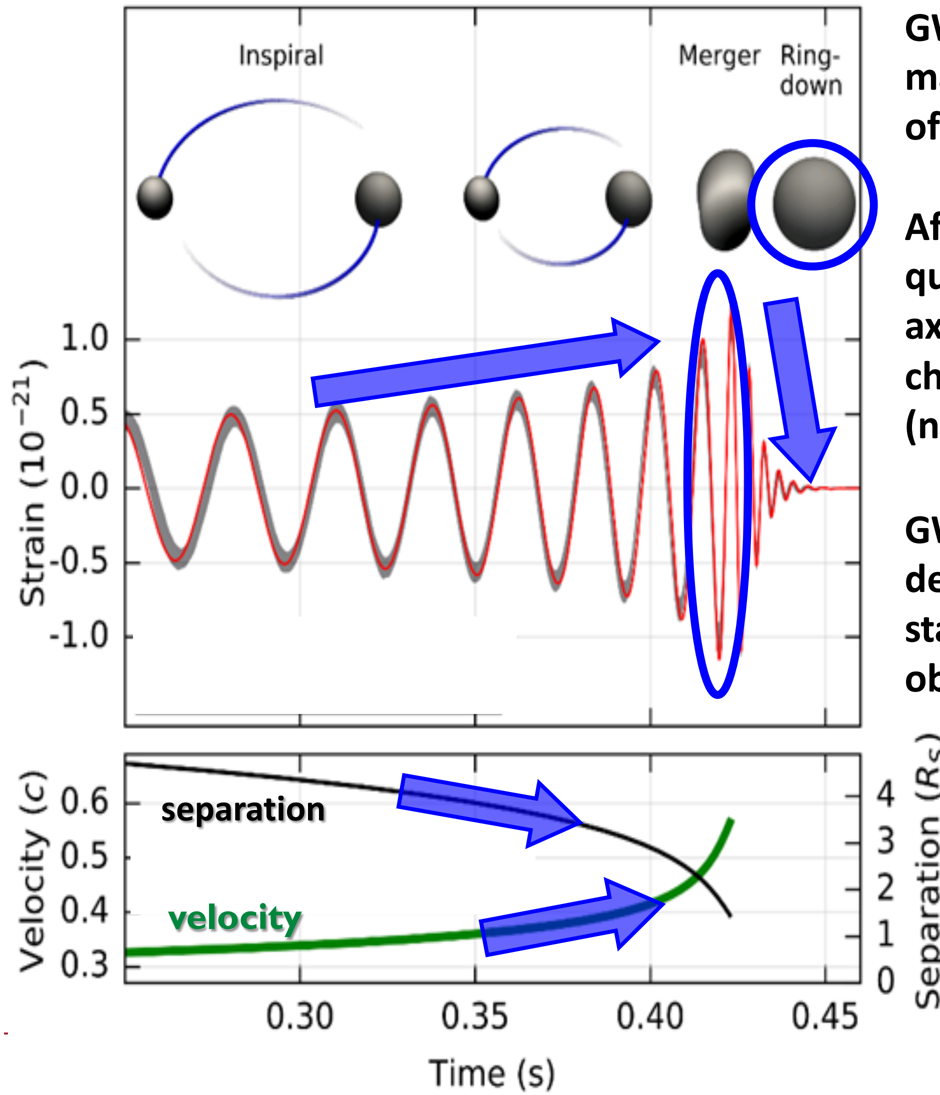


GWs from BH-BH merger

Also, GW frequency and, accordingly amplitude, increases as shown in the linearized theory

Orbital separation decreases due to GW emission

Orbital velocity increases basically according to Kepler's law

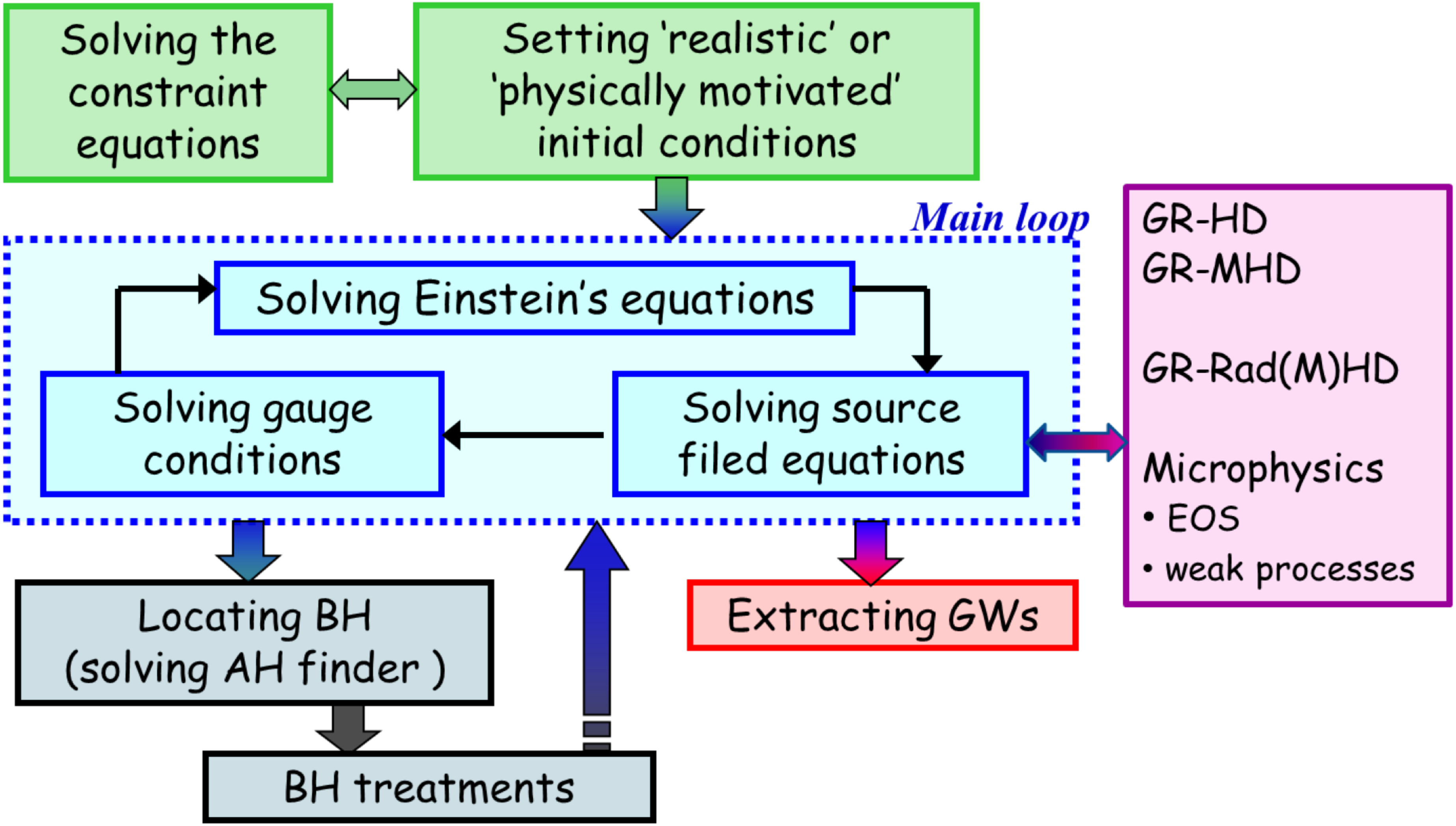


GW Amplitude takes maximum at the moment of the merger

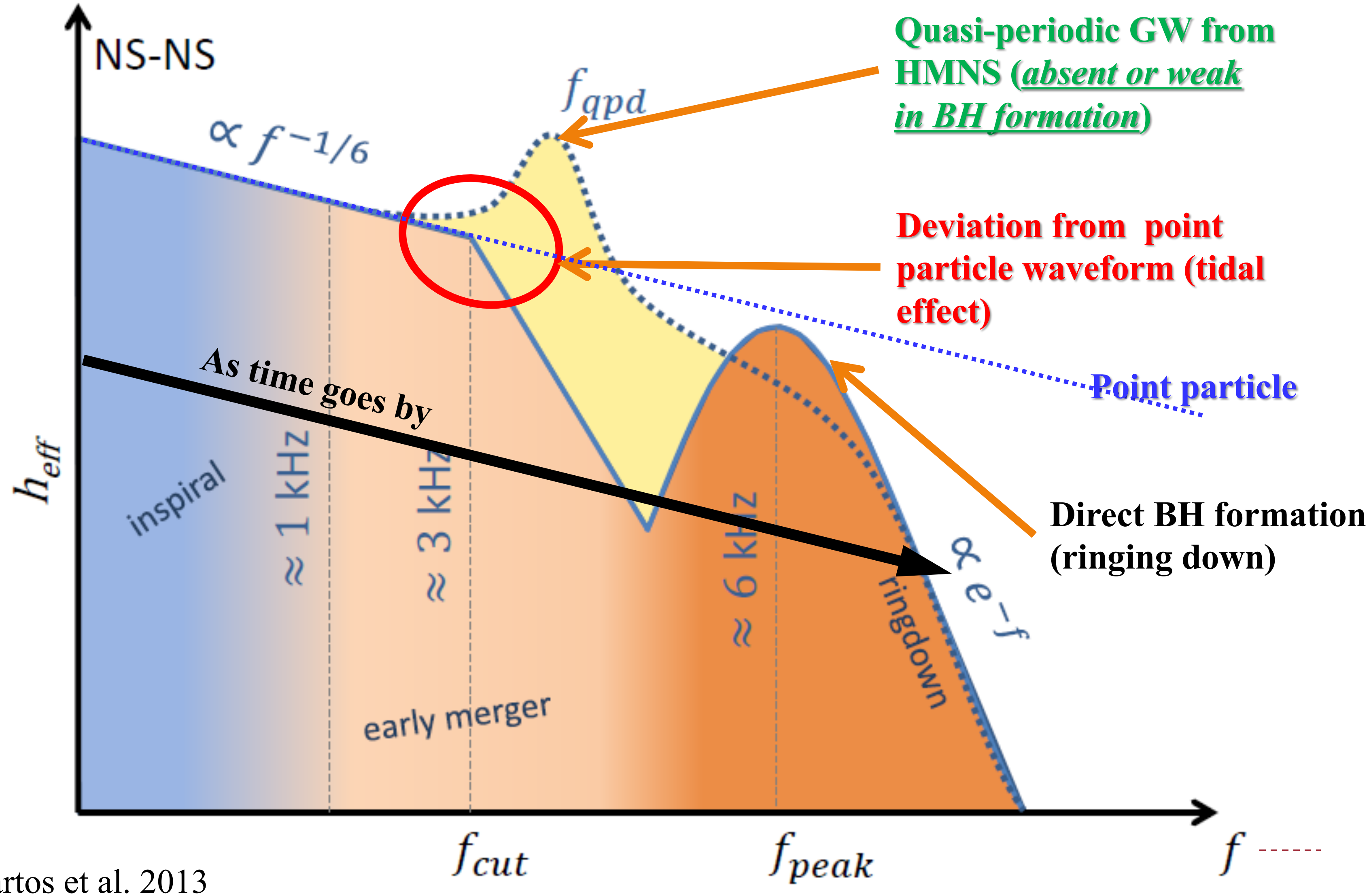
After the merger, BH quickly becomes axisymmetric due to its characteristic property (no hair theorem)

GW amplitude quickly decreases because stationary axisymmetric object does not emit GW

Summary of Numerical Relativity

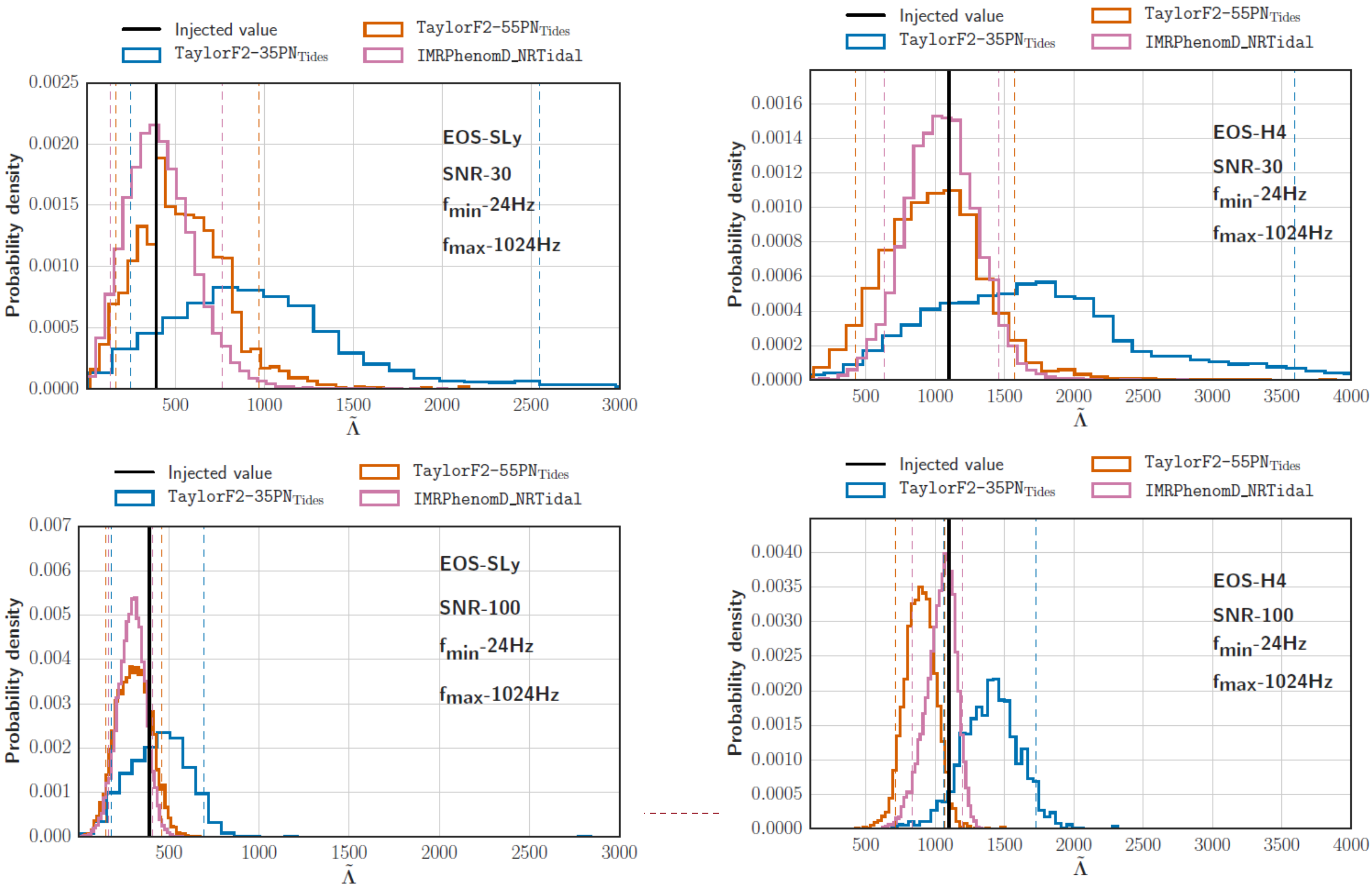


Schematic picture of GW spectra

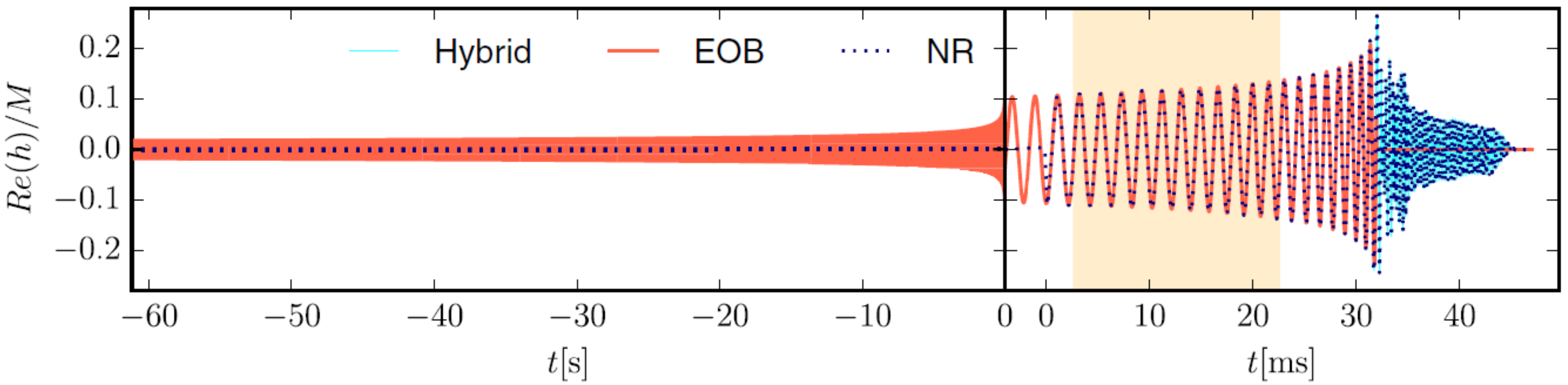


- Bartos et al. 2013

Messina et al. (2019) 1904.09558



Dudi et al. (2019) PRD 98 084061



GW Modelling based on NR

- A Model for frequency-domain GW waveforms constructed from our numerical relativity simulations (GW phase as an example)
  - Parameters φ_i are determined by fitting the NR results with the model

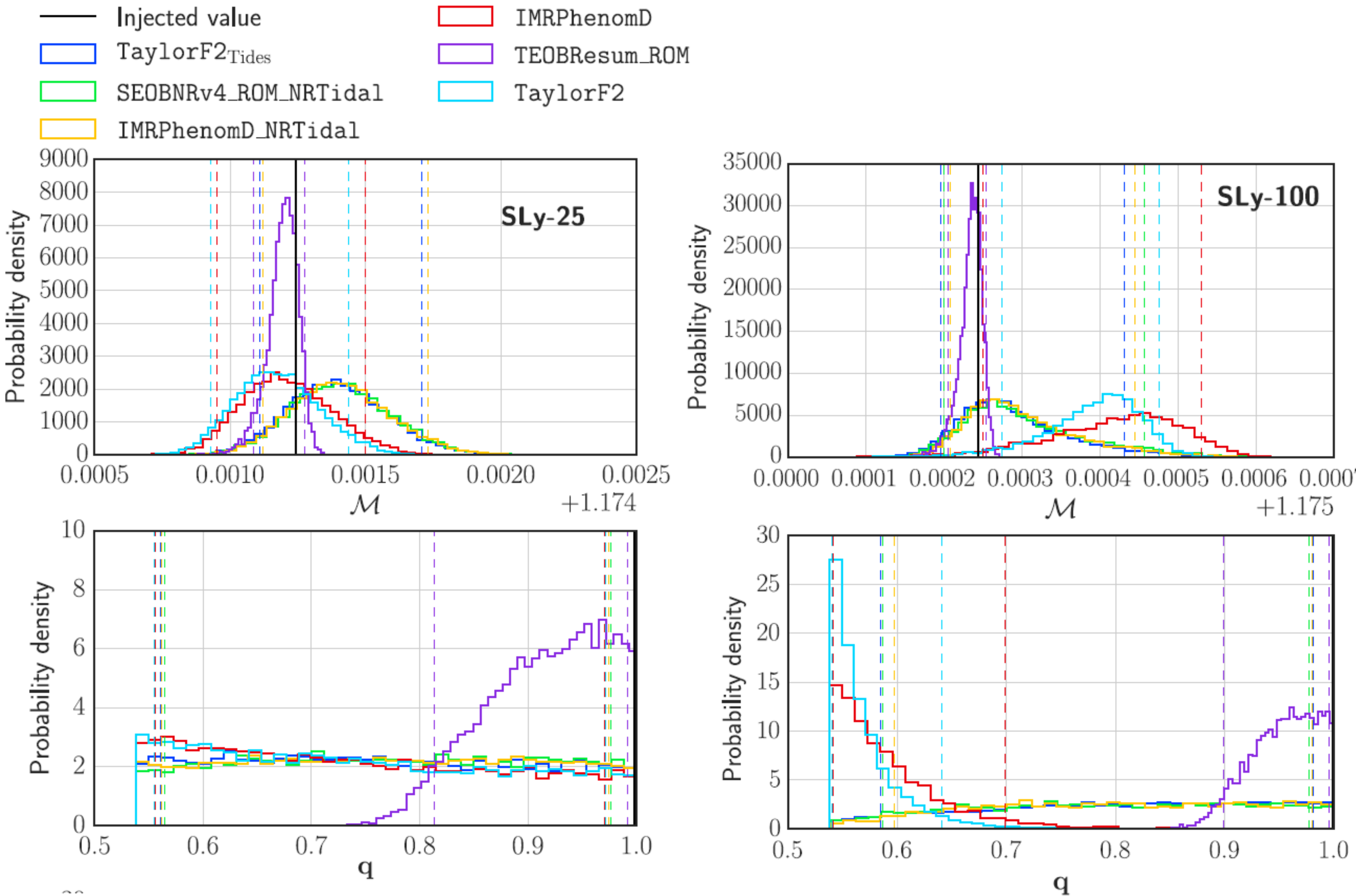
$$\phi_{\rm NR} = \phi_{\rm TaylorF2} + \frac{3}{128\eta}(\pi f M)^{-5/3}\sum_{i=9}^{12}\varphi_i(\Xi)(\pi f M)^{i/3}$$

  - The results are

$$\varphi_9 = -31639 - 57538(1-4\eta)$$
$$\varphi_{10} = 115409 + 234839(1-4\eta)$$
$$\varphi_{11} = -206911 - 525206(1-4\eta)$$
$$\varphi_{12} = -161911 + 431837(1-4\eta)$$

  - Similarly, we constructed the correction due to the tidal deformation
  - Our model can be used for $300 \lesssim \Lambda \lesssim 1900$ with less than 0.1 rad error
  - Such a model is crucial to extract information of Λ from GW

Dudi et al. (2019) PRD 98 084061



Dudi et al. (2019) PRD 98 084061