Summer School on "Frontiers in Lattice QCD"
June 24—July 12 2019, Peking University

Renormalization of lattice operators & non-perturbative matching of 3/4-flavor Wilson coefficients using position-space correlators

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Outline

- Renormalization of lattice operators
  - Motivation
  - Theoretical foundation
  - Example: quark mass renormalization from Z_S
  - Renormalization of mixing operators?
- NP matching of 3/4-flavor Wilson coefficients
  - Motivation
  - Theoretical foundation
  - Result of an exploratory calculation

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  - Motivation
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  - Result of an exploratory calculation

※ **Both use correlation functions in position-space**

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Renormalization

- Necessary step before continuum limit

$$\left.\begin{array}{c} X^{\mathrm{lat}}(a_1) \to Z^{R/\mathrm{lat}}(\mu; a_1) X^{\mathrm{lat}}(a_1) \\ \vdots \\ X^{\mathrm{lat}}(a_n) \to Z^{R/\mathrm{lat}}(\mu; a_n) X^{\mathrm{lat}}(a_n) \end{array}\right\} \xrightarrow{a \to 0} X^R(\mu)$$

- Example
  - Quark mass

$$m_q^{\mathrm{lat}}(a) \to Z_m(\mu; a) m_q^{\mathrm{lat}}(a)$$

  - Matrix Elements

$$\langle f | O_i^{\mathrm{lat}}(a) | i \rangle \to Z_{ij}^{R/\mathrm{lat}}(\mu; a) \langle f | O_j^{\mathrm{lat}}(a) | i \rangle$$

Position-space renormalization

- X-space scheme

$$O^{\mathrm{lat}}(a) \to Z^{X/\mathrm{lat}}(\mu;a)O^{\mathrm{lat}}(a) \xrightarrow{a\to 0} O^{X}(\mu)$$

  - Renormalization condition

$$\langle O^X(\mu;x)O^X(\mu;y)^\dagger\rangle = \langle O(x)O(y)^\dagger\rangle\,|_{\alpha_s\to 0} \qquad \mu = 1/|x-y|$$

  - Z factor

$$\widetilde{Z}^{X/\mathrm{lat}}(\mu;a;x) = \sqrt{\frac{G^{\mathrm{free}}(x)}{G^{\mathrm{lat}}(a;x)}}$$

  - Non-perturbative treatment: Step scaling

$$\Sigma^X(\mu,2\mu) = \lim_{a\to 0}\frac{Z^{X/\mathrm{lat}}(2\mu;a)}{Z^{X/\mathrm{lat}}(\mu;a)}$$

Cichy-Jansen-Korcyl, 2016

Matching to other schemes

$$O^R(\mu_R) = Z^{R/X}(\mu_R;\mu_X)O^X(\mu_X)$$

- Usually used for $\overline{\text{MS}}$
- Matching condition by 2pt function

$$Z^{R/X}(\mu_R;\mu_X)^2 G^X(\mu_X;x) = G^R(\mu_R;x)$$

$$\Rightarrow Z^{R/X}(\mu_R;\mu_X) = \sqrt{\frac{G^R(\mu_R;x)}{G^X(\mu_X;x)}}$$

- Direct renormalization: lat → R

$$O^{lat}(a) \to Z^{R/lat}(\mu;a)O^{lat}(a) \xrightarrow{a\to 0} O^R(\mu)$$

$$\widetilde{Z}^{R/lat}(\mu;a;x) = \sqrt{\frac{G^R(\mu;x)}{G^{lat}(a;x)}}$$

Gauge invariant!

Mass renormalization w/ 2pt func.

- Scalar current renormalization useful

  $Z_m = Z_S^{-1} (= Z_P^{-1}$ if chiral symmetry in lattice fermions)

- Renormalization using 2pt functions

$$\tilde{Z}_{S/P}^{\overline{MS}/lat}(\mu; a; x) = \sqrt{\frac{G_{S/P}^{\overline{MS}}(\mu; x)}{G_{S/P}^{lat}(a; x)}}$$

$$G_S(x) = \langle \bar{u}d(x)\bar{d}u(0) \rangle$$

$$G_P(x) = \langle \bar{u}i\gamma_5 d(x)\bar{d}i\gamma_5 u(0) \rangle$$

– Advantages

  ✦ Renormalization in a gauge invariant manner

  ✦ $\overline{MS}$ calculation available to $O(\alpha_s^4)$

Chetyrkin & Maier, 2011

x-dependence of $\widetilde{Z}$

$$\widetilde{Z}^{\overline{\mathrm{MS}}/\mathrm{lat}}_{S/P}(\mu; a; x) = \sqrt{\frac{G^{\overline{\mathrm{MS}}}_{S/P}(\mu; x)}{G^{\mathrm{lat}}_{S/P}(a; x)}}$$

- Discretization error (SDs, lat)
  - $(a/x)^{2n}$
  - O(4) violation
- Truncation of perturbative calculation (LDs, $\overline{\mathrm{MS}}$)
- NP effects: not encoded in $\overline{\mathrm{MS}}$ (LDs)
  - OPE effects: $m\langle\bar{q}q\rangle x^4$, $\langle GG\rangle x^4$, $\langle\bar{q}\bar{q}qq\rangle x^6$, …
  - Instanton effects

☆ Window problem: $a \ll x \ll 1/\Lambda_{QCD}$

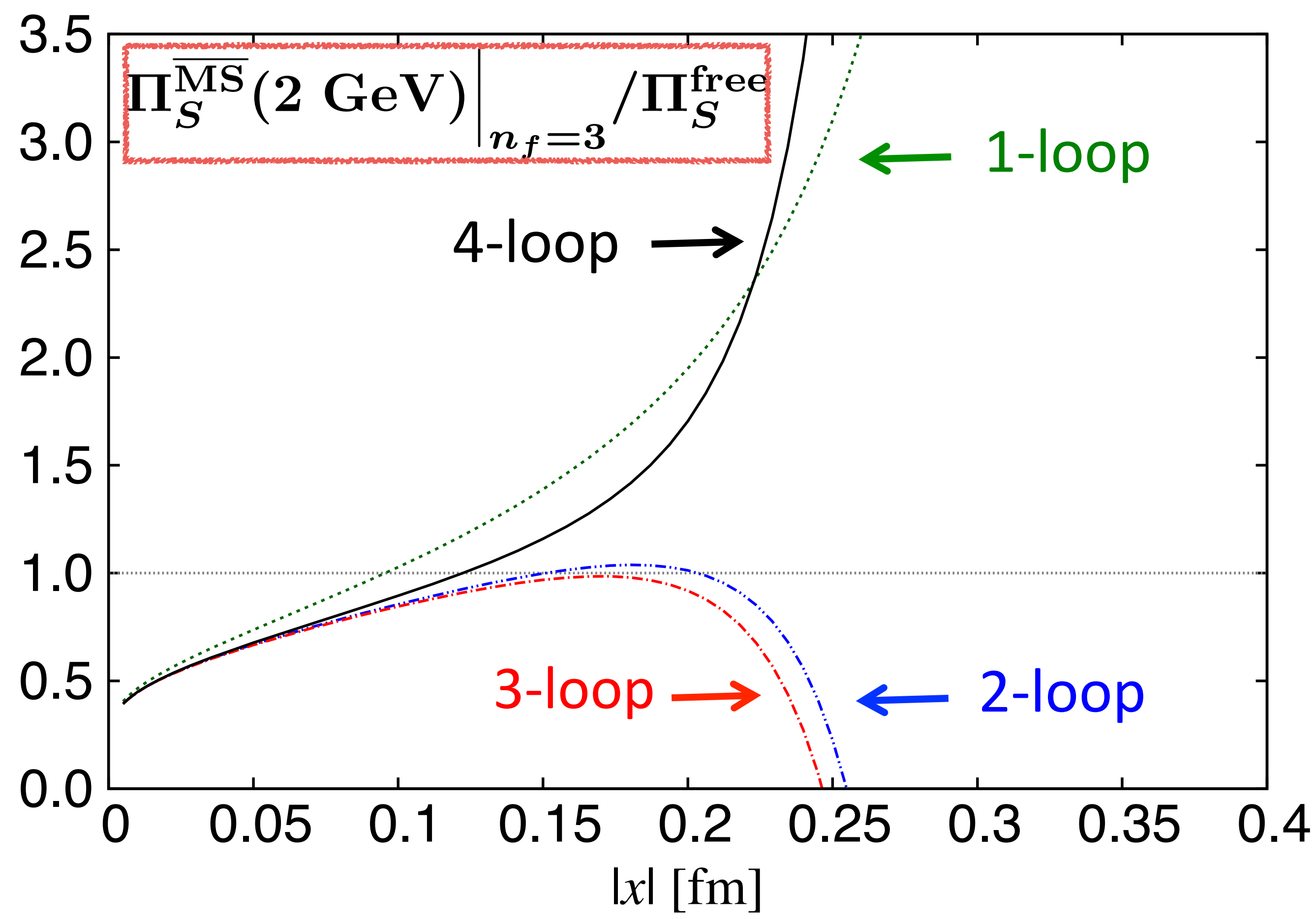
$\overline{\text{MS}}$ correlator

- Available up to $O(\alpha_s^4)$ [Chetyrkin, Maier (2011)]

- Bad convergence of original perturbative series

$$G_S^{\overline{\text{MS}}}(\mu_x = 1/|x|; x)\Big|_{n_f=3} = \frac{3}{\pi^4 x^6}\left(1 + 0.20a_s - 19a_s^2 - 11a_s^3 + 579a_s^4\right)$$

$$a_s = \alpha_s(\mu_x = 1/|x|)/\pi$$



  - large coef. at $O(a_s^4)$
  - a_s blows up at ~ 0.2 fm

- No window for recently available lattice spacings

Resummation of perturbation

$$C_0 + C_1 a_s(\mu) + C_2 a_s(\mu)^2 + \ldots$$

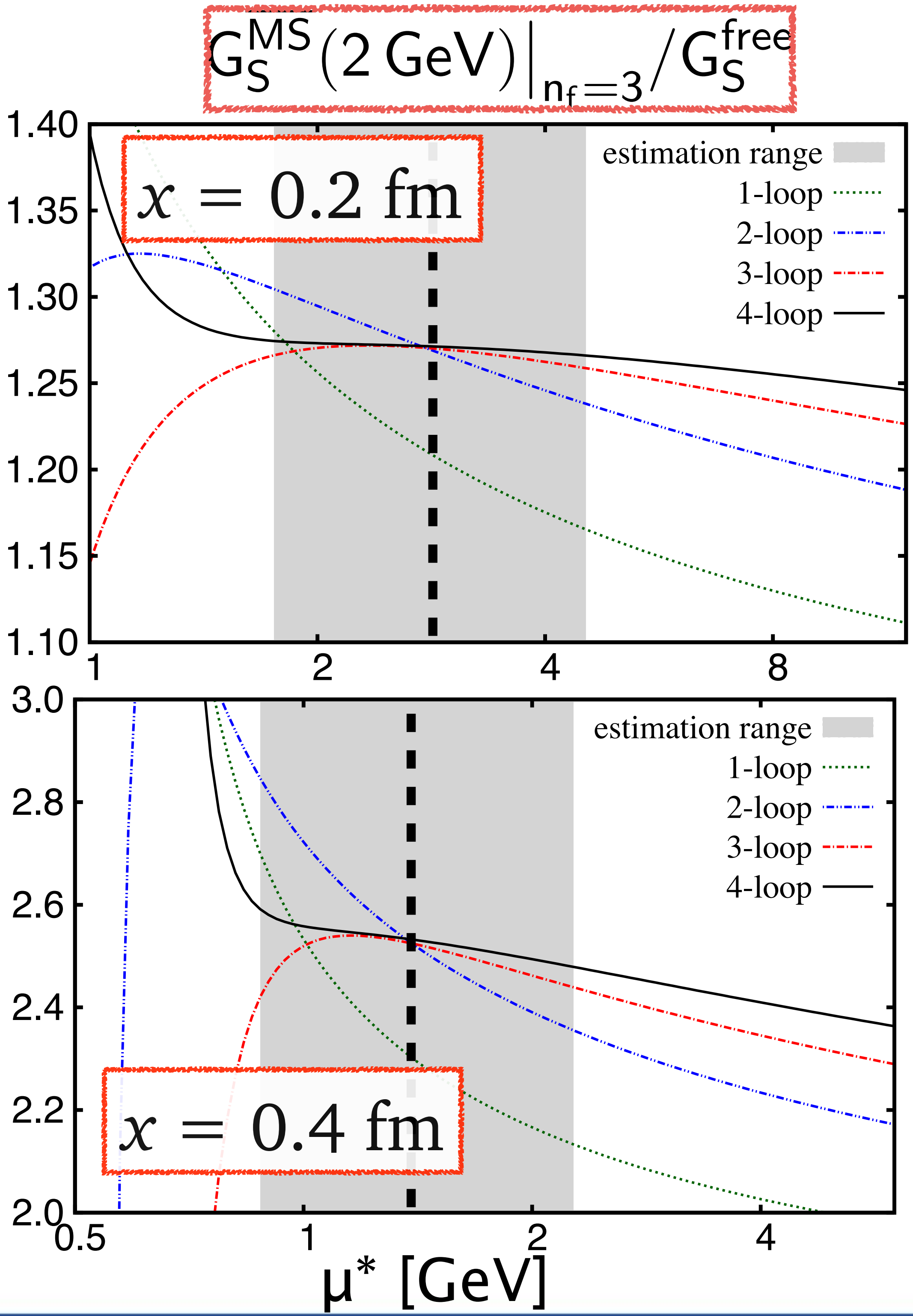
$$\Vert$$

RG equation of a_s:

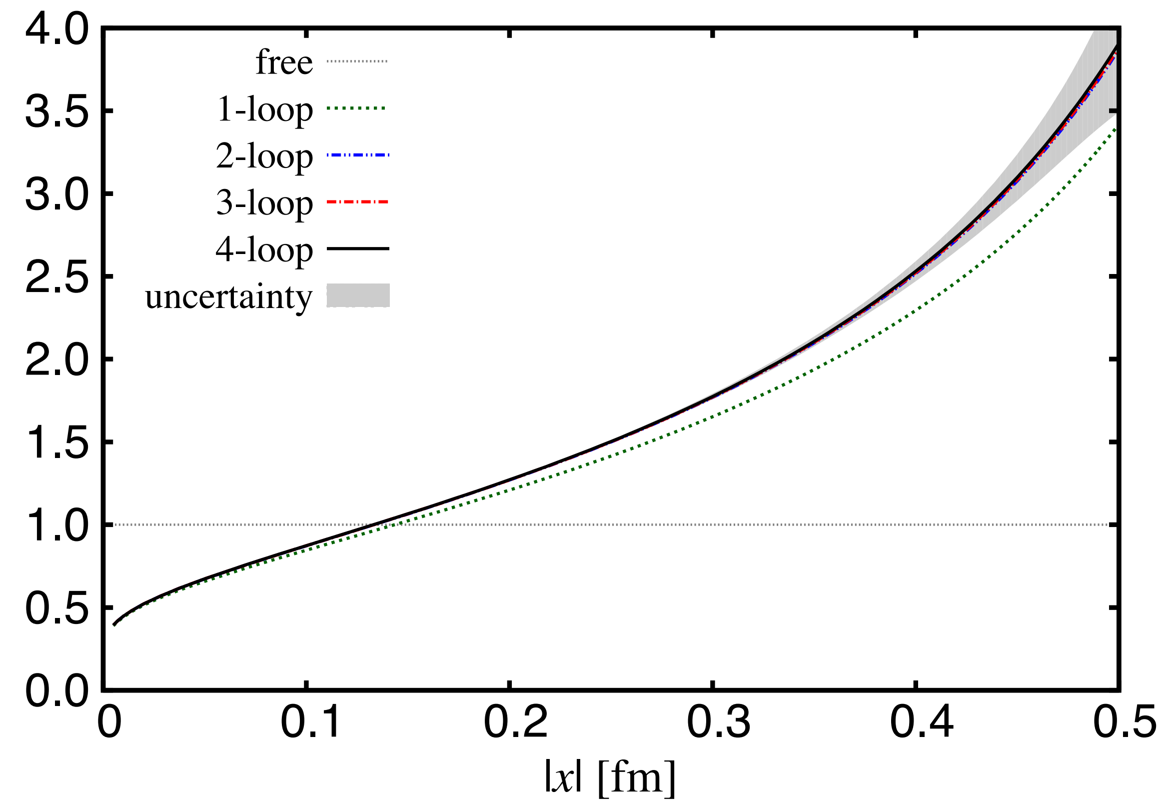
$$a_s(\mu) = a_s(\mu^*)\left(1 + a_s(\mu^*)\beta_0 \ln \mu^2/\mu^{*2} + \ldots\right)$$

$$C_0 + C_1 a_s(\mu^*) + C_2^* a_s(\mu^*)^2 + \ldots$$

μ*-dependence



- We use $\mu_x^* = 2.9/x$
- Truncation error estimated by gray band



[MT etal (2015)]

x-dependence of $\widetilde{Z}$

$$\widetilde{Z}_{S/P}^{\overline{MS}/\mathrm{lat}}(\mu; a; x) = \sqrt{\frac{G_{S/P}^{\overline{MS}}(\mu; x)}{G_{S/P}^{\mathrm{lat}}(a; x)}}$$

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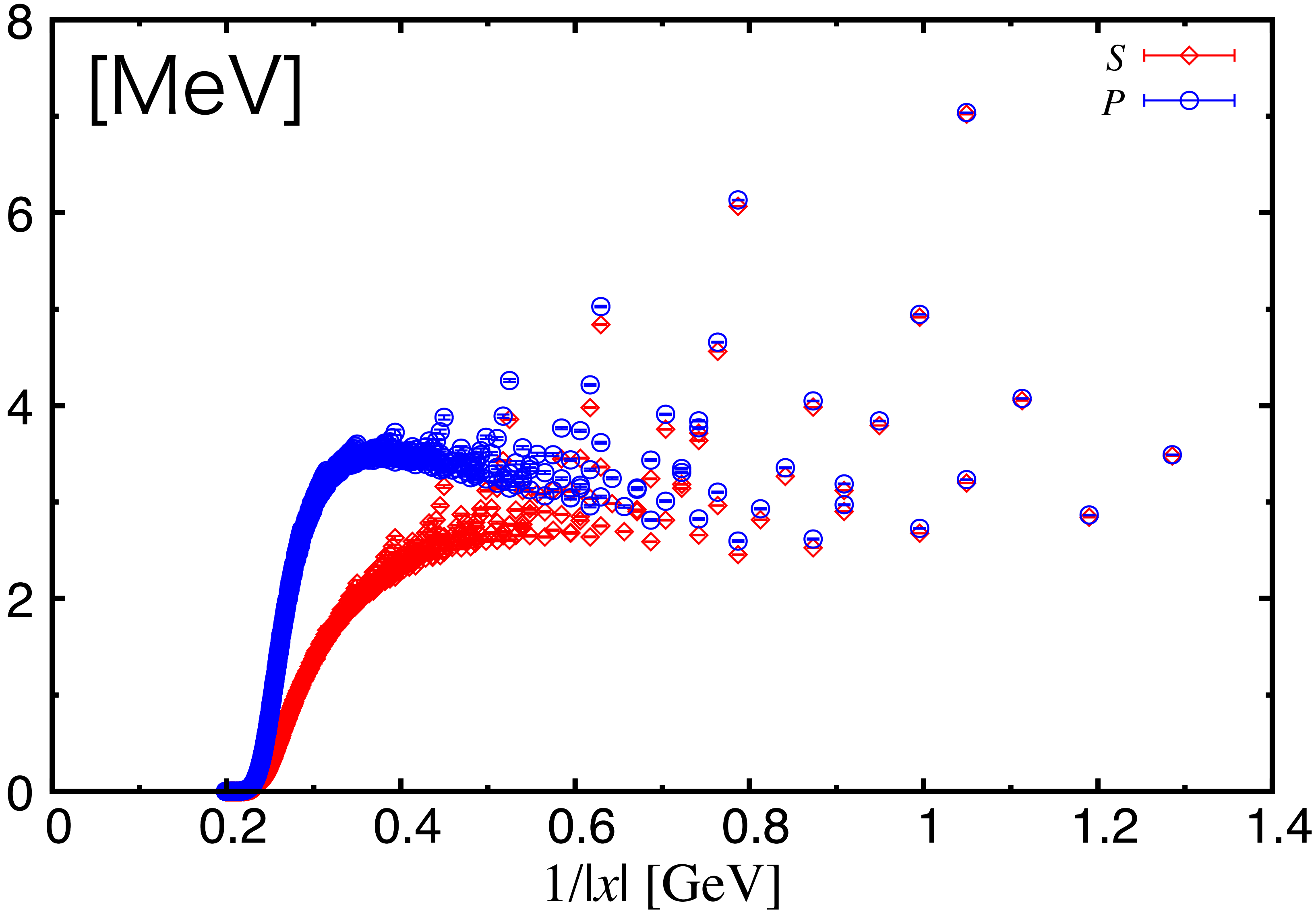
  - Instanton effects

☆ Window problem: $a \ll x \ll 1/\Lambda_{QCD}$

Renormalized mass

$$\widetilde{m}^{\overline{MS}}_{q,S/P}(3\,\mathrm{GeV}; a; x) = \widetilde{Z}^{\overline{MS}/\mathrm{lat}}_{S/P}(3\,\mathrm{GeV}; a; x)^{-1}\, m^{\mathrm{bare,phys}}_{q}(a)$$

RBC/UKQCD (2016)



Ensemble (RBC/UKQCD)

- 2+1 DW fermions
- $a^{-1} \approx 3.1$ GeV
- $M_\pi \approx 300$ MeV

Renormalized mass

$$\widetilde{m}^{\overline{MS}}_{q,S/P}(3\,\mathrm{GeV}; a; x) = \widetilde{Z}^{\overline{MS}/\mathrm{lat}}_{S/P}(3\,\mathrm{GeV}; a; x)^{-1} \underline{m_q^{\mathrm{bare,phys}}(a)}$$

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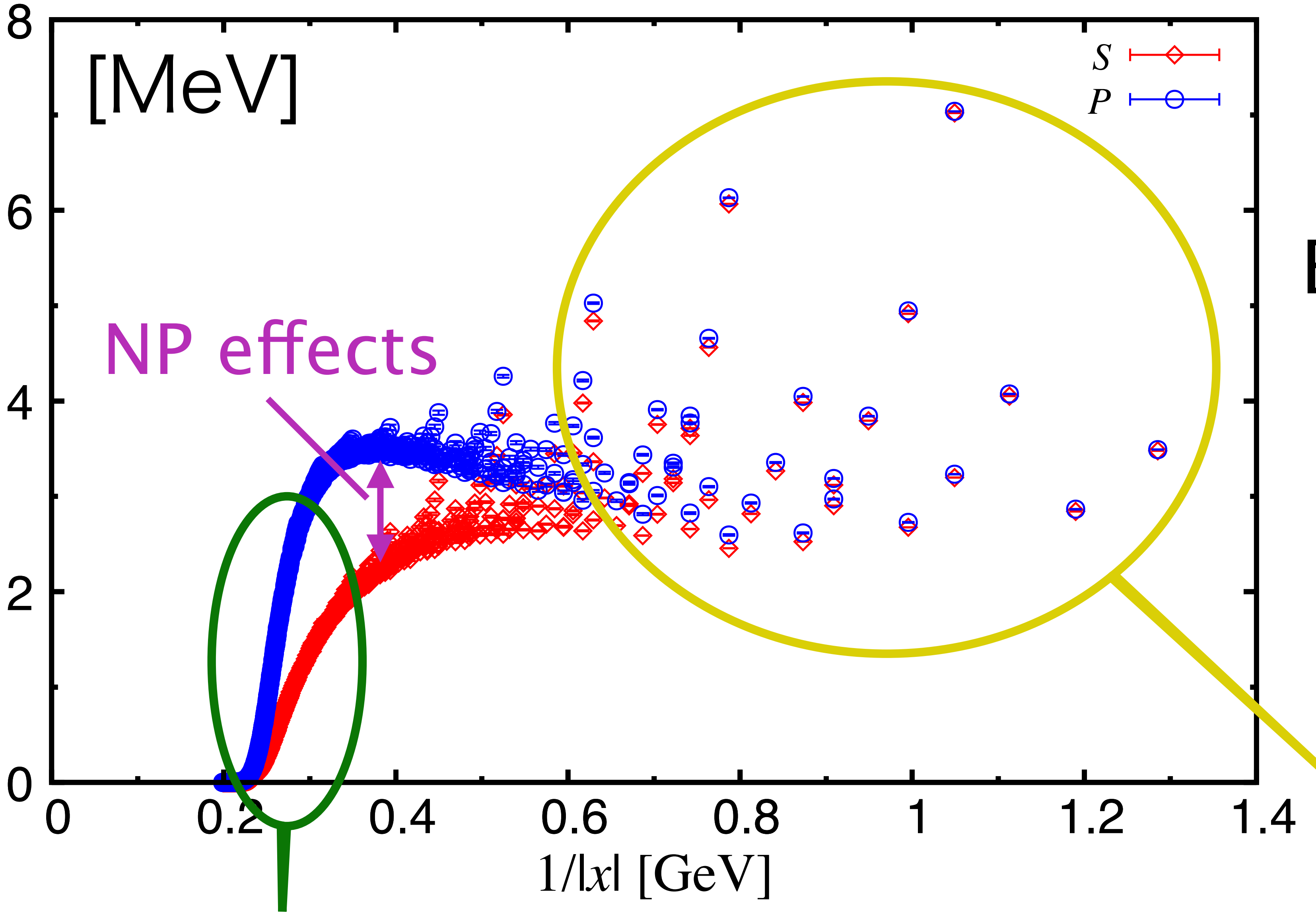

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Discretization error

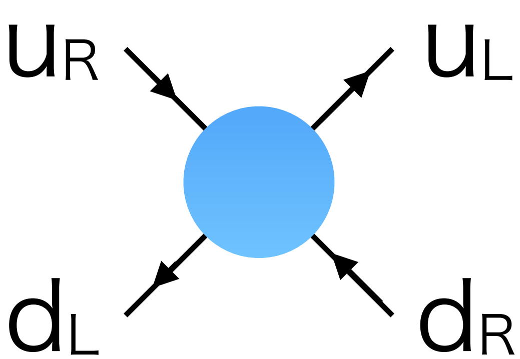
(1,1,1,1) ≠ (0,0,0,2) on lat

Truncation of perturbation

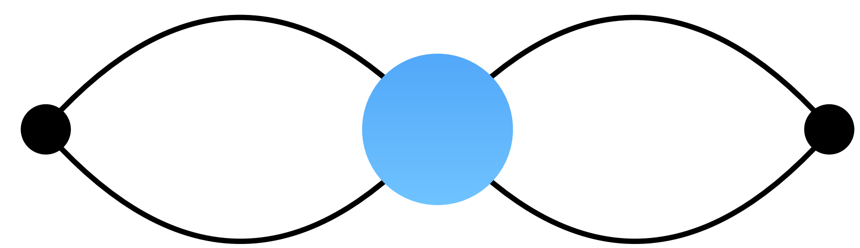
treated as much as possible

NP effects

- 3% difference b/w S & P at $1/|x| \approx 1$ GeV
- "Instanton effects ≫ OPE effects" for S & P Shuryak 1993
- 't Hooft vertex & single instanton effect



't Hooft 1979



- Chirality flip → difference b/w S & P
- Same magnitude but different sign for S & P

- Average can reduce such effects (applicable for DWF)

$$\widetilde{Z}^{\overline{\mathrm{MS}}/\mathrm{lat}}_{S+P}(\mu; a; x) = \sqrt{\frac{G_S^{\overline{\mathrm{MS}}}(\mu; x)}{\frac{1}{2}\left(G_S^{\mathrm{lat}}(a; x) + G_P^{\mathrm{lat}}(a; x)\right)}}$$

x-dependence of $\tilde{Z}$

$$\tilde{Z}_{S+P}^{\overline{MS}/lat}(\mu; a; x) = \sqrt{\frac{G_S^{\overline{MS}}(\mu; x)}{\frac{1}{2}\left(G_S^{lat}(a; x) + G_P^{lat}(a; x)\right)}}$$

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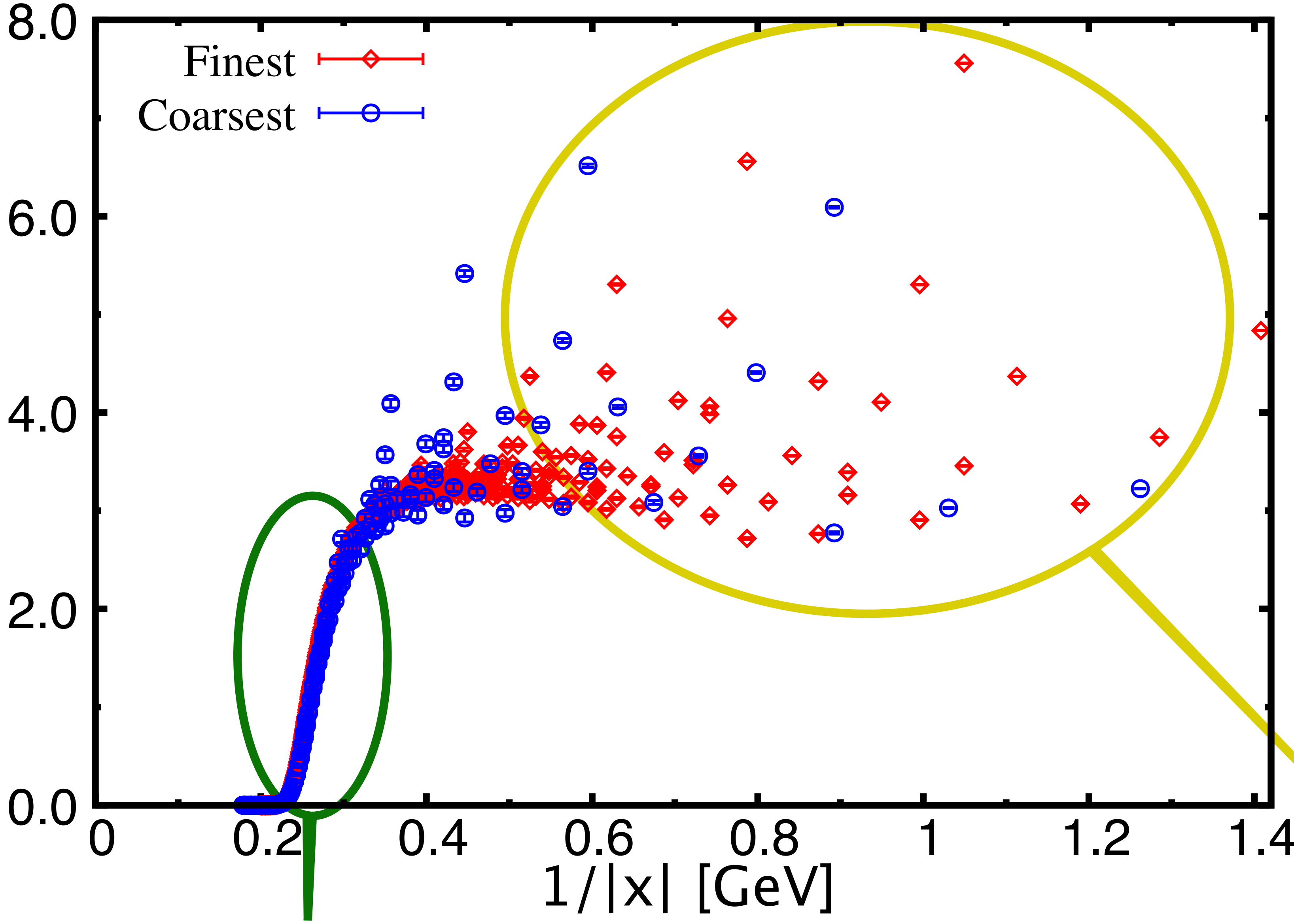
⭐ Window problem: $a \ll x \ll 1/\Lambda_{QCD}$

Renormalized mass

$$\widetilde{m}^{\overline{MS}}_{q,S/P}(3\,\text{GeV};a;x) = \widetilde{Z}^{\overline{MS}/\text{lat}}_{S/P}(3\,\text{GeV};a;x)^{-1}\, m_q^{\text{bare,phys}}(a)$$

[RBC/UKQCD (2016)]



Ensemble (RBC/UKQCD)

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Discretization error

(1,1,1,1) ≠ (0,0,0,2) on lat

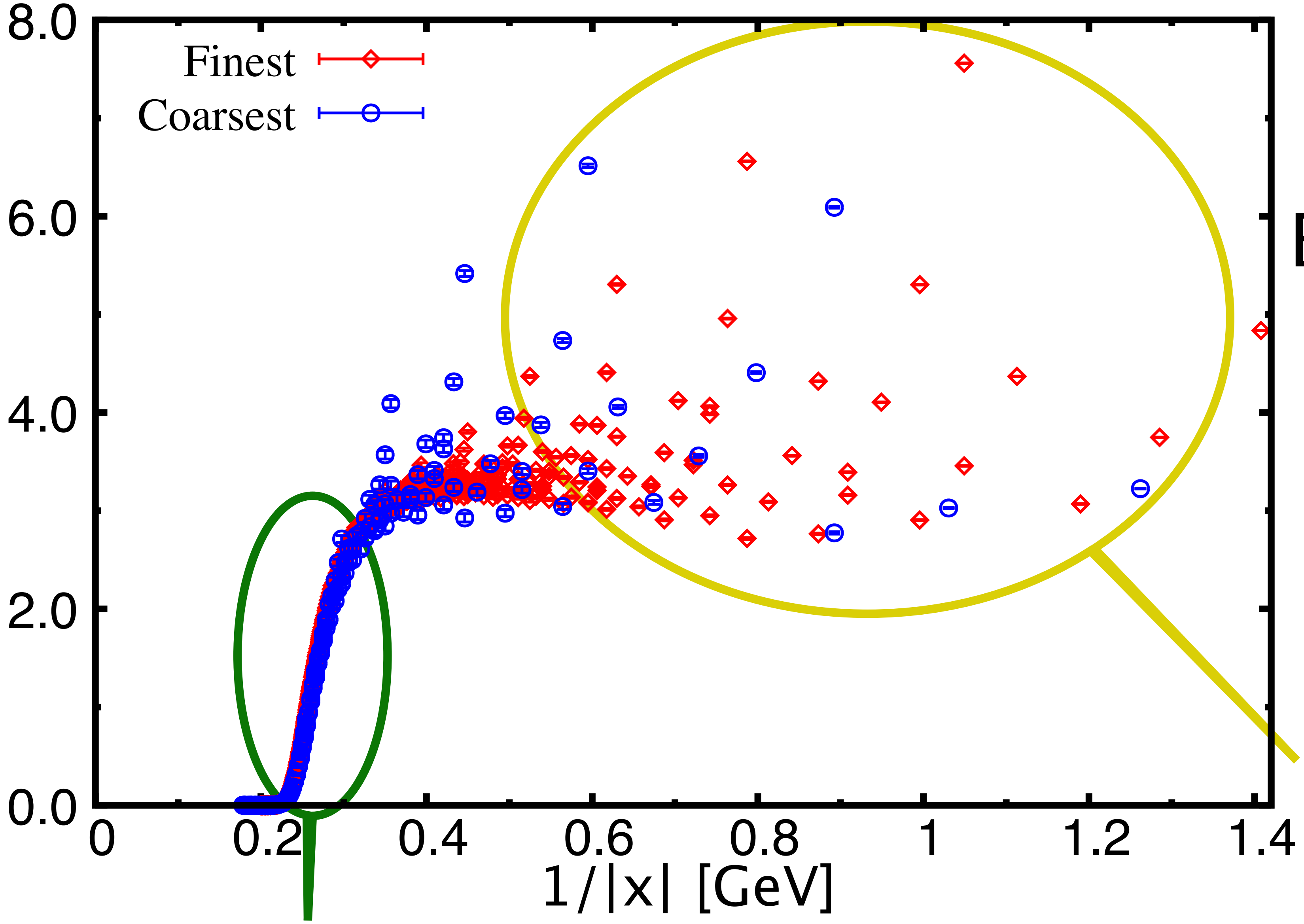
Truncation of perturbation

treated as much as possible

Renormalized mass

$$\widetilde{m}^{\overline{MS}}_{q,S/P}(3\,\text{GeV};a;x) = \widetilde{Z}^{\overline{MS}/\text{lat}}_{S/P}(3\,\text{GeV};a;x)^{-1}\, m^{\text{bare,phys}}_q(a)$$

[RBC/UKQCD (2016)]



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Discretization error

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☆ How to take $a \to 0$?

Truncation of perturbation

treated as much as possible

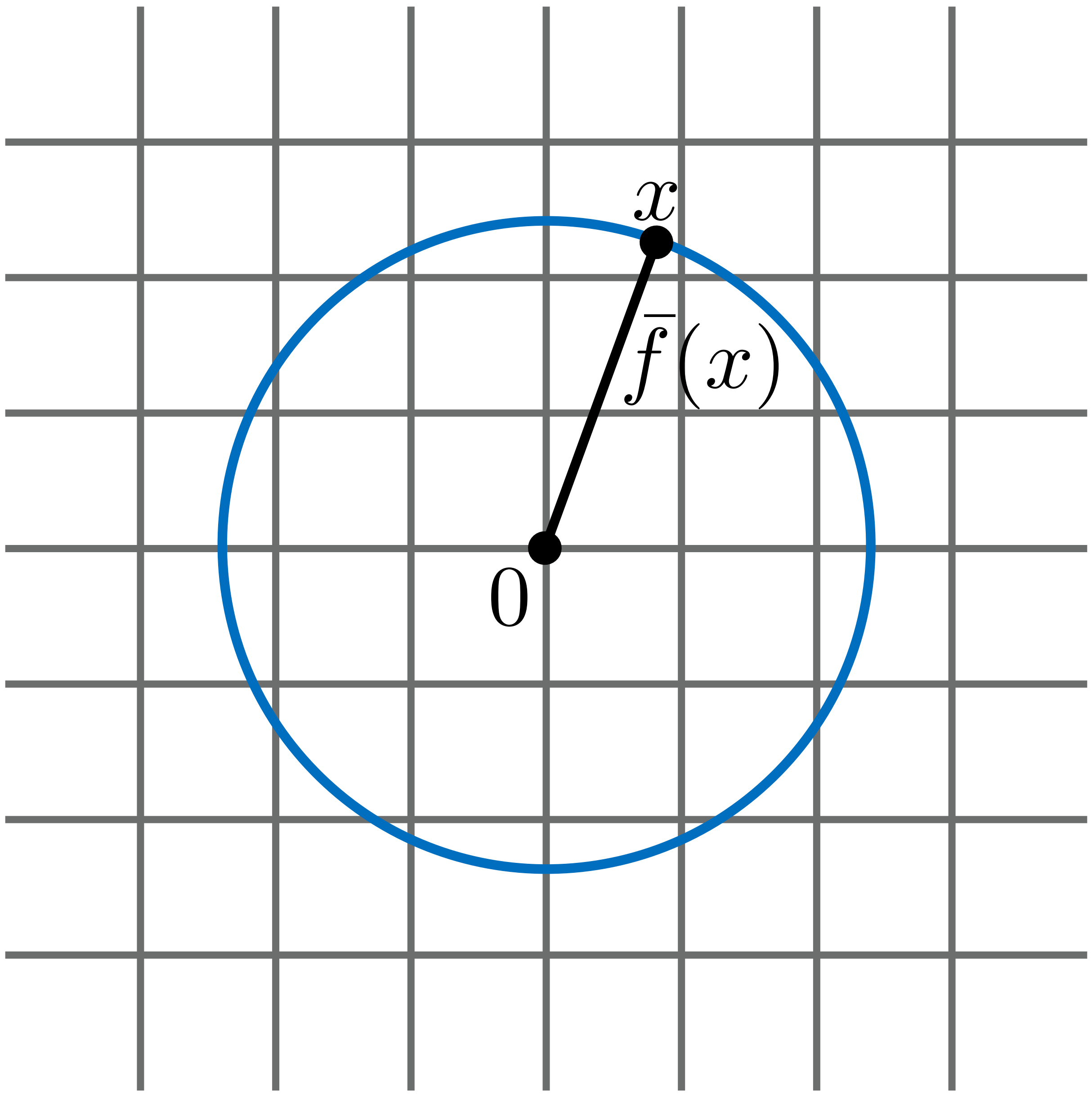
Average over spheres

MT & N.Christ, PRD99 014515

- Estimate the value of a quantity at each 4d point from values at lattice points, with a guideline

$$\bar{f}(x) = \eta(f^{\mathrm{lat}}; x)$$

※ details in following slides



- Take the average over the sphere for each distance |x|

$$\hat{f}(|x|) = \frac{1}{2\pi^2} \oint_{S^3(|x|)} \mathrm{d}\Omega\, \bar{f}(x)$$

✦ More sophisticated approach in Miyamoto etal 1906.01987

Potential $O(a^1)$ error (1-dim)

- Defs:
  - f_n: lattice value at site n
  - $f(x)$: "continuum limit" : $f_n = f(an) + O(a^2)$

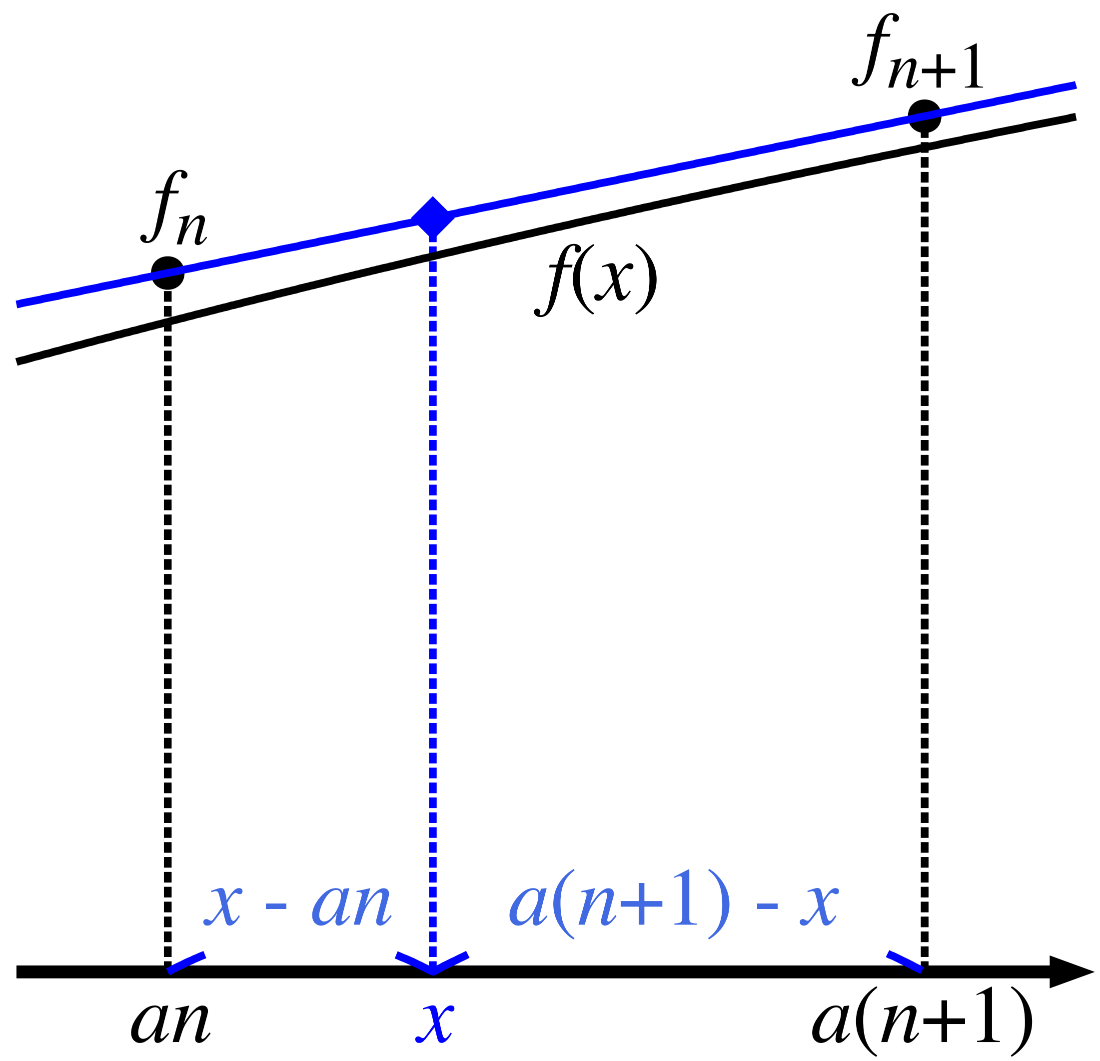
- Estimation $\bar{f}(x)$ should satisfy
  - $\bar{f}(x) = f(x) + O(a^2)$

- Potential $O(a^1)$ error in $\bar{f}(x)$
  - $f_n = f(an) + O(a^2)$

    $= f(x) + \underbrace{f'(x)\cdot(an-x)}_{O(a^1)} + O(a^2)$
  - $\bar{f}(x)$ is calculated using f_n's $\Rightarrow$ $O(a^1)$ can appear
  - Balanced combination needed

Evaluation of $\bar{f}(x)$ (1-dim)

- Linear interpolation

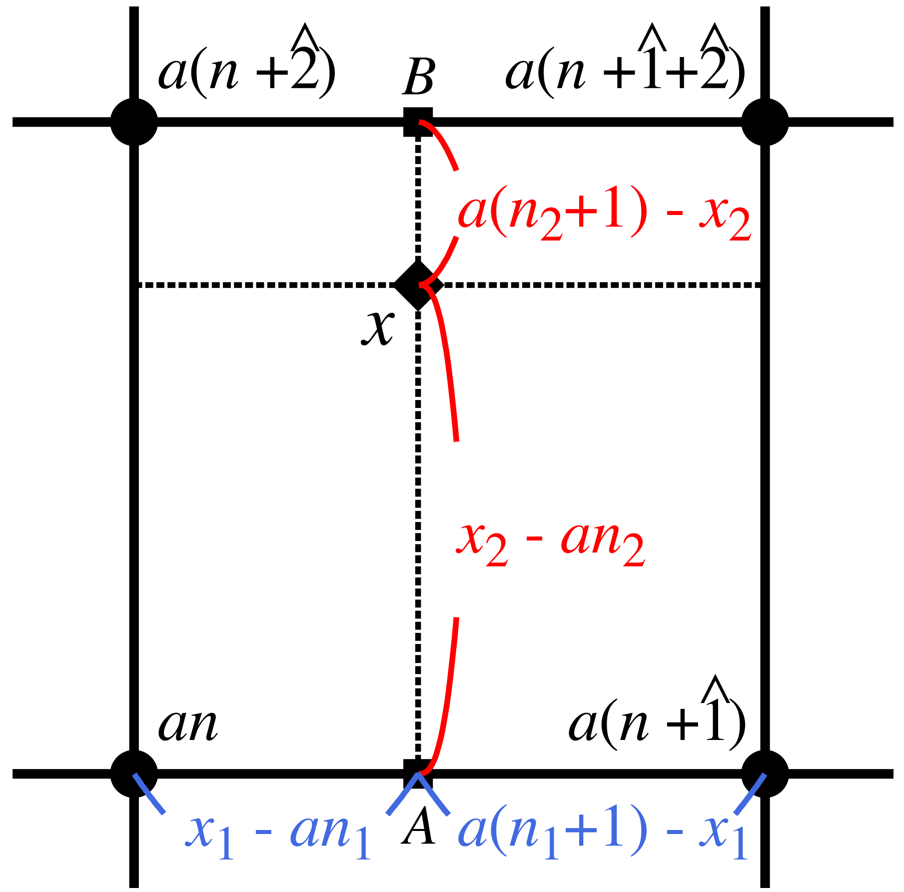


$$\bar{f}(x) = \frac{(a(n+1)-x)f_n + (x-an)f_{n+1}}{a} = f(x) + O(a^2)$$

Accurate up to O(a²)

Evaluation of $\bar{f}(x)$ (2-dim)

- Bilinear interpolation



$$\bar{f}(x) = \frac{(a(n_2+1)-x_2)\bar{f}(A) + (x_2 - an_2)\bar{f}(B)}{a}$$

$$= a^{-2}\left(\ a(n_1+1)-x_1 \quad x_1 - an_1\ \right)\begin{pmatrix} f_n & f_{n+\hat{2}} \\ f_{n+\hat{1}} & f_{n+\hat{1}+\hat{2}} \end{pmatrix}\begin{pmatrix} a(n_2+1)-x_2 \\ x_2 - an_2 \end{pmatrix}$$

$$= f(x) + O(a^2)$$

Evaluation of $\bar{f}(x)$ (4-dim)

- Quadrilinear interpolation

$$\bar{f}(x) = a^{-4} \sum_{i,j,k,l=0}^{1} \Delta_{1,i}\Delta_{2,j}\Delta_{3,k}\Delta_{4,l} f_{n+i\hat{1}+j\hat{2}+k\hat{3}+l\hat{4}}$$

$$\Delta_{\mu,i} = |a(n_\mu + 1 - i) - x_\mu|$$

– Accurate up to $O(a^2)$

x-dependence of $\tilde{Z}$

$$\tilde{Z}^{\overline{\mathrm{MS}}/\mathrm{lat}}_{S+P}(\mu; a; x) = \sqrt{\frac{G^{\overline{\mathrm{MS}}}_S(\mu; x)}{\frac{1}{2}\left(G^{\mathrm{lat}}_S(a; x) + G^{\mathrm{lat}}_P(a; x)\right)}}$$

✅ • Discretization error (SDs, lat)
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- OPE effects: $m\langle\bar{q}q\rangle x^4$, $\langle GG\rangle x^4$, $\langle\bar{q}\bar{q}qq\rangle x^6$, …
- Instanton effects

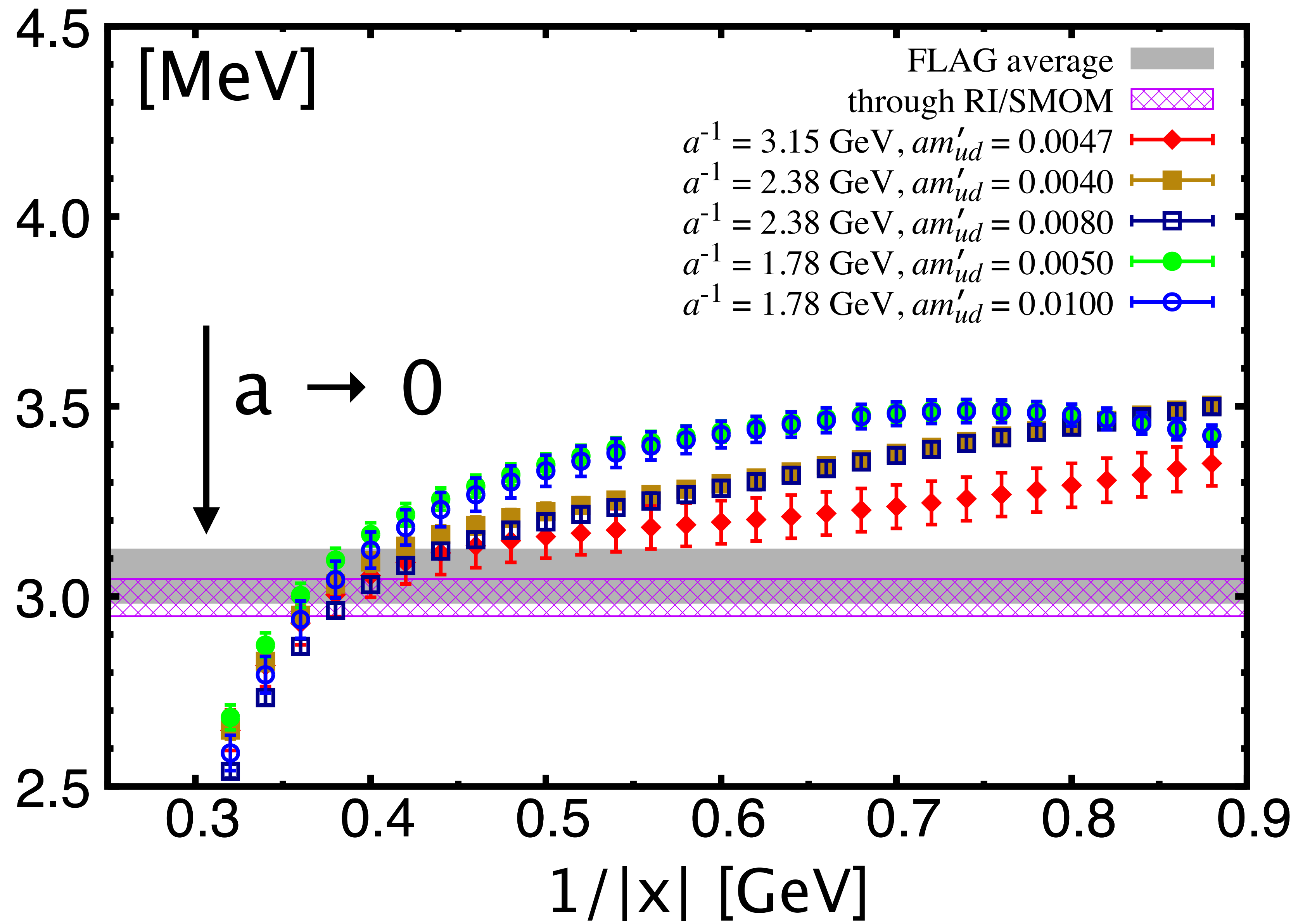
☆ Window problem: $a \ll x \ll 1/\Lambda_{QCD}$

Result of spherical average

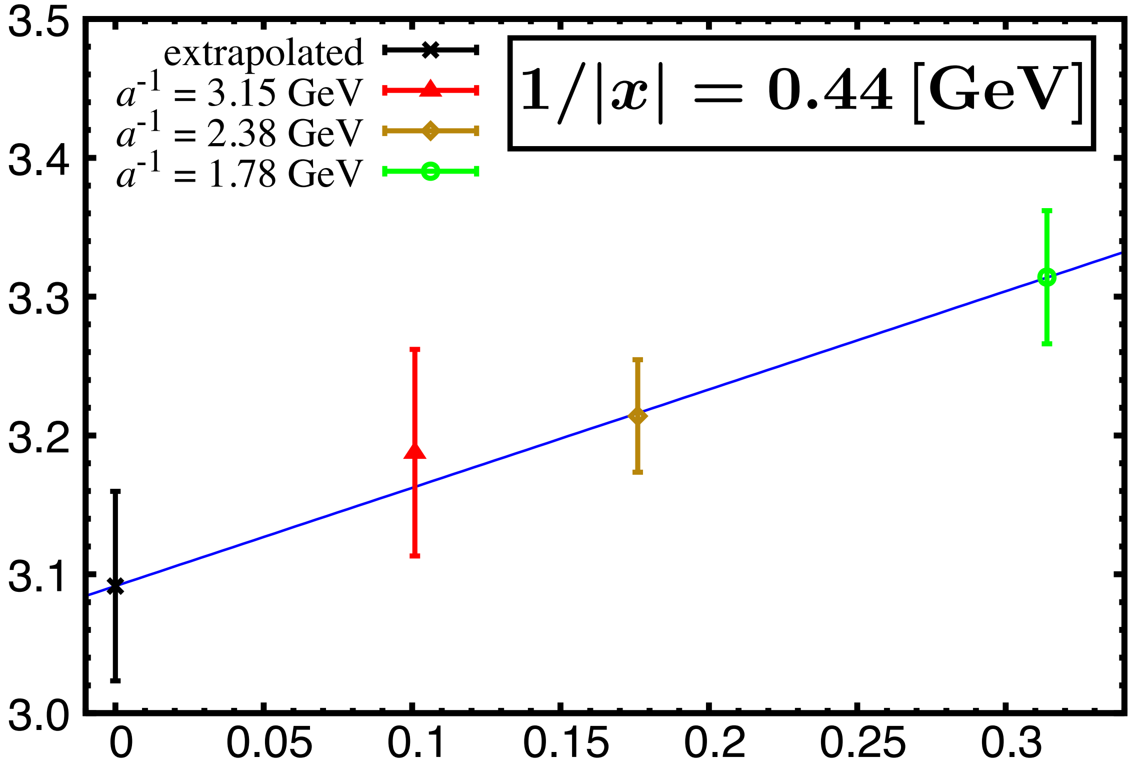
- Sphere average of

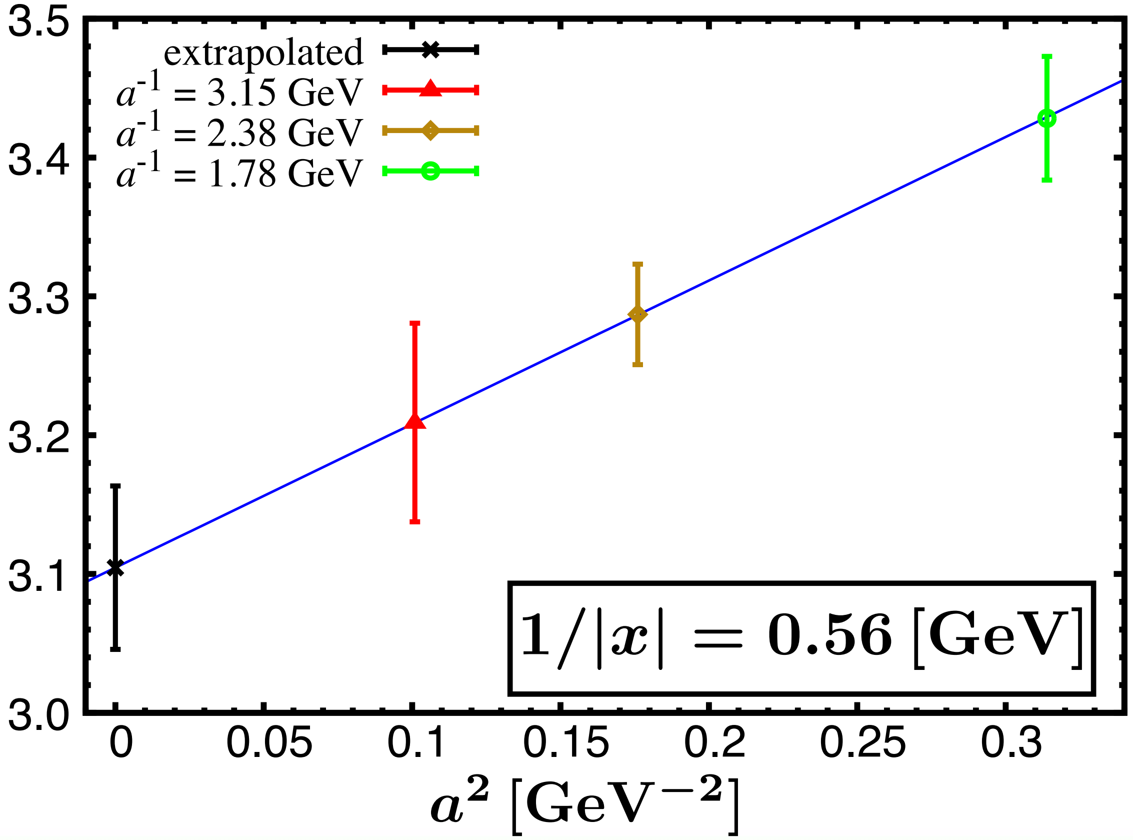
$$\widetilde{m}_q^{\overline{MS}}(\mu; a; x) = \widetilde{Z}_{S+P}^{\overline{MS}/lat}(\mu; a; x)^{-1} m_q^{bare,phys}(a)$$

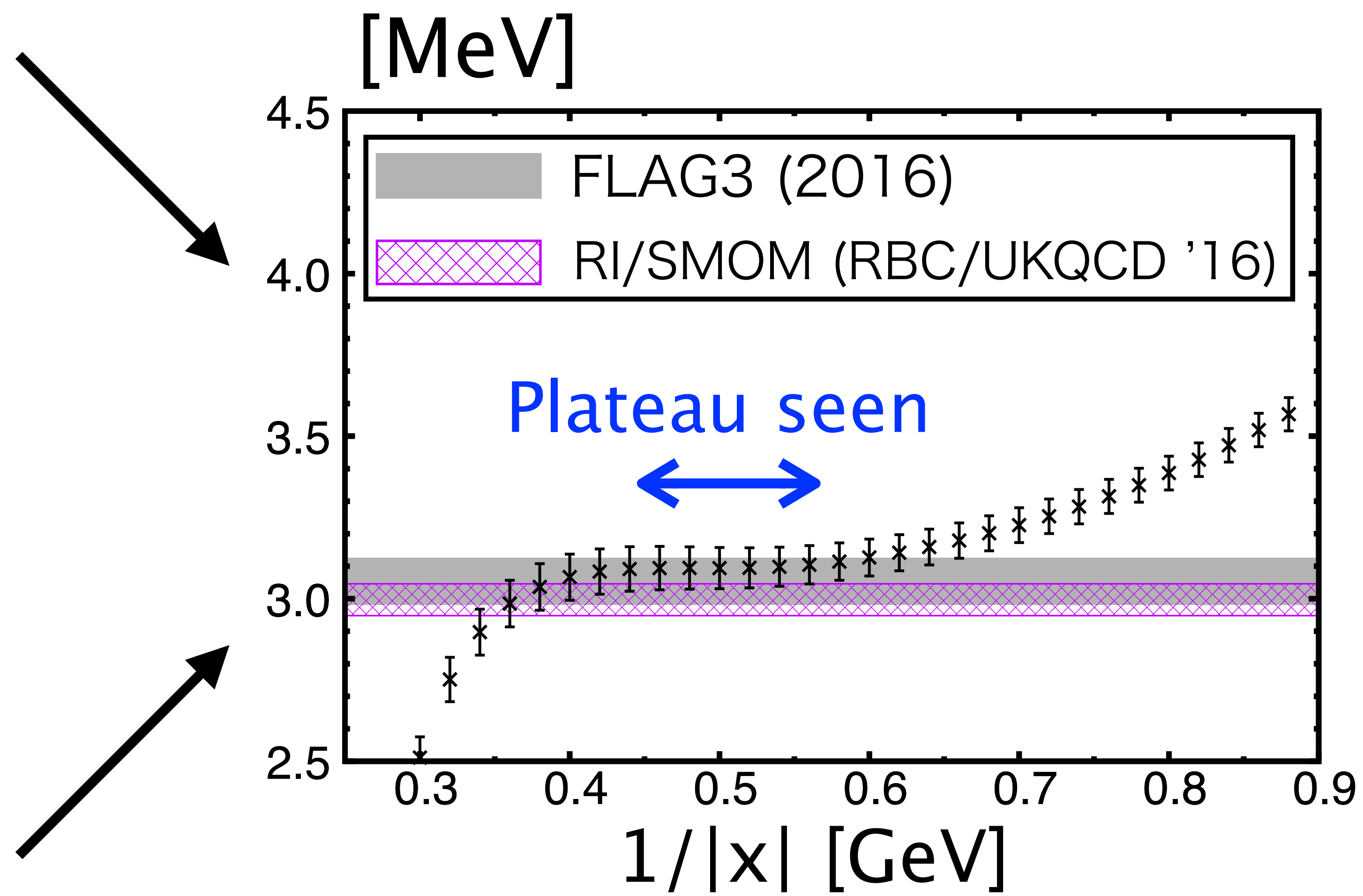
- Able to calculate at any distance
- Plateau seen better for finer lattices



Continuum limit







$$m_{ud}^{\overline{MS}}(3\ \mathrm{GeV})|_{2+1f} = 3.09\ (6)_{stat}(6)_{sys}$$

$$m_{s}^{\overline{MS}}(3\ \mathrm{GeV})|_{2+1f} = 85.3\ (1.6)_{stat}(1.7)_{sys}$$

Extension to mixing operators

- 2pt functions

$$G_{ij}(x) = \langle O_i(x) O_j(0)^\dagger \rangle$$

- Naïve extension of renormalization condition

$$\widetilde{Z}^{R/lat}_{ik}(\mu; a; x) G^{lat}_{ik}(a; x) \left(\widetilde{Z}^{R/lat}(\mu; a; x) \right)^T_{kj} = G^R_{ij}(\mu; x)$$

How to solve this?

Ambiguity of $\widetilde{Z}$

$$\widetilde{Z}^{R/lat}_{ik}(\mu; a; x) G^{lat}_{ik}(a; x) \left(\widetilde{Z}^{R/lat}(\mu; a; x)\right)^{T}_{kj} = G^{R}_{ij}(\mu; x)$$

- G_{ij}: positive definite & real symmetric

  $^{\exists}A_{ij}$ s.t. $G_{ij} = A_{ik}(A^{T})_{kj}$

  proven by diagonalizing G

- One solution of the condition:

$$\widetilde{Z}^{R/lat}_{ij}(\mu; a; x) = A^{R}_{ik}(\mu; x) \left(A^{lat}(a; x)\right)^{-1}_{kj}$$

- General solution:

$$\widetilde{Z}^{R/lat}_{ij}(\mu; a; x) = A^{R}_{ik}(\mu; x) P_{kl} \left(A^{lat}(a; x)\right)^{-1}_{lj}, \quad P \in O(N)$$

Ambiguity of $\tilde{Z}$

- Interpretation of ambiguity

  G_{ij}: symmetric

  Effective number of conditions: $N(N+1)/2$

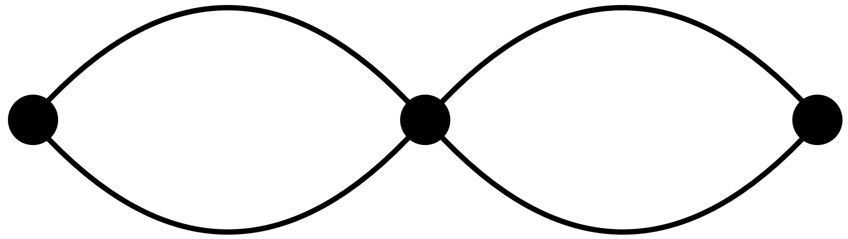
  Z_{ij}: N^2 independent values

  $N(N-1)/2$ conditions missing

- Alternative tool (for 4-quark operators)

  $\langle J_\alpha(x) O_i(0) J_\beta(y) \rangle$

  w/ quark bilinears J_α



Ishikawa, Lat19

Ambiguity of $\tilde{Z}$

- Interpretation of ambiguity

  G_{ij}: symmetric

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  Dimension of O(N)

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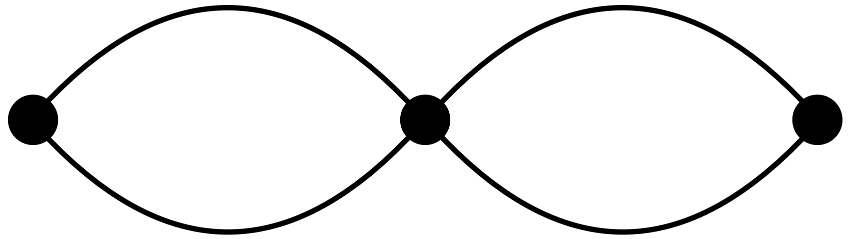

Ishikawa, Lat19

Section Summary

- Idea

$$Z^{R/\mathrm{lat}}(\mu;a)^2 G^{\mathrm{lat}}(a;x) = G^{R}(\mu;x)$$

Gauge invariant!

- Practically

$$\widetilde{Z}^{R/\mathrm{lat}}(\mu;a;x) = \sqrt{\frac{G^{R}(\mu;x)}{G^{\mathrm{lat}}(a;x)}}$$

$$= Z^{R/\mathrm{lat}}(\mu;a) + \underline{\text{irrelevant x–dependence}}$$

1. Discretization error (SDs)
2. Truncation of perturbative series (LDs)
3. Other non-perturbative effects (LDs)

- Applied to m_q using Z_S & Z_P
- Extension to the case of mixing operators?

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  - Example: quark mass renormalization from Z_S
  - Renormalization of mixing operators?
- NP matching of 3/4-flavor Wilson coefficients
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  - Theoretical foundation
  - Result of an exploratory calculation

Neutral Kaon System

- Kaons
  - Charged: $|K^+\rangle = |\bar{s}u\rangle,\ |K^-\rangle = |\bar{u}s\rangle$
  - Neutral: $|K^0\rangle = |\bar{s}d\rangle,\ |\overline{K}^0\rangle = |\bar{d}s\rangle$ * strong eigenstates

  Not CP eigenstates: $CP|K^0\rangle = |\overline{K}^0\rangle,\ CP|\overline{K}^0\rangle = |K^0\rangle$

- CP eigenstates
  - $|K_1\rangle = \frac{1}{\sqrt{2}}\left(|K^0\rangle + |\overline{K}^0\rangle\right),\quad CP|K_1\rangle = +|K_1\rangle$: CP even
  - $|K_2\rangle = \frac{1}{\sqrt{2}}\left(|K^0\rangle - |\overline{K}^0\rangle\right),\quad CP|K_2\rangle = -|K_2\rangle$: CP odd

- CP nature of decay modes
  - |ππ>: CP even
  - |πππ>: CP odd

K_S & K_L in CP limit

- Large difference in lifetimes:
  - $m_K - 2m_\pi \approx 220$ MeV $\gg m_K - 3m_\pi \approx 80$ MeV
  - $K_S \equiv K_1$, $K_L \equiv K_2$ (in CP limit)
  - $\tau_S \approx 9 \times 10^{-11}$s, $\tau_L \approx 5 \times 10^{-8}$s

- Possible processes:
  - $K_S \rightarrow \pi\pi$ (CP even to CP even)
  - $K_L \rightarrow \pi\pi\pi$ (CP odd to CP odd)

CP-violation in K → ππ

- $K_L \to \pi\pi$ discovered (1964) 
- Scenarios of CP-violation

CP odd | CP even

$$|K_L\rangle = |K_2\rangle + \varepsilon|K_1\rangle$$

direct CPV: ε' ($|K_2\rangle \to |\pi\pi\rangle$)

indirect CPV: ε ($|K_1\rangle \to |\pi\pi\rangle$)

$$|\pi\pi\rangle$$

CP even

$$\eta_{00} \equiv \frac{A(K_L \to \pi^0\pi^0)}{A(K_S \to \pi^0\pi^0)} \equiv \epsilon - 2\epsilon' \qquad \eta_{+-} \equiv \frac{A(K_L \to \pi^+\pi^-)}{A(K_S \to \pi^+\pi^-)} \equiv \epsilon + \epsilon'$$

direct CPV

- discovered in 1993
- NA48 (CERN), KTeV (FNAL): $\mathrm{Re}\left(\frac{\epsilon'}{\epsilon}\right) \approx \frac{1}{6}\left(1 - \left|\frac{\eta_{00}}{\eta_{+-}}\right|^2\right) = 16.6(2.3)\times 10^{-4}$

SM prediction

- SM vs Exp.

$$\mathrm{Re}\left(\frac{\epsilon'}{\epsilon}\right) = \mathrm{Re}\left\{\frac{i\omega e^{i(\delta_2-\delta_0)}}{\sqrt{2}\epsilon}\left[\frac{\mathrm{Im}\,A_2}{\mathrm{Re}\,A_2} - \frac{\mathrm{Im}\,A_0}{\mathrm{Re}\,A_0}\right]\right\} = 1.38(5.15)(4.59)\times 10^{-4}$$

RBC & UKQCD, 2015

↓ Kitahara et al, 2016 Improvement of RG

$1.06(5.07)\times 10^{-4}$

Exp: $16.6(2.3) \times 10^{-4}$ ⟷ 2.8σ

- 27% systematic uncertainties from various sources in A_0

$$A_I = \langle(\pi\pi)_I|H_W|K\rangle$$

$$= \frac{G_F}{2}V_{us}^* V_{ud}\sum_i \underbrace{[z_i(\mu)+\tau y_i(\mu)]}_{\text{pQCD}}\underbrace{Z_{ij}(\mu, a^{-1})}_{\text{LQCD (+pQCD)}}\underbrace{\langle(\pi\pi)_I|Q_i^{lat}(a^{-1})|K\rangle}_{\text{LQCD}}$$

Other sources
- ✦ Discretization (12%)
- ✦ Finite Volume (7%)

SM prediction

Lellouch–Lüscher factor (11%)

- SM vs Exp.

$$\mathrm{Re}\left(\frac{\epsilon'}{\epsilon}\right) = \mathrm{Re}\left\{\frac{i\omega e^{i(\delta_2-\delta_0)}}{\sqrt{2}\epsilon}\left[\frac{\mathrm{Im}\,A_2}{\mathrm{Re}\,A_2} - \frac{\mathrm{Im}\,A_0}{\mathrm{Re}\,A_0}\right]\right\} = 1.38(5.15)(4.59)\times 10^{-4}$$

RBC & UKQCD, 2015

↓ Kitahara et al, 2016
Improvement of RG

$1.06(5.07)\times 10^{-4}$

Exp: $16.6(2.3)$ x 10^{-4} ⟷ 2.8σ

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$$A_I = \langle(\pi\pi)_I|H_W|K\rangle$$

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Operator renormalization (15%)
Window problem: $\Lambda_{QCD} \ll \mu \ll a^{-1}$
Solved by "step scaling"

3f/4f matching (12%)
⭐ Target of this talk

Other sources
✦ Discretization (12%)
✦ Finite Volume (7%)

N_f in Weak Hamiltonian

$$H_W = \Sigma_i \, w_i^{4f}(\mu) \, O_i^{4f}(\mu)$$

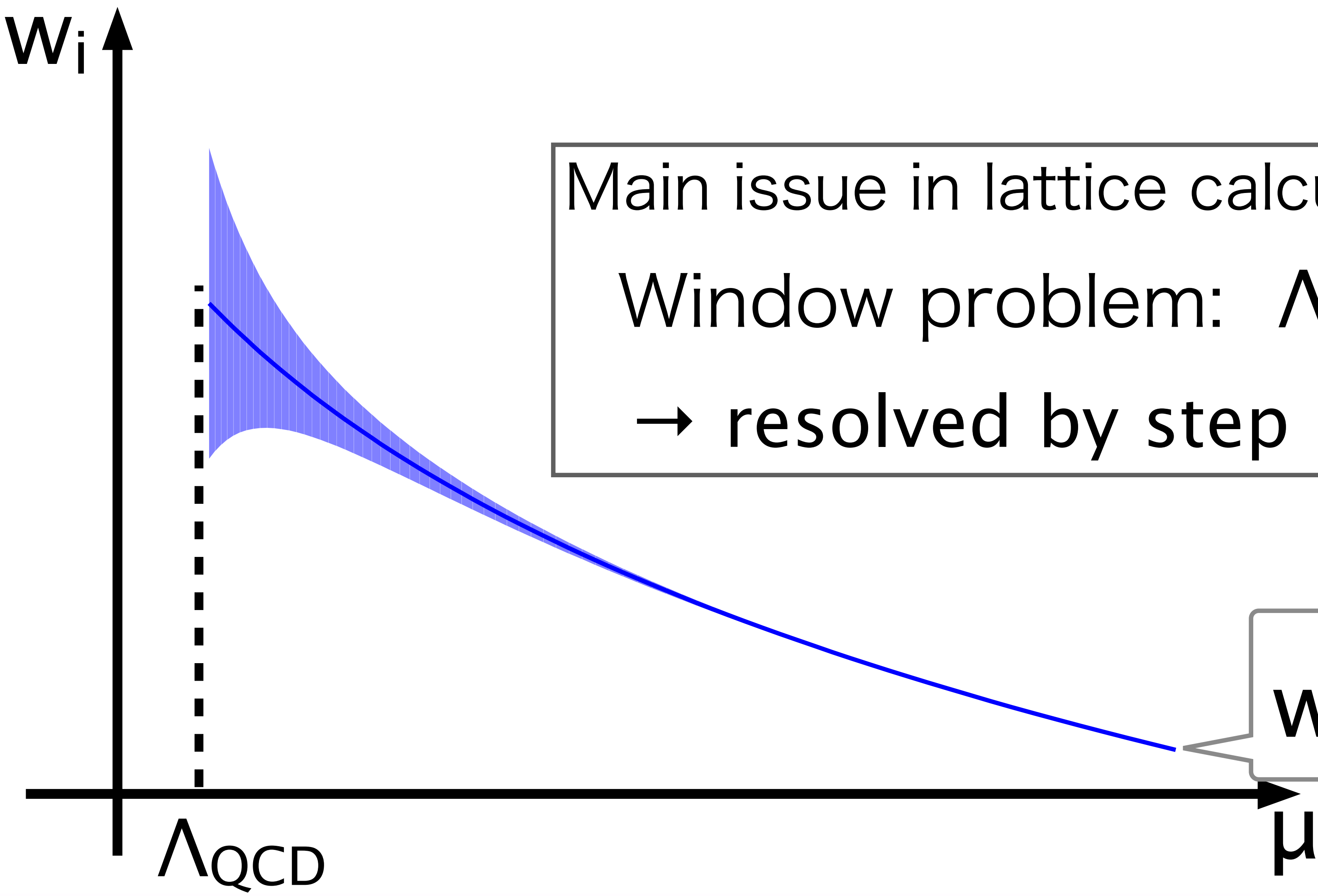
$$= \Sigma_i \, w_i^{3f}(\mu) \, O_i^{3f}(\mu)$$

$$= \ldots$$

We can use either 3f or 4f for WMEs

WMEs w/ 4-flavor operators

$$<f|H_W|i> = \Sigma_i \underbrace{w_i^{4f}(\mu)}_{\text{pQCD}} \underbrace{<f|O_i^{4f}(\mu)|i>}_{\text{LQCD}}$$



Main issue in lattice calculation is

Window problem: $\Lambda_{QCD} \ll \mu \ll a^{-1}$

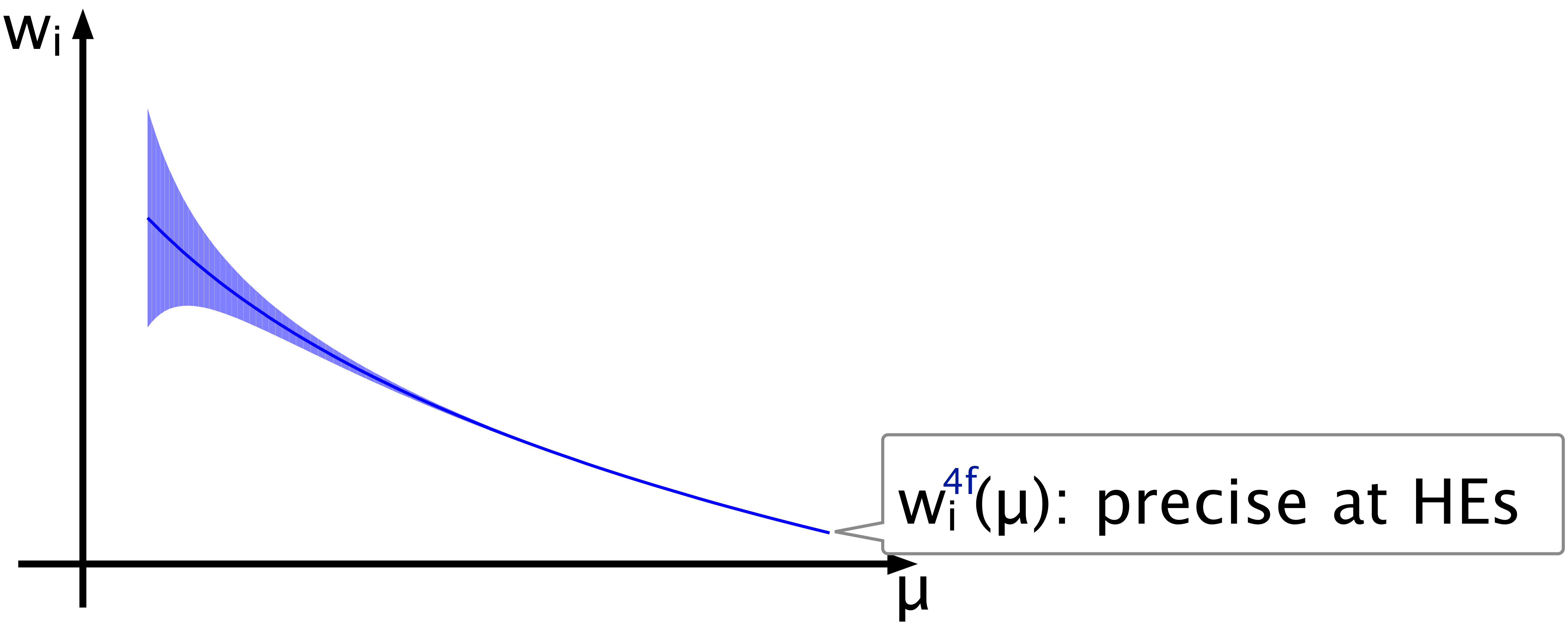
→ resolved by step scaling, etc…

$w_i^{4f}(\mu)$: precise at HEs

WMEs w/ 3-flavor operators

$$\langle f|H_W|i\rangle = \Sigma_i\, w_i^{3f}(\mu)\, \langle f|O_i^{3f}(\mu)|i\rangle$$

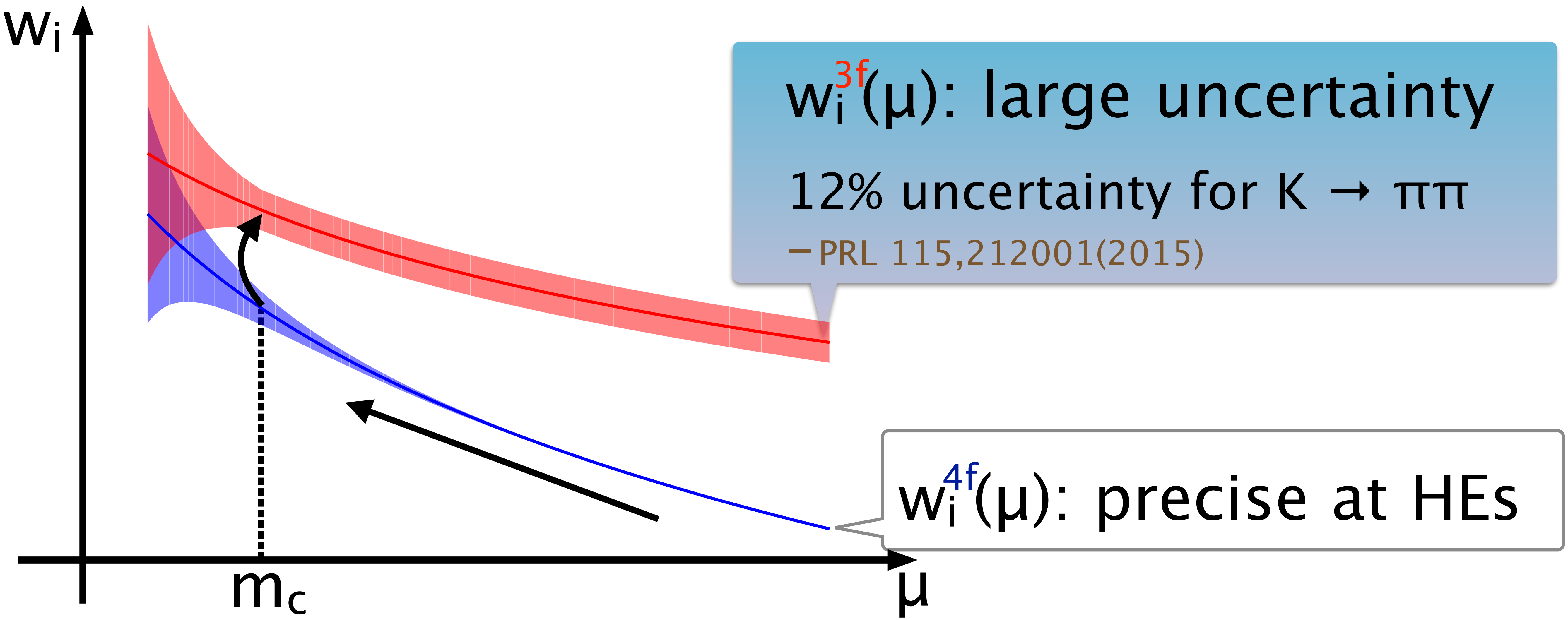
pQCD LQCD



WMEs w/ 3-flavor operators

$$\langle f|H_W|i\rangle = \Sigma_i\, w_i^{3f}(\mu)\, \langle f|O_i^{3f}(\mu)|i\rangle$$

pQCD LQCD



WMEs w/ 3-flavor operators

$$<f|H_W|i> = \Sigma_i\, w_i^{3f}(\mu)\, <f|O_i^{3f}(\mu)|i>$$

~~pQCD~~ LQCD

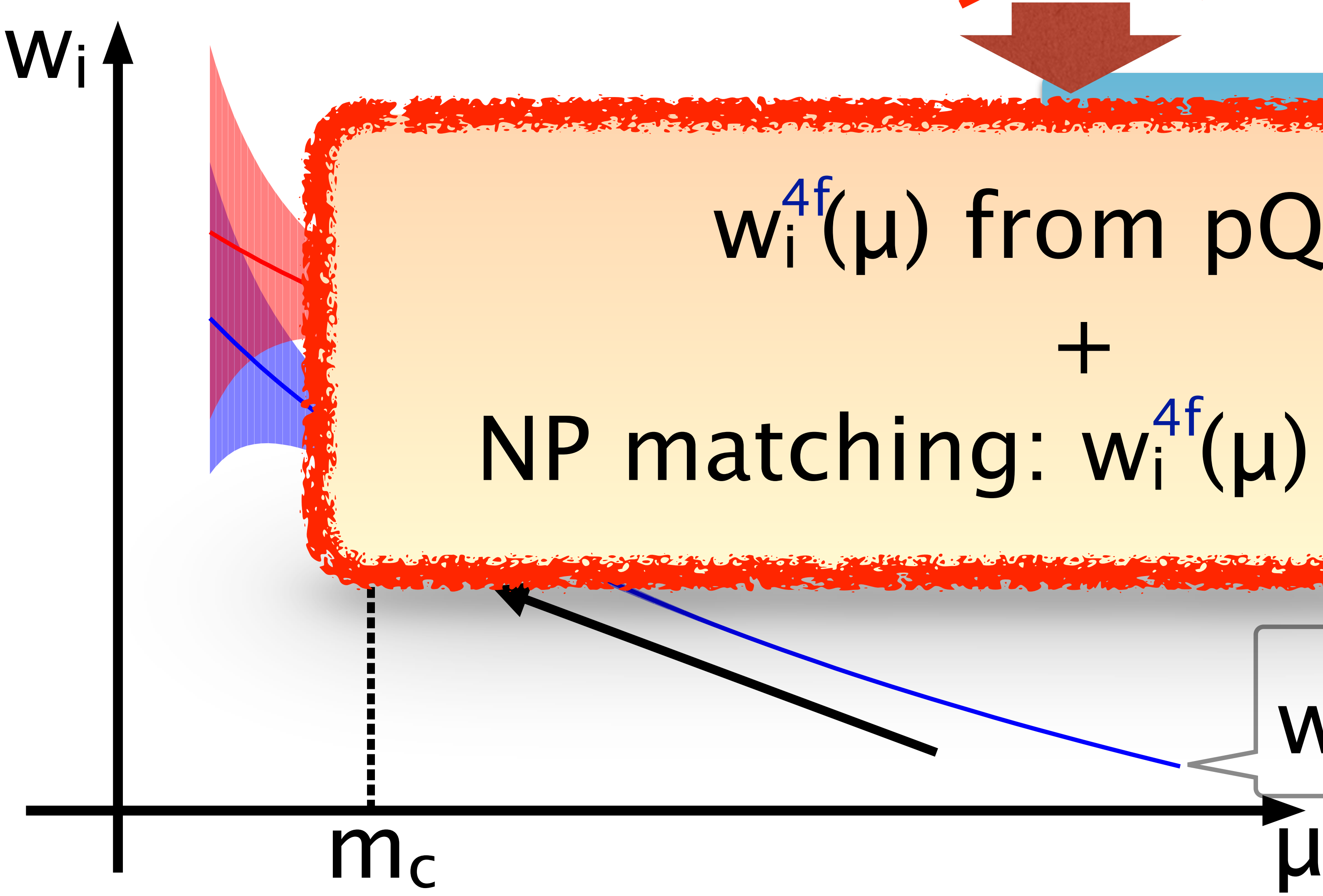
rtainty

→ ππ

$w_i^{4f}(\mu)$ from pQCD

\+

NP matching: $w_i^{4f}(\mu) \rightarrow w_i^{3f}(\mu)$

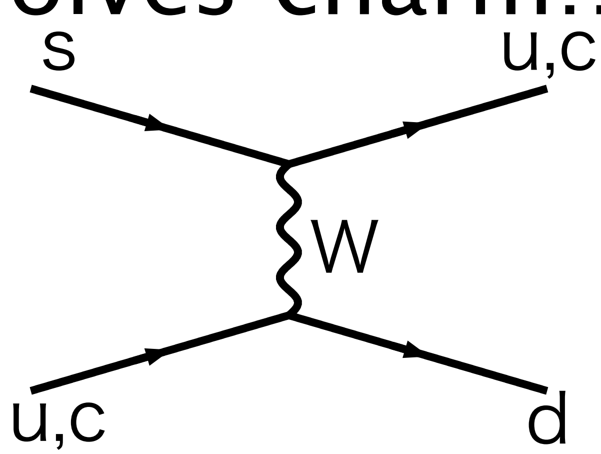


$w_i^{4f}(\mu)$: precise at HEs

$w_i^{3f}(\mu) \neq w_i^{4f}(\mu)$?

- Of coarse sea charm effects ⇒ $w_i^{3f}(\mu) \neq w_i^{4f}(\mu)$
  - Maybe small difference → neglect in this project
- If O_i^{4f} involves charm…

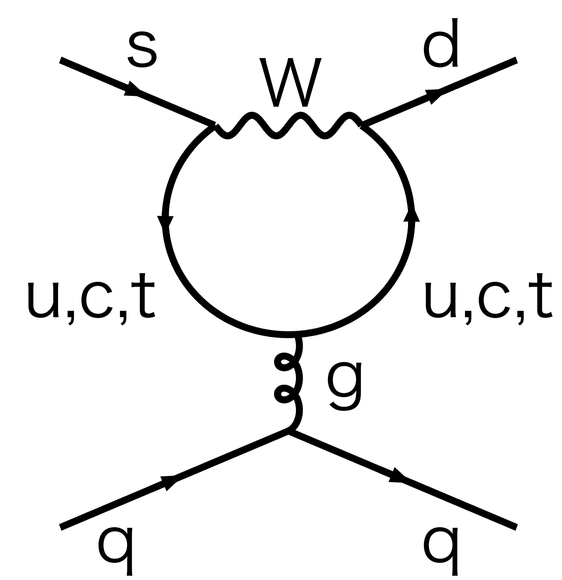
  Ex)



current–current

$O_i^u = (\bar{s}d)_{V-A}(\bar{u}u)_{V-A}$

$O_i^c = (\bar{s}d)_{V-A}(\bar{c}c)_{V-A}$



QCD penguin

$O_i = (\bar{s}d)_{V-A}\Sigma_q(\bar{q}q)_{V\pm A}$

  - Corresponding w_i's in 3f & 4f different
  - w_i^{3f} necessary if MEs calculated with O_i^{3f}

$K \to \pi\pi$ by RBC/UKQCD (2015)

- 2+1 DWF
- a^{-1} = 1.38 GeV

  ⇒ too coarse to introduce charm

  ⇒ 3-flavor operators for MEs
  & perturbative 4/3-flavor matching

  ⇒ 12% systematic uncertainty in A_0

- NP matching (obtained from finer lattices) is desired

NP 4f-3f matching in position Sp.

$$H_W = \Sigma_i\, w_i^{4f}(\mu)\, O_i^{4f}(\mu) = \Sigma_i\, w_i^{3f}(\mu)\, O_i^{3f}(\mu)$$

- This means:

  $$\Sigma_i \langle \bar{O}(x)\, O_i^{4f}(\mu;y)^\dagger \rangle\, w_i^{4f}(\mu) = \Sigma_i \langle \bar{O}(x)\, O_i^{3f}(\mu;y)^\dagger \rangle\, w_i^{3f}(\mu)$$

  for any operator $\bar{O}(x)$

  at LDs: $1/|x-y| \ll m_c$

- Relation b/w w_i^{4f} & w_i^{3f} can be obtained by choosing appropriate number of $\bar{O}(x)$'s

  ⇒ We choose

  $$\bar{O}(x) = O_i^{3f}(\mu;x)$$

NP 4f-3f matching in position Sp.

$$H_W = \Sigma_i\, w_i^{4f}(\mu)\, O_i^{4f}(\mu) = \Sigma_i\, w_i^{3f}(\mu)\, O_i^{3f}(\mu)$$

$$\langle O_j^{3f}(\mu;x)\, H_W(y)^\dagger \rangle$$

$$\Sigma_j\, G_{ij}^{3f-4f}(\mu;x-y)\, w_j^{4f}(\mu) = \Sigma_j\, G_{ij}^{3f-3f}(\mu;x-y)\, w_j^{3f}(\mu)$$

$$G_{ij}^{nf-n'f}(\mu;x-y) = \langle O_i^{nf}(\mu;x)\, O_j^{n'f}(\mu;y)^\dagger \rangle$$

$$w_i^{3f}(\mu) = \Sigma_{jk}\, (G^{3f-3f}(\mu;x-y))^{-1}_{ij}\, G_{jk}^{3f-4f}(\mu;x-y)\, w_k^{4f}(\mu)$$

☆ Gauge invariant & free from contact terms

⇒ can prevent mixing with irrelevant operators

Matching Mtx & 3f operators

- $M_{ik} = \Sigma_j \, (G^{3f-3f}(\mu;x-y))^{-1}_{ij} \, G^{3f-4f}_{jk}(\mu;x-y)$
- Inverse matrix $(G^{3f-3f}(\mu;x-y))^{-1}_{ij}$ exists
  ONLY IF O_i^{3f}'s are independent with each other
  - Ex: $\Delta S = 1$ weak operators not the case!

Type	Q_i
current-current	$Q_1 = (\bar{s}_\alpha d_\alpha)_L (\bar{u}_\beta u_\beta)_L$ $Q_2 = (\bar{s}_\alpha d_\beta)_L (\bar{u}_\beta u_\alpha)_L$
QCD penguin	$Q_3 = (\bar{s}_\alpha d_\alpha)_L \sum_q^{3f} (\bar{q}_\beta q_\beta)_L$ $Q_4 = (\bar{s}_\alpha d_\beta)_L \sum_q^{3f} (\bar{q}_\beta q_\alpha)_L$ $Q_5 = (\bar{s}_\alpha d_\alpha)_L \sum_q^{3f} (\bar{q}_\beta q_\beta)_R$ $Q_6 = (\bar{s}_\alpha d_\beta)_L \sum_q^{3f} (\bar{q}_\beta q_\alpha)_R$
EW penguin	$Q_7 = \frac{3}{2} (\bar{s}_\alpha d_\alpha)_L \sum_q^{3f} e_q (\bar{q}_\beta q_\beta)_R$ $Q_8 = \frac{3}{2} (\bar{s}_\alpha d_\beta)_L \sum_q^{3f} e_q (\bar{q}_\beta q_\alpha)_R$ $Q_9 = \frac{3}{2} (\bar{s}_\alpha d_\alpha)_L \sum_q^{3f} e_q (\bar{q}_\beta q_\beta)_L$ $Q_{10} = \frac{3}{2} (\bar{s}_\alpha d_\beta)_L \sum_q^{3f} e_q (\bar{q}_\beta q_\alpha)_L$

Fierz symmetry
→ 3 relations among Q_i's
→ 7 independent operators

Matching matrix

- If we choose
  - $O_i^{3f} = (Q_1, Q_2, \ldots, Q_{n3})$
  - $O_i^{4f} = (Q_1, Q_2, \ldots, Q_{n3}, P_1, P_2, \ldots, P_{nc})$
  - P_i's contain charm / Q_i's don't

We now redefine Q_i's as independent operators

- Then $M = (G^{3f-3f}(x))^{-1}(G^{3f-3f}(x) \,\vdots\, \langle Q(x)\, P(0)^\dagger \rangle)$

$$= \left(1_{n3\times n3} \;\vdots\; \underbrace{\begin{matrix} \bullet & \bullet & \cdots & \bullet \\ \bullet & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \bullet & \cdots & \cdots & \bullet \end{matrix}}_{nc} \right)$$

Represents how P_i's turn to Q_i's below charm threshold

ΔS = 1 4-quark operators

3-flavor

Type	Q_i
current-current	$Q_1 = (\bar{s}_\alpha d_\alpha)_L(\bar{u}_\beta u_\beta)_L$ $Q_2 = (\bar{s}_\alpha d_\beta)_L(\bar{u}_\beta u_\alpha)_L$
QCD penguin	$Q_3 = (\bar{s}_\alpha d_\alpha)_L \sum_q^{3f} (\bar{q}_\beta q_\beta)_L$ $Q_4 = (\bar{s}_\alpha d_\beta)_L \sum_q^{3f} (\bar{q}_\beta q_\alpha)_L$ $Q_5 = (\bar{s}_\alpha d_\alpha)_L \sum_q^{3f} (\bar{q}_\beta q_\beta)_R$ $Q_6 = (\bar{s}_\alpha d_\beta)_L \sum_q^{3f} (\bar{q}_\beta q_\alpha)_R$
EW penguin	$Q_7 = \frac{3}{2}(\bar{s}_\alpha d_\alpha)_L \sum_q^{3f} e_q(\bar{q}_\beta q_\beta)_R$ $Q_8 = \frac{3}{2}(\bar{s}_\alpha d_\beta)_L \sum_q^{3f} e_q(\bar{q}_\beta q_\alpha)_R$ $Q_9 = \frac{3}{2}(\bar{s}_\alpha d_\alpha)_L \sum_q^{3f} e_q(\bar{q}_\beta q_\beta)_L$ $Q_{10} = \frac{3}{2}(\bar{s}_\alpha d_\beta)_L \sum_q^{3f} e_q(\bar{q}_\beta q_\alpha)_L$

7 independent operators

4-flavor

Type	P_i
current-current	$P_1 = (\bar{s}_\alpha d_\alpha)_L(\bar{u}_\beta u_\beta)_L$ $P_1^c = (\bar{s}_\alpha d_\alpha)_L(\bar{c}_\beta c_\beta)_L$ $P_2 = (\bar{s}_\alpha d_\beta)_L(\bar{u}_\beta u_\alpha)_L$ $P_2^c = (\bar{s}_\alpha d_\beta)_L(\bar{c}_\beta c_\alpha)_L$
QCD penguin	$P_3 = (\bar{s}_\alpha d_\alpha)_L \sum_q^{4f} (\bar{q}_\beta q_\beta)_L$ $P_4 = (\bar{s}_\alpha d_\beta)_L \sum_q^{4f} (\bar{q}_\beta q_\alpha)_L$ $P_5 = (\bar{s}_\alpha d_\alpha)_L \sum_q^{4f} (\bar{q}_\beta q_\beta)_R$ $P_6 = (\bar{s}_\alpha d_\beta)_L \sum_q^{4f} (\bar{q}_\beta q_\alpha)_R$
EW penguin	$P_7 = \frac{3}{2}(\bar{s}_\alpha d_\alpha)_L \sum_q^{4f} e_q(\bar{q}_\beta q_\beta)_R$ $P_8 = \frac{3}{2}(\bar{s}_\alpha d_\beta)_L \sum_q^{4f} e_q(\bar{q}_\beta q_\alpha)_R$ $P_9 = \frac{3}{2}(\bar{s}_\alpha d_\alpha)_L \sum_q^{4f} e_q(\bar{q}_\beta q_\beta)_L$ $P_{10} = \frac{3}{2}(\bar{s}_\alpha d_\beta)_L \sum_q^{4f} e_q(\bar{q}_\beta q_\alpha)_L$

9 independent operators

Color trivialization by Fierz trf.

- Def: $(\bar{s}d)_L(\bar{q}q)_{R/L} = \bar{s}\gamma_\mu(1-\gamma_5)d \cdot \bar{q}\gamma_\mu(1\pm\gamma_5)q$

- Left-Left operators

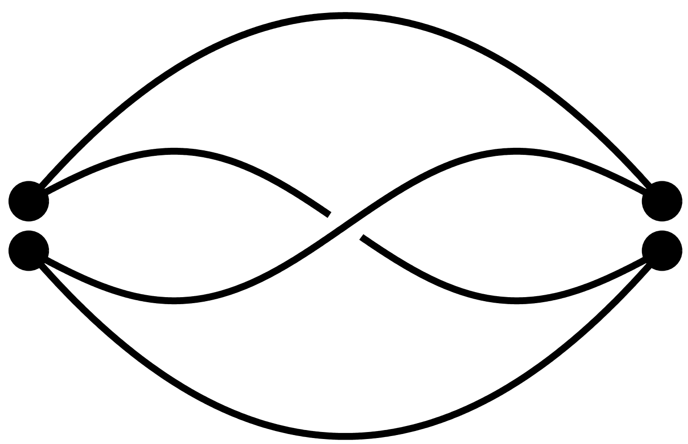
$$(\bar{s}_\alpha d_\beta)_L(\bar{q}_\beta q_\alpha)_L = (\bar{s}_\alpha q_\alpha)_L(\bar{q}_\beta d_\beta)_L$$

- Left-Right operators

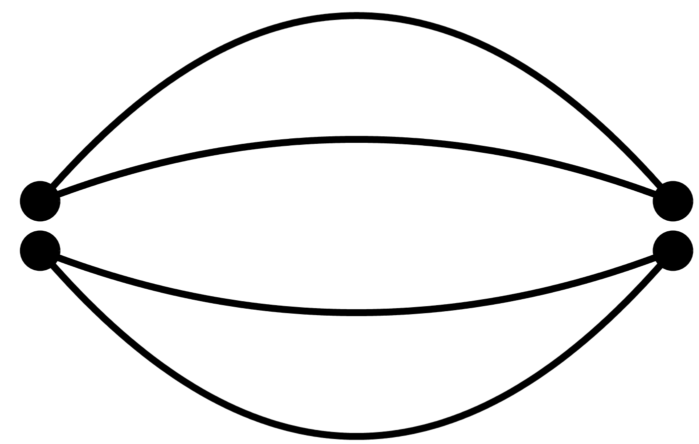
$$(\bar{s}_\alpha d_\beta)_L(\bar{q}_\beta q_\alpha)_R = -2\bar{s}_\alpha(1+\gamma_5)q_\alpha \cdot \bar{q}_\beta(1-\gamma_5)d_\beta$$

Contractions

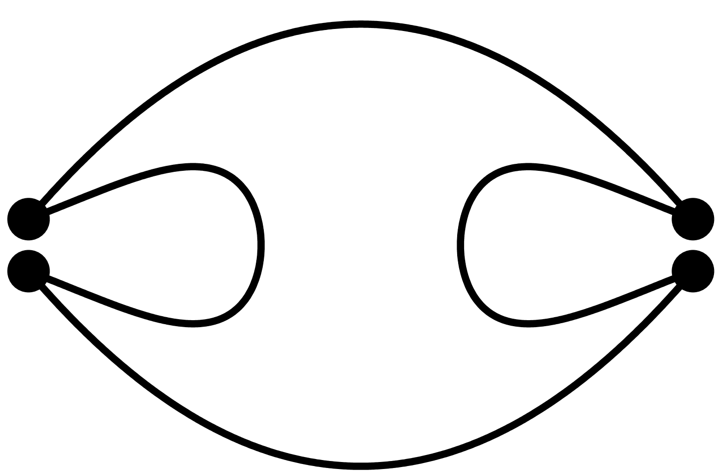
- 4/3-flavor matching should be independent of m_{ud} & m_s
  ⇒ Calculate w/ SU(3) valence quarks + 1 heavier quark



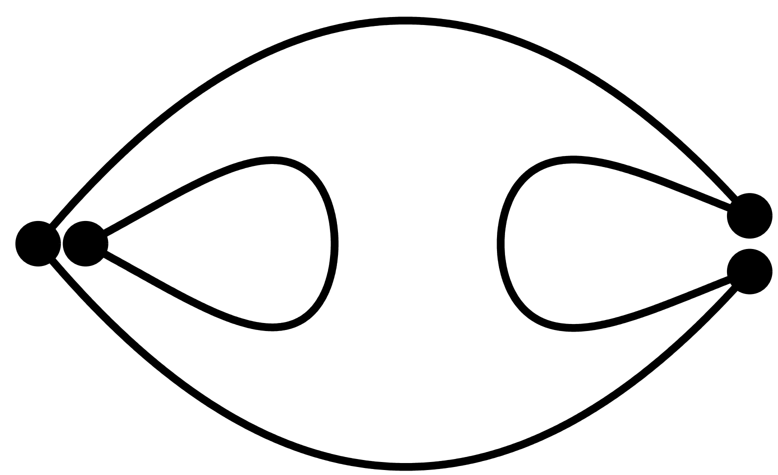
6 contractions



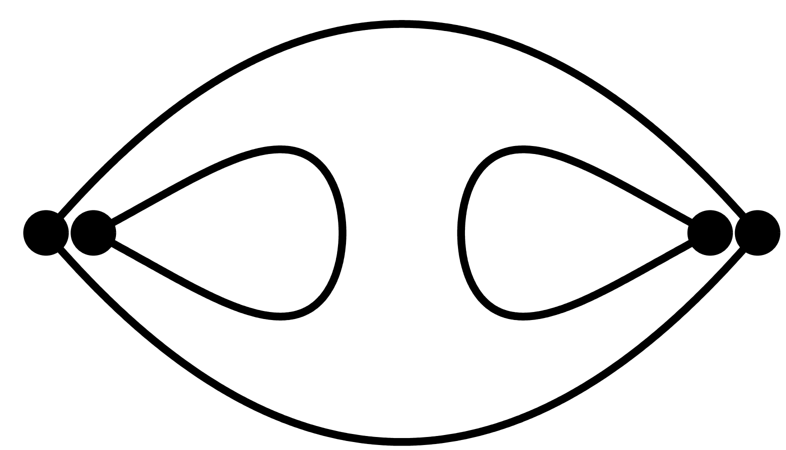
6 contractions



18 contractions



32 contractions



18 contractions

Subtraction of power divergence

- Loop diagram can contain power divergence
  - from power divergence of operators

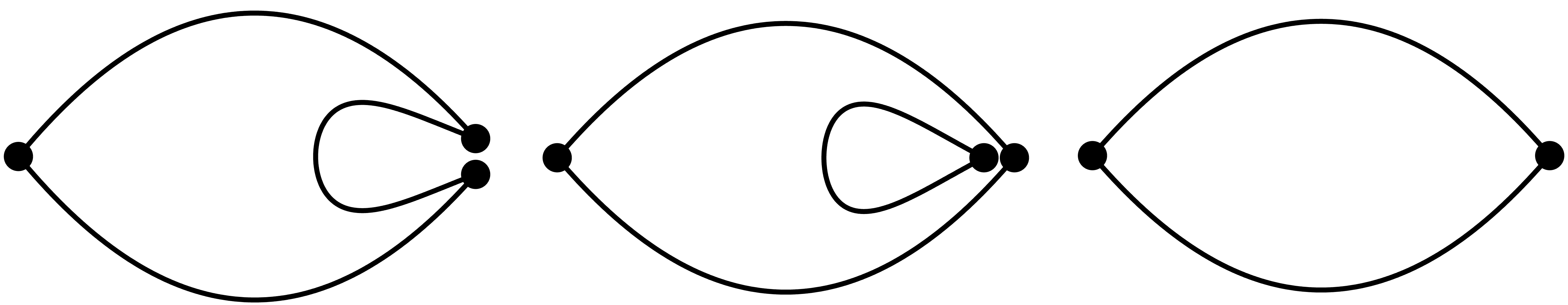
$$O_i \sim \frac{m_q}{a^2}$$

  - Eliminate by redefining

$$O_i' = O_i - C_- \bar{s}(1-\gamma_5)d - C_+ \bar{s}(1+\gamma_5)d$$

with a condition

$$\left\langle \bar{s}(1\pm\gamma_5)d(x)\cdot O_i'(y)^\dagger \right\rangle \Big|_{x-y=x_0} = 0$$



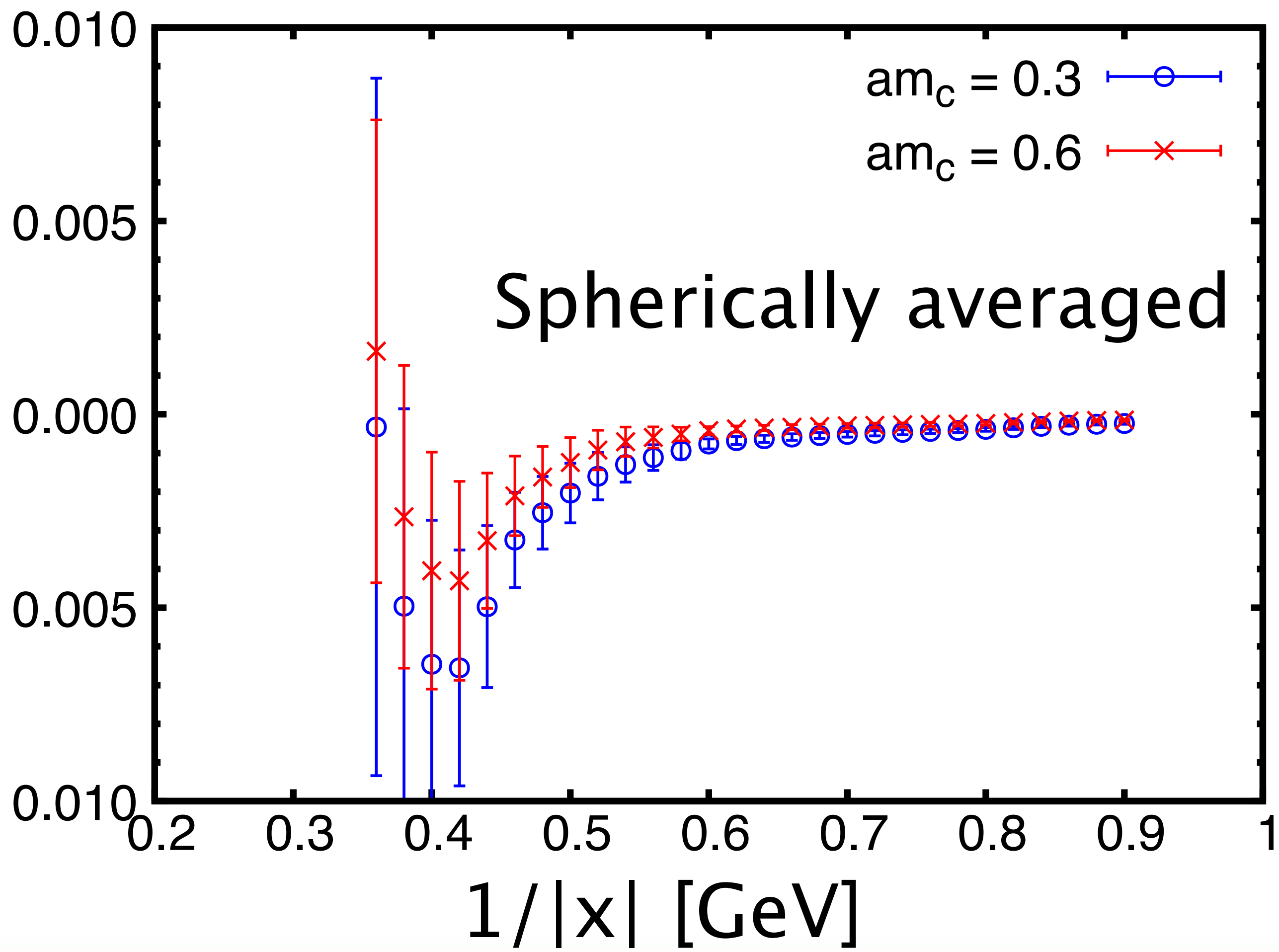
12 contractions 12 contractions 3 contractions

Result for M_{ij}

$$M_{ik} = \Sigma_j \, (G^{3f-3f}(x))^{-1}_{ij} G^{3f-4f}_{jk}(x)$$

$$= \left(1_{n3\times n3} \;\vdots\; \begin{matrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \bullet & \cdots & \cdots & \bullet \end{matrix} \right)$$

– Valid and should be independent of x at LDs $|x| \gg 1/m_c$



- 16^3 x 32
- a^{-1} = 1.78 GeV
- 88 confs in 3,500 MD time
- $m_{ud}^{val} = m_s^{val} = m_s^{sea}$
- Unrenormalized

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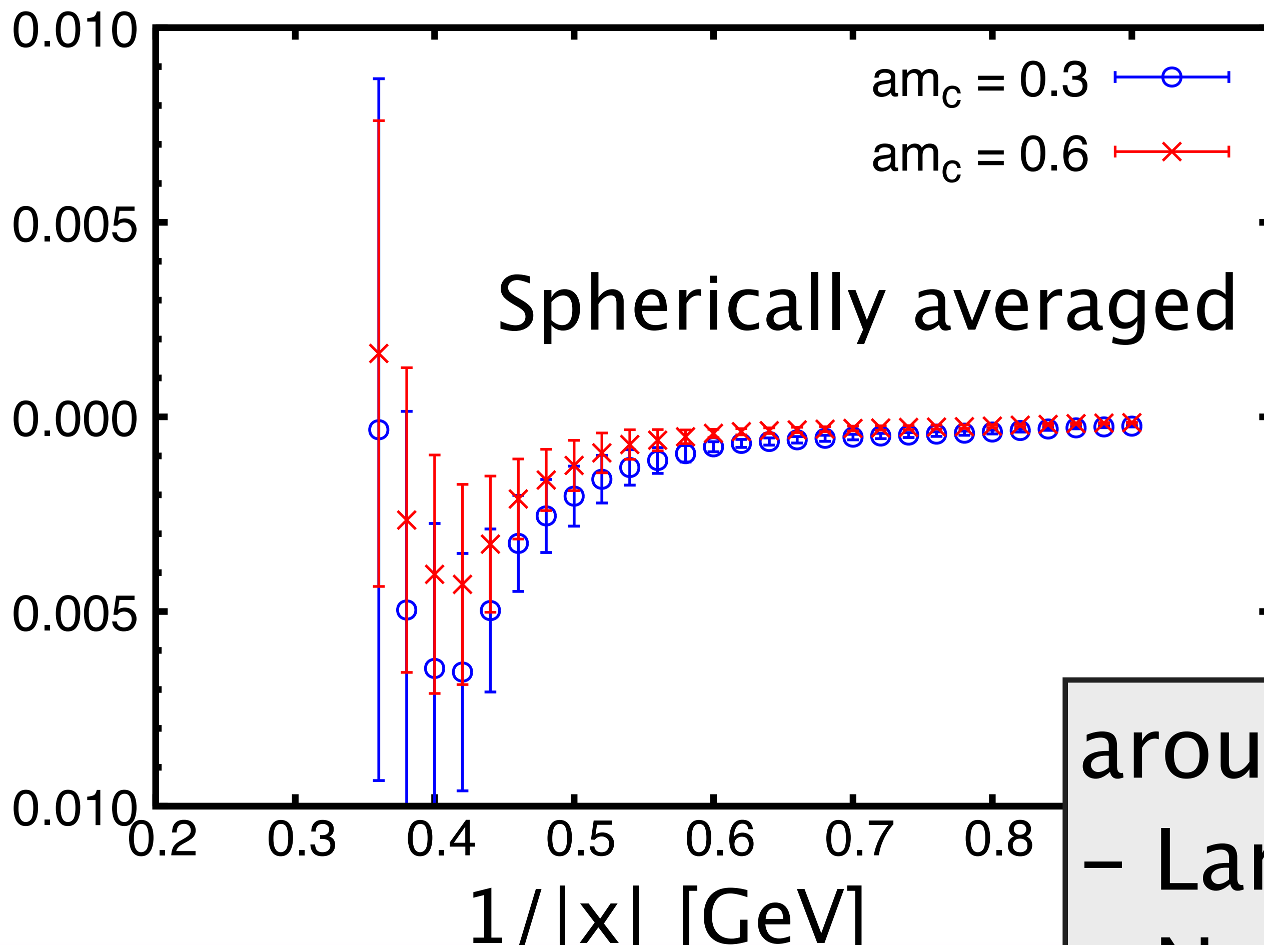
– Valid and should be independent of x at LDs $|x| \gg 1/m_c$



- $16^3 \times 32$
- $a^{-1} = 1.78$ GeV
- 88 confs in 3,500 MD time
- $m_{ud}^{val} = m_s^{val} = m_s^{sea}$
- Unrenormalized

around 1/|x| = 0.4 GeV…
– Large statistical error
– No clear plateau

Section Summary

- Purpose

$$w_i^{4f} \xrightarrow{\text{NP method}} w_i^{3f} \quad (\text{for } K \to \pi\pi)$$

- Idea

$$w_i^{4f} O_i^{4f} = w_i^{3f} O_i^{3f}$$

$$\longrightarrow w_i^{3f}(\mu) = \left(G^{3f\text{-}3f}(\mu; x)\right)_{ij}^{-1} G_{jk}^{3f\text{-}4f}(x) w_k^{4f}(\mu)$$

- Large statistical error in an exploratory calculation on 16^{-1} lattice at $a^{-1} = 1.78$ GeV
  - Trying to improve using Lanczos A2A, …
- Main calculation will be at $a^{-1} = 2.35$ GeV, 3.15 GeV, …

Previous effort in mom Sp.

- Condition

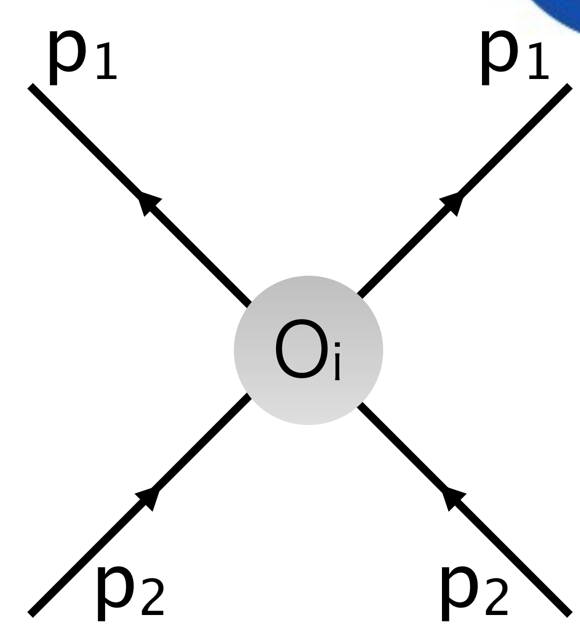
$$P^{abcd}_{\alpha\beta\gamma\delta}\ \Lambda^{abcd}_{\alpha\beta\gamma\delta}\,(O^{3f}_i(\mu);p_1,p_2)\ w^{3f}_i(\mu)$$

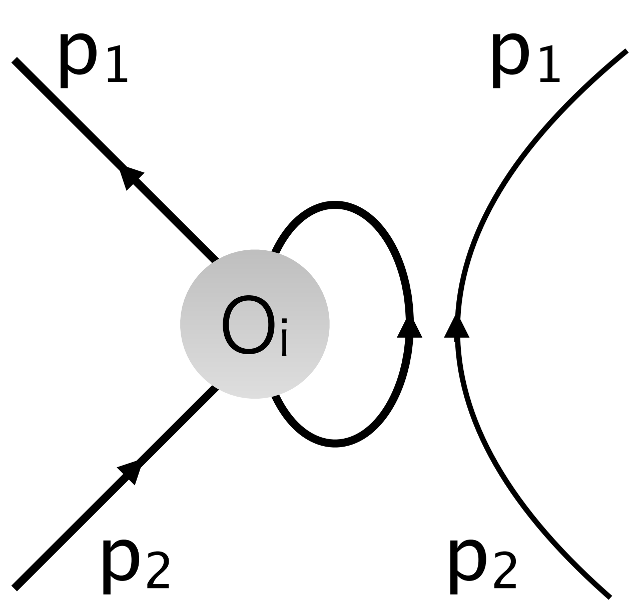
$$\|$$

$$P^{abcd}_{\alpha\beta\gamma\delta}\ \Lambda^{abcd}_{\alpha\beta\gamma\delta}\,(O^{4f}_i(\mu);p_1,p_2)\ w^{4f}_i(\mu)$$

G-fixed amputated Green's function

flavor, color and spin projector





- Condition valid in $|p_{1,2}| \ll m_c$
- Statistical error
  - $|p_{1,2}| = 1.2$ GeV → 10%
  - $|p_{1,2}| = 0.6$ GeV → 50%
-

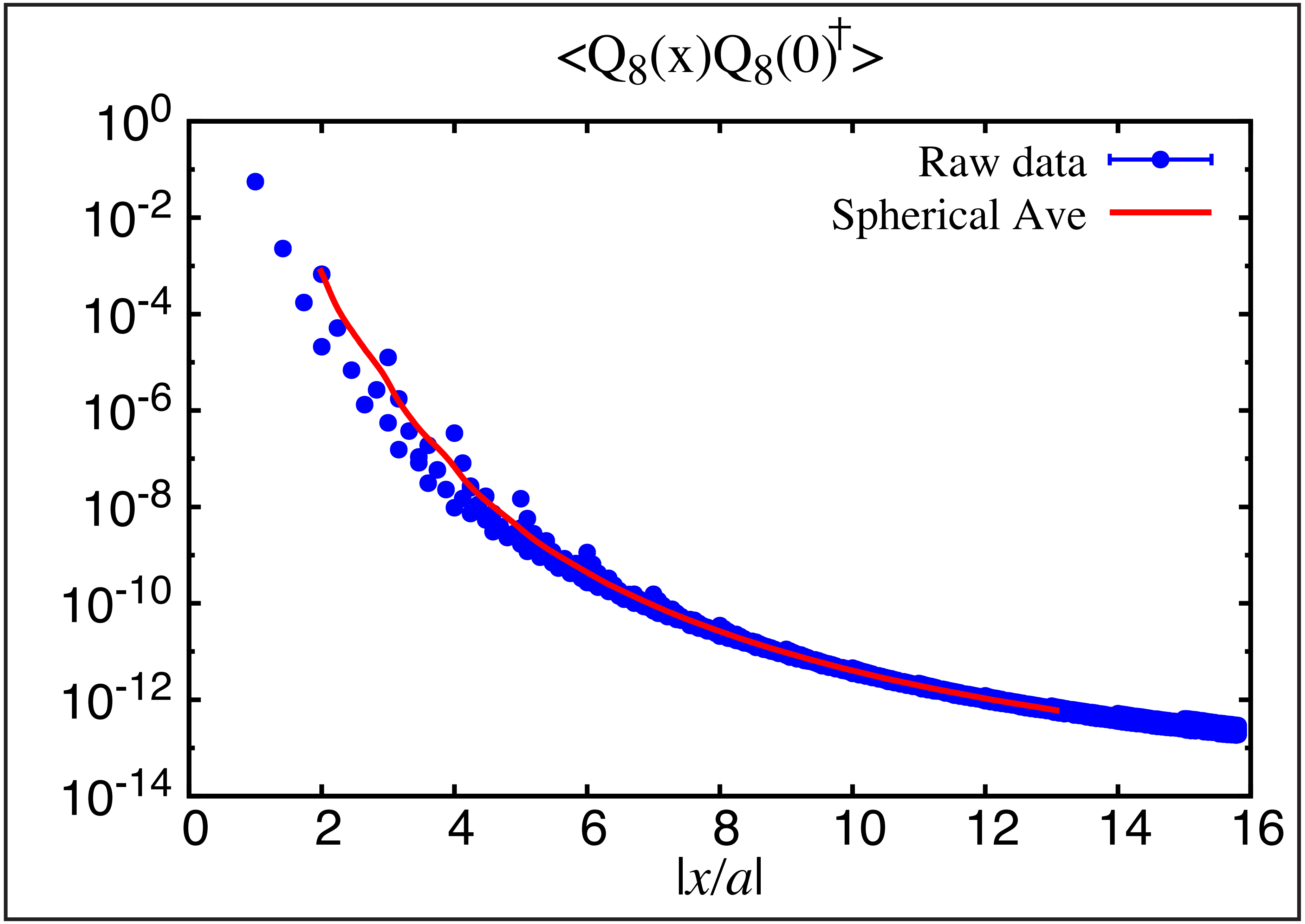
Why mom procedure so bad?

- Gauge fixing
  - Large Gribov noise
    - Gauge condition does not have a unique solution on the gauge orbit
    - Gauge-dependent quantities have some ambiguity
  - Mixing with gauge-noninvariant operators
- Off-shell condition
  - Mixing with operators that vanish by EoM

☆ All significant at small $p_{1,2}$

☆ Position-space procedure is free from all of these

Spherical average of a correlator



Euclidean correlators

$$\langle 0|O(x)O(y)^{\dagger}|0\rangle$$

- Good tool to extract information on O in QCD vacuum
  - $\sum_{\vec{x}}$: projection to $\vec{p} = 0$ state $\Rightarrow$ mass spectrum $\sim e^{-Mt}$
  - $E(\vec{p})$
- 3pt, 4pt functions also useful
  - scattering amplitudes
  - weak matrix elements
  - form factors
  - ...

Path integral

$$\langle\mathcal{O}\rangle = \frac{1}{Z}\int \mathcal{D}\overline{\psi}\mathcal{D}\psi\mathcal{D}U\,\mathcal{O}\ \mathrm{e}^{-S} = \frac{1}{Z}\int \mathcal{D}U\,\det D\ \mathrm{e}^{-S_G}\,\overline{\mathcal{O}}$$

$$Z = \int \mathcal{D}\overline{\psi}\mathcal{D}\psi\mathcal{D}U\ \mathrm{e}^{-S} = \int \mathcal{D}U\,\det D\ \mathrm{e}^{-S_G}$$

- $\mathcal{O} = \mathcal{O}(\overline{\psi}, \psi, U)$: Operator of interest
- $\overline{\mathcal{O}} = \overline{\mathcal{O}}(D^{-1}(U), U)$: Wick-contracted form of $\mathcal{O}$

Ex: $\mathcal{O} = \overline{\psi}(n)\mathrm{i}\gamma_5\psi(n)\cdot\overline{\psi}(0)\mathrm{i}\gamma_5\psi(0)$

$$\overline{\mathcal{O}} = n_f \underline{\mathrm{Tr}[(D^{-1})_{n,0}\gamma_5(D^{-1})_{0,n}\gamma_5]} - n_f^2 \underline{\mathrm{Tr}[(D^{-1})_{n,n}\gamma_5]\cdot\mathrm{Tr}[(D^{-1})_{0,0}\gamma_5]}$$

2 diagrams: