Lattice QCD at *T>0*

Chemical potential on the lattice

Improved action for QCD at *T>0*

Center symmetry and deconfinement in SU(N) gauge theory

Polyakov loop correlators and the free energy of static quark anti-quark pair

Integral method: Equation of State in SU(3) gauge theory

Chiral crossover in QCD

QCD Equation of State

QCD at finite baryon density

The naive continuum prescription of introducing chemical potential by adding a term $\mu \int d^4 x_E \bar{\psi}\gamma_0\psi$ does not work !

$$S = a^3 \sum_x \left[ma\bar{\psi}_x\psi_x + \mu a\bar{\psi}_x\gamma_0\psi_x + \frac{1}{2}\sum_\mu (\bar{\psi}_x\gamma_\mu\psi_{x+\mu} - \bar{\psi}_{x-\mu}\gamma_\mu\psi_x) \right]$$

$$\Downarrow$$

$\epsilon(\mu) \sim \mu^2/a^2$ instead of $\epsilon(\mu) \sim \mu^4$

The correct prescription is

$$U_0(x) \to e^{\mu a} U_0(x), \quad U_0^\dagger(x) \to e^{-\mu a} U_0^\dagger(x)$$

Hasenfratz, Karsch, PLB 125 (1983) 308

$$S = \frac{1}{2}\sum_x \left(\bar{\psi}_x e^{\mu a} U_{0,x} \psi_{x+\hat{0}} - \bar{\psi}_x e^{-\mu a} U_{0,x}^\dagger \psi_{x-\hat{0}} + \sum_i \eta_{i,x} \left(\bar{\psi}_x U_{i,x} \psi_{x+\hat{i}} - \bar{\psi}_x U_{i,x}^\dagger \psi_{x-\hat{i}} \right) + 2am\bar{\psi}_x\psi_x \right)$$

$$\Downarrow$$

det M is complex => sign problem *det M exp(-S)* cannot be a probability

Improved gauge action

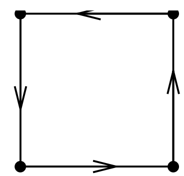
$$S^{(1,1)}_{\mu,\nu}(x) = 1 - \frac{1}{N}\mathrm{Re\ Tr}\,U_{x,\mu}U_{x+\hat{\mu},\nu}U^{+}_{x+\hat{\nu},\mu}U^{+}_{x,\nu}$$

$$= -\frac{1}{2N}g^2a^4\Big(F_{\mu,\nu}F_{\mu,\nu} + \frac{1}{12}a^2F_{\mu,\nu}(\partial_\mu^2 + \partial_\nu^2)F_{\mu,\nu} + O(a^4)\Big) .$$

can be eliminated by adding larger loops

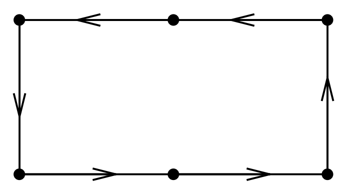
Luescher and Weisz,
Commun. Math. Phys. 97 (1985) 59

$$S^{\mathrm{imp}}_{\mathrm{gluon}} = \beta\ \Big[\ c_0(g^2)\sum_{x,\mu<\nu}\Big(1 - \frac{1}{N}\mathrm{ReTr}$$



$$\Big)_{\mu,\nu}(x)\Big)$$

$$+\ c_1(g^2)\sum_{x,\mu<\nu}\Big(1 - \frac{1}{N}\mathrm{ReTr}$$



$$\Big)_{\mu,\nu}(x)\Big)\Big]$$

$$\mathcal{O}(g^0a^2)$$

$$c_0^{(0)} = 5/3 \quad c_1^{(0)} = -1/6$$

Improved staggered fermion actions

Heller, Karsch, Sturm, PRD60 (1999) 114502

Standard staggered action has discretization errors ~ a^2

Eliminate those using higher order difference scheme

$$S_F = m\sum_x \bar\chi(x)\chi(x) + \sum_x\sum_\mu \eta_\mu(x)\bar\chi(x)[\ c_{1,0}\,(\chi(x+\mu)-\chi(x-\mu)) +$$

$$+c_{3,0}\,(\chi(x+3\mu)-\chi(x-3\mu))$$

$$\sum_{\nu\neq\mu} c_{1,2}\,(\chi(x+\mu+2\nu)-\chi(x-\mu+2\nu))+\chi(x+\mu-2\mu)-\chi(x-\mu-2\mu)\ \ ]$$

Free quark propagator: $\dfrac{-i\sum_\mu \gamma_\mu h_\mu(p)+m}{D^{(0)}(p)+m^2}$

$$s_\mu(p) \equiv \sin(p_\mu) \qquad c_\mu(p) \equiv \cos(p_\mu)$$

$$h_\mu(p) = 2\ s_\mu(p)\Big[c^{(0)}_{1,0} + 2\ c^{(0)}_{1,2}\sum_{\nu\neq\mu} c_\nu(2p)\Big] + 2\ c^{(0)}_{3,0} s_\mu(3p)$$

$$D^{(0)}(p) = \sum_\mu h_\mu(p)h_\mu(p) = \sum_\mu A p_\mu^2 A \left(A + 2B_1 p_\mu^2 + 2B_2\sum_{\nu\neq\mu} p_\nu^2\right) + \mathcal{O}(p^6)$$

$$A = 2c^{(0)}_{1,0} + 12c^{(0)}_{1,2} + 6c^{(0)}_{3,0} \qquad B_1 = -\frac{1}{3}c^{(0)}_{1,0} - 2c^{(0)}_{1,2} - 9c^{(0)}_{3,0} \qquad B_2 = -8c^{(0)}_{1,2}$$

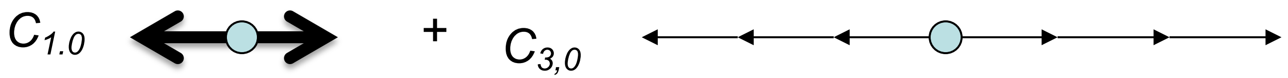
The different staggered which flavors sit in different corners of the Brillouin zone

Are completely equivalent in the free theory => flavor symmetry

Not the case in the interacting theory:

exchange with gluons with momenta ~ *π/a* can change the quark flavor (taste) as it brings it to another corner of the Brillouin zone

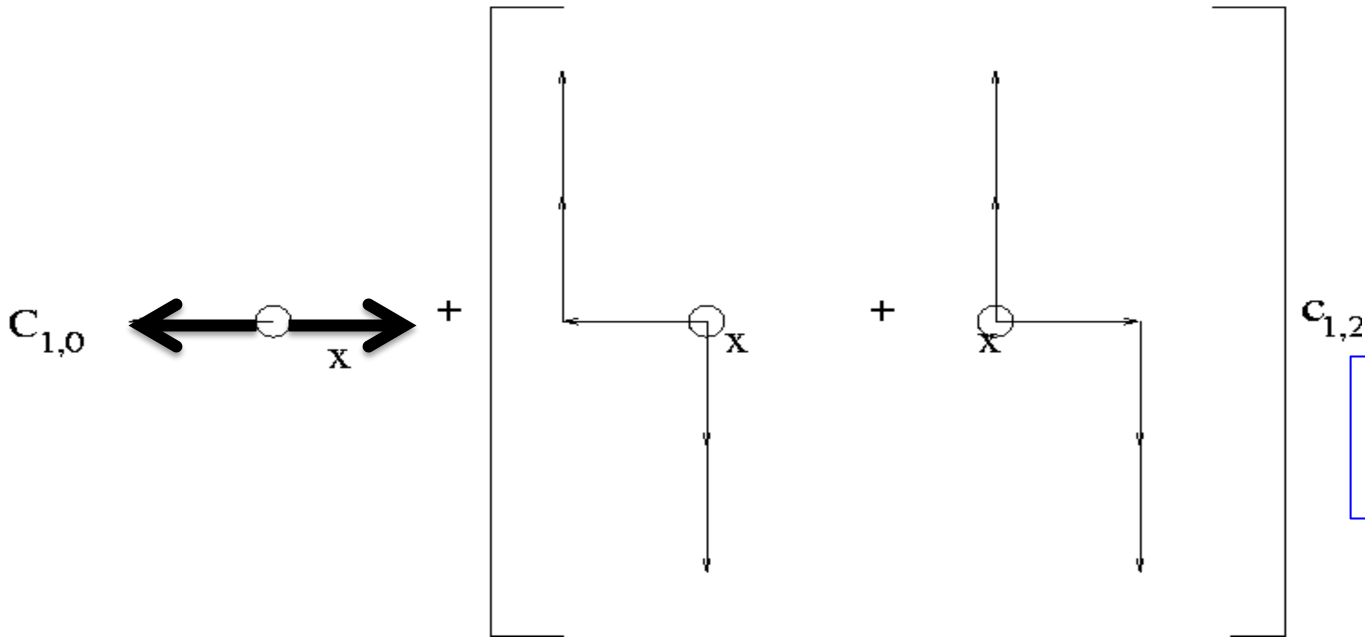
Rotational symmetry at order p^4 : $B_1 = B_2$ ⇨ $c^{(0)}_{1,0} + 27\ c^{(0)}_{3,0} + 6\ c^{(0)}_{1,2} = 24\ c^{(0)}_{1,2}$



Naik action:

$$c^{(0)}_{1,0} = \frac{9}{16} \quad , \quad c^{(0)}_{3,0} = -\frac{1}{48}$$

Normalization: $c^{(0)}_{1,0} + 3\ c^{(0)}_{3,0} + 6\ c^{(0)}_{1,2} \quad = \quad 1/2$



p4 action:

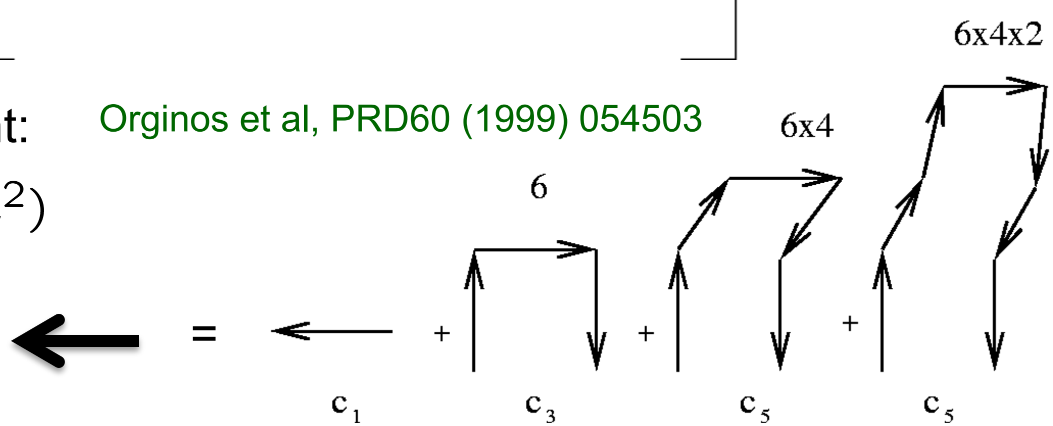
$$c^{(0)}_{1,0} = \frac{3}{8} \quad , \quad c^{(0)}_{1,2} = \frac{1}{48}$$

Taste symmetry improvement:

Orginos et al, PRD60 (1999) 054503

no taste breaking at $\mathcal{O}(g^2 a^2)$

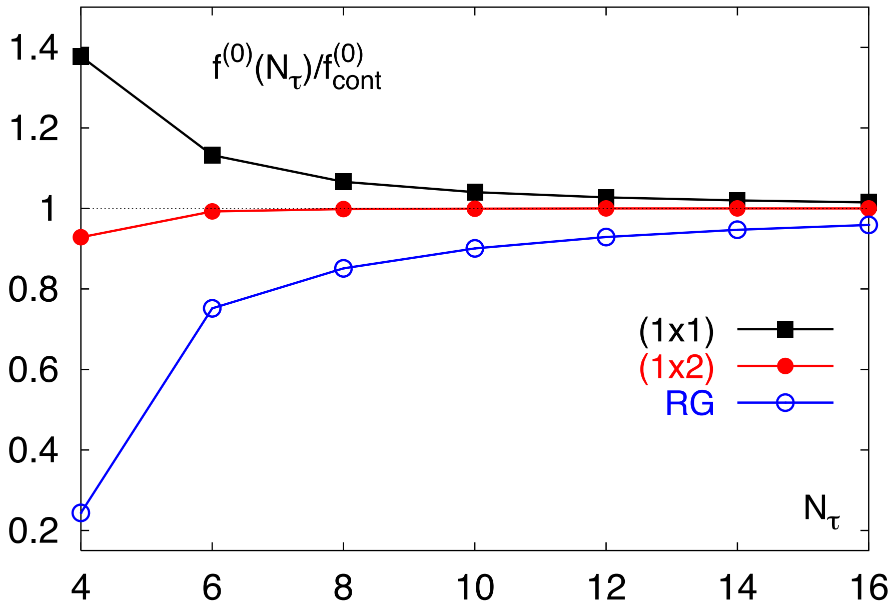
Fat (smeared) link:



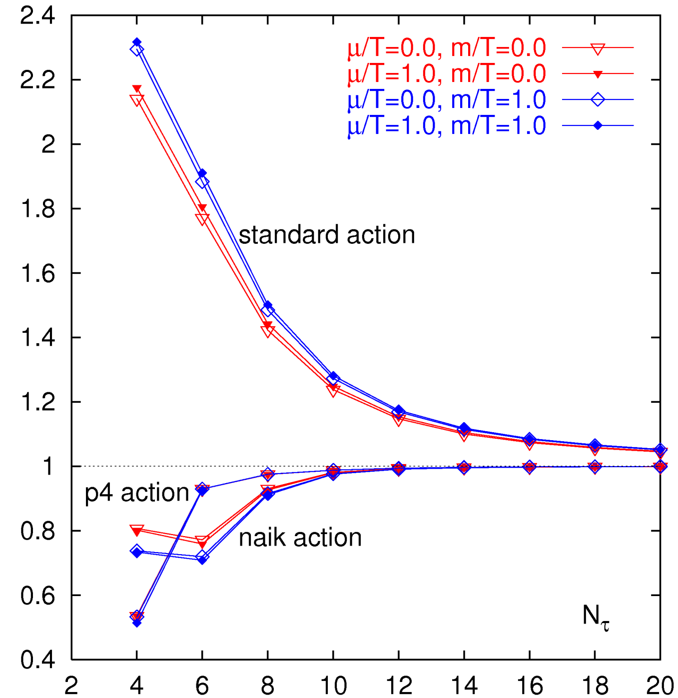
Projection to U(3) => HISQ action

Why improved actions ?

Pressure of the ideal gluon gas



Pressure of the ideal quark gas



$$T = 1/(N_\tau a)$$

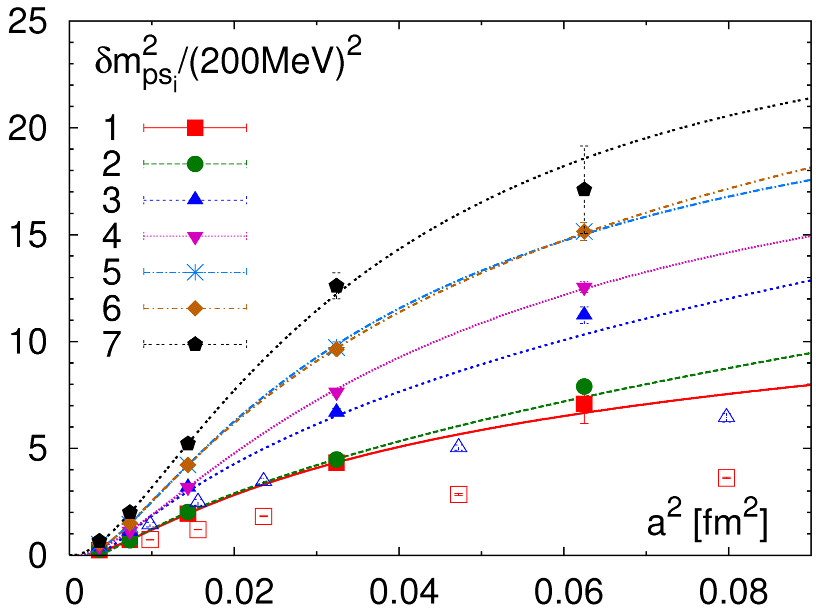
$$\frac{p(N_\tau)}{p_{cont}} = 1 - \frac{1143}{980}\left(\frac{\pi}{N_\tau}\right)^4 + \frac{73}{2079}(1+6528 c_{30})\left(\frac{\pi}{N_\tau}\right)^6 + \mathcal{O}(N_\tau^{-8})$$

$c_{30} = 0$ for p4, $c_{30} = -1/48$ for Naik

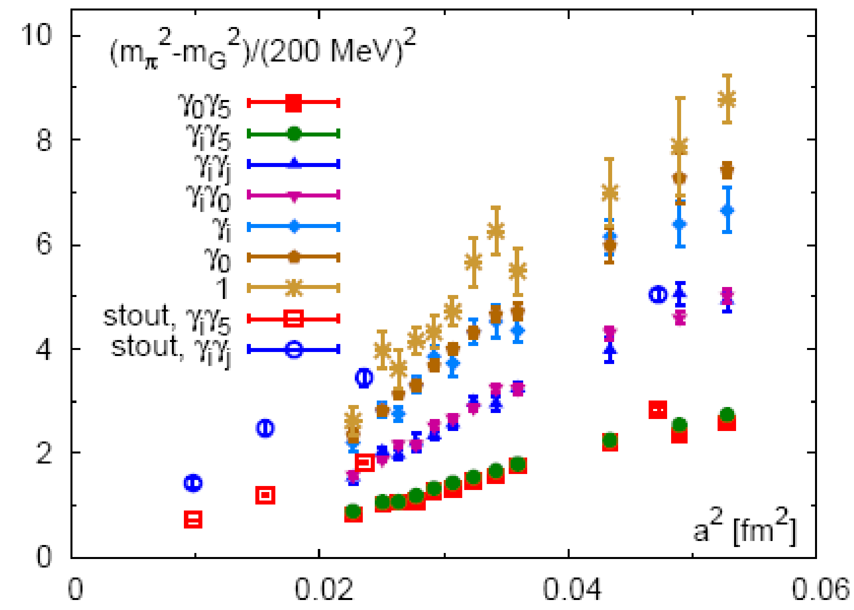
Mass splitting of pseudo-scalar mesons

Only one out of 16 PS mesons has zero mass in the chiral limit, the quadratic mass splitting is the measure of flavor symmetry breaking

asqtad



HISQ



PS meson splittings in HISQ calculations are reduced by factor ~ 2.5 compared to asqtad at the same lattice spacing and are even smaller than for stout action => discretizations effects for $N_\tau=8$ HISQ calculations are similar to those in $N_\tau=12$ asqtad calculations

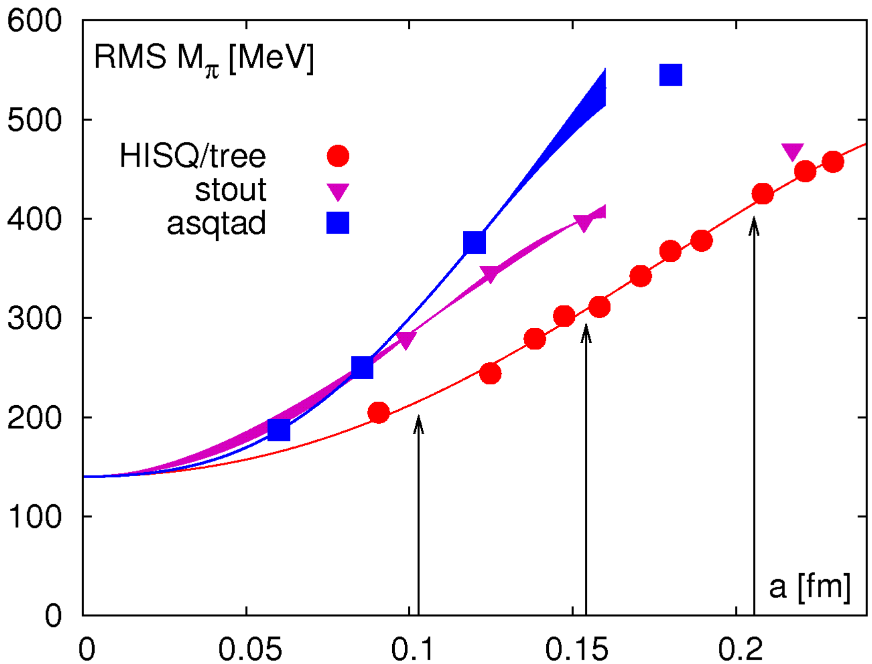
Glossary of improved staggered actions

p4 = std. staggered Dslash with 3-step (fat3) link +p4 term

asqtad = std. staggered Dslash with 7-step (fat7) link + Naik term

HISQ = std. staggered Dslash with re-unitarized doubly smeared 7-step (fat7) link

stout = std. staggered Dslash with re-unitarized doubly smeared 3-step (fat7) link



p4, asqtad, HISQ, stout

Center symmetry and deconfinement transition

Above the phase transition temperature Z(N) (center) symmetry of SU(N) gauge theory is broken
Quarks transform non-trivially under Z(N) symmetry group
=> static charges in fundamental representations can be screened by gluons !

Lattice set-up:

$U_\mu(\tau,x) = e^{igA_\mu(\tau,x)},\ N_\sigma^3 \times N_\tau,\ T = 1/(N_\tau a)$

Thermodynamic limit: $N_\sigma/N_\tau \to \infty$; Continuum limit : $N_\tau \to \infty$, T-fixed Temperature is set by $a \leftrightarrow \beta = 2N_c/g^2$; allowable gauge transformations: $U_\mu(x) \to \Omega(x+\mu)U_\mu(x)\Omega^\dagger(x)$

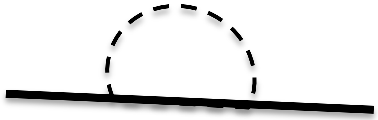
$$\Omega(0,\vec{x}) = \Omega(\beta,\vec{x})C, \quad C = e^{2\pi i n/N_c}I \quad \to Z(N) - \text{symmetry}$$

$$L(\vec{x}) = \frac{1}{N_c}\mathrm{tr}\prod_{\tau=1}^{N_\tau} U_0(\tau,\vec{x})$$

Polyakov loop is changed $L(\vec{x}) \to e^{2\pi n i/N_c}L(\vec{x})$

$< L > \neq 0 \ \to Z(N)$ spontaneously broken; $< L >= e^{-F_Q/T}$ -free energy of an isolated static quark is finite $\Rightarrow$ deconfinement

L is order parameter



The free energy of static quark is infinite in the Continuum limit due to linear *1/a* divergence => needs renormalization

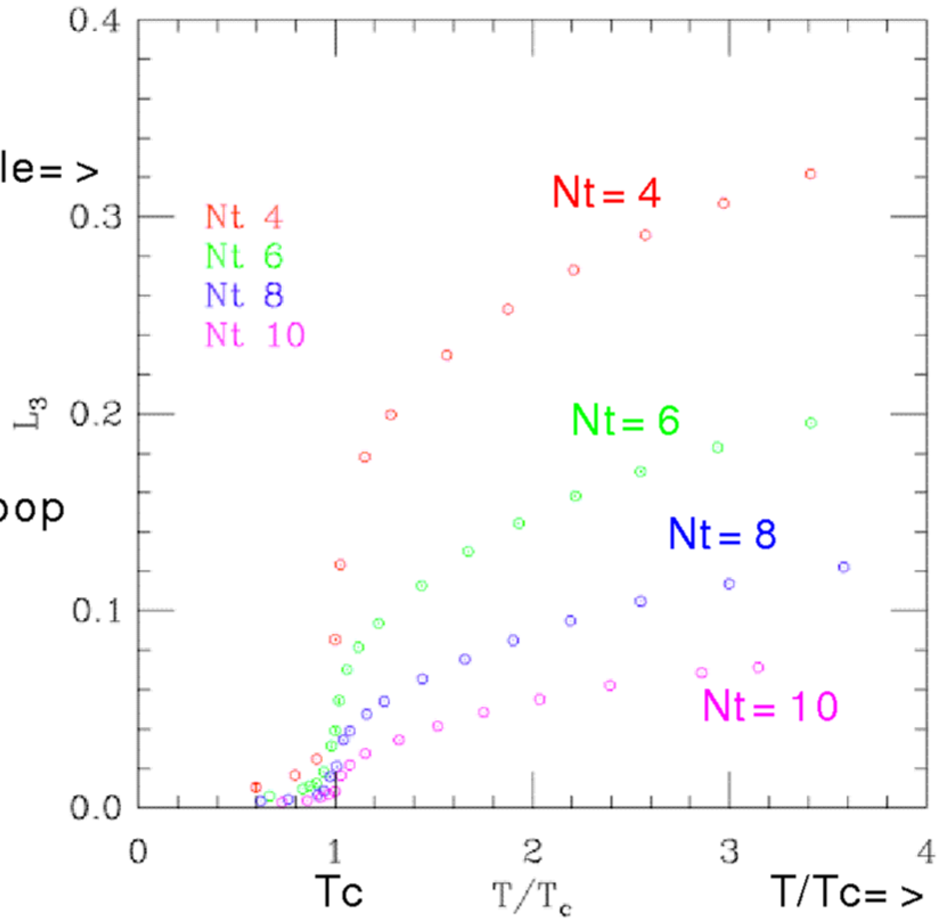
Continuum limit for *L* ?

Dumitru et al, hep-th/0311223

Bare triplet loop vs T, at different Nt

Note scale=> ~ .3

Triplet loop



Nt = # time steps.

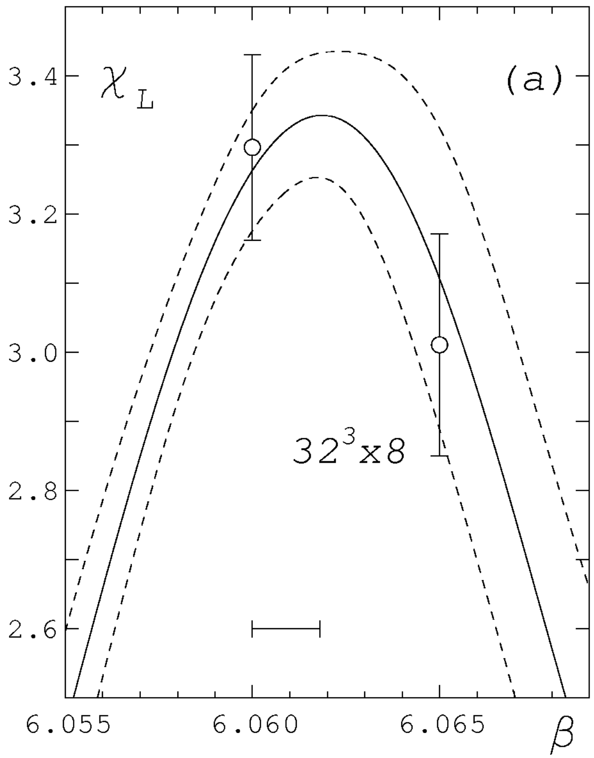
Bare loop vanishes as Nt -> ∞

needs renormalization !

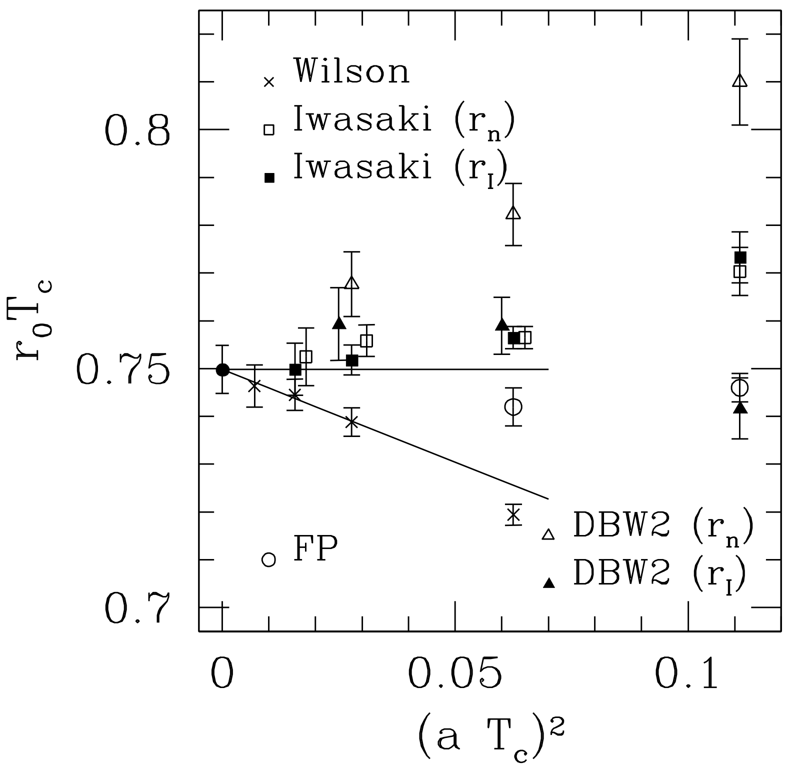
How to determine the deconfinement transition temperature ?

$$\frac{\chi_L}{T^2} = N_\sigma^3 \left(\langle L^2 \rangle - \langle L \rangle^2 \right) = \langle (\delta L)^2 \rangle \text{ has a peak at } \beta_c$$

Boyd et al., Nucl. Phys. B496 (1996) 167



Necco, Nucl. Phys. B683 (2004) 167



- Use different volumes and Ferrenberg-Swedsen re-weighting to combine information collected at different gauge couplings

Finite volume behavior can tell the order of the phase transition, e.g. for 1st order transition the peak height scales as spatial volume !

Correlator of Polyakov loops and deconfinement

The correlation function of Polyakov loops defines the free energy of static quark anti-quark pair (also an order parameter)

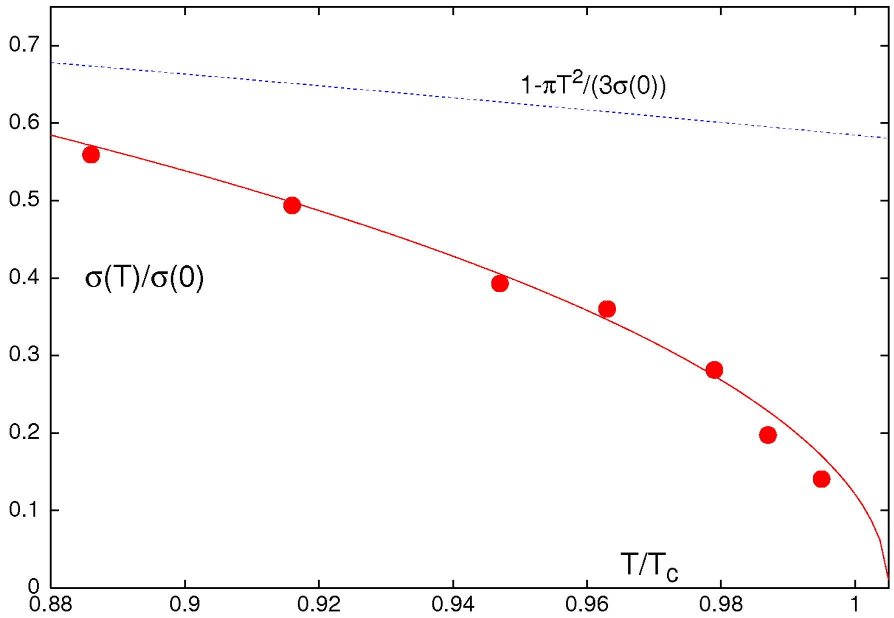
$T < T_c$:

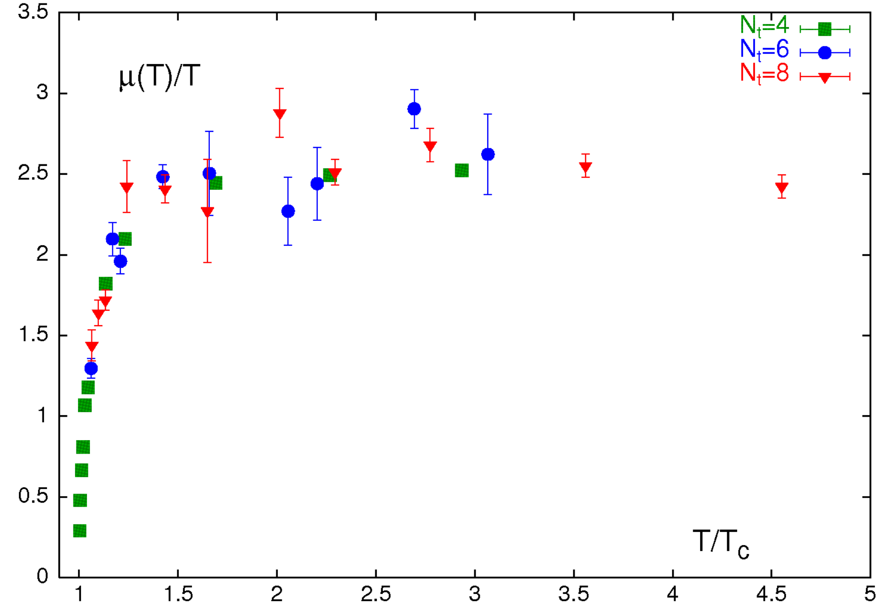
$$< L(r)L^\dagger(0) > \sim e^{-\sigma(T)r/T}$$

$T > T_c$:

$$\ln\left(\frac{< L(r)L^\dagger(0) >}{|< L >|^2}\right) \sim e^{-\mu(T)r}$$

Kaczmarek, Phys. Rev. D62 (2000) 034021



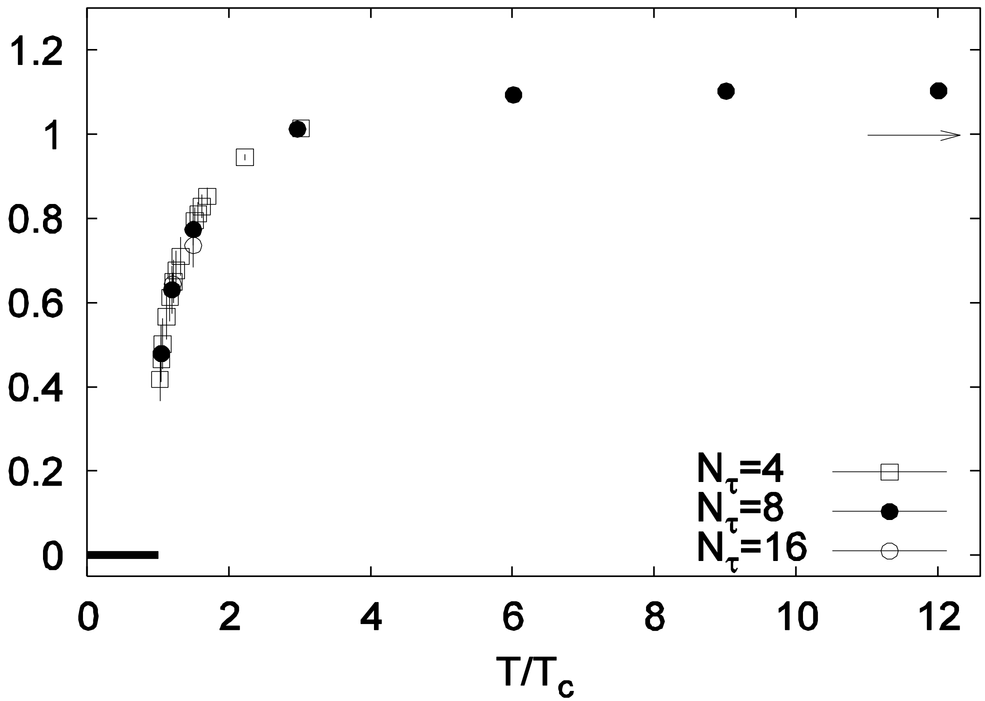


small inverse correlation length => weak 1st order phase transition SU(3) gauge theory is far from the large N-limit !

The renormalized Polyakov loop in pure glue theory

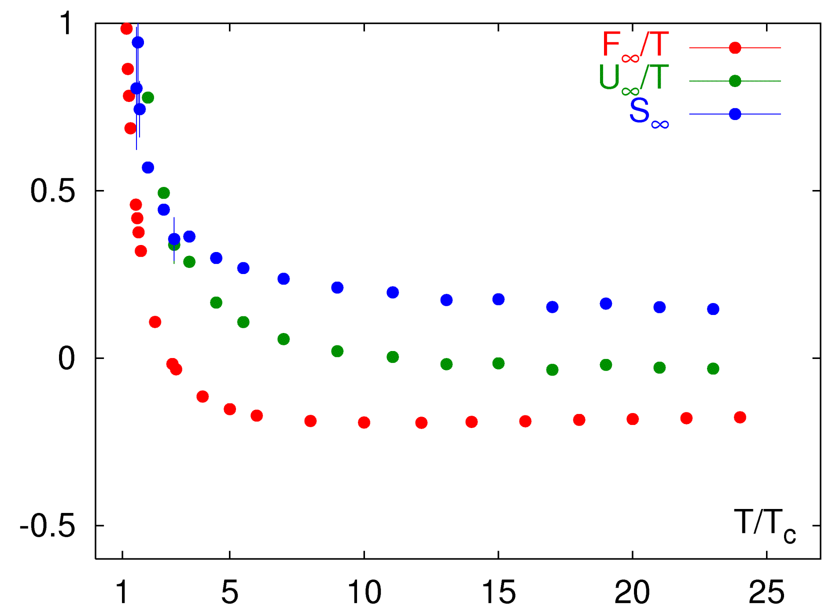
$$r \ll 1/T: \;\; F_{Q\bar{Q}}(r,T) = V(r, T=0) + T \ln 9$$
$$\Rightarrow \text{normalize the } Q\bar{Q} \text{ free energy to the } T=0 \text{ potential}$$

$$\lim_{r\to\infty} \langle L(r) L^\dagger(0)\rangle = \exp(-F_{Q\bar{Q}}(r\to\infty, T)/T) = \exp(-F_\infty/T) = |\langle L\rangle|^2, F_Q = F_\infty/2$$



$$L_{ren} = \exp(-F_\infty(T)/(2T))$$

Kaczmarek et al, PLB 543 (2002) 41,
PRD 70 (2004) 074505, hep-lat/0309121



$$\text{LO}: F_\infty = -T S_\infty = -\frac{4}{3}\alpha_s m_D$$

Integral method: equation of state in SU(3) gauge theory

In Monte-Carlo simulations ln $Z(T)$ cannot be determined but only its derivatives

Boyd et al., NPB 496 (1996) 167

$$\frac{\partial \ln Z(\beta)}{\partial \beta} = \frac{1}{Z(\beta)} \frac{\partial}{\partial \beta} \int \mathcal{D}U e^{-\beta S_g(U)} = - < S_g >$$

$$\frac{p(T)}{T^4} = \int_{\beta_0}^{\beta(T)} d\beta' \left(\frac{\partial \ln Z(T)}{\partial \beta'} - \frac{\partial \ln Z(T=0)}{\partial \beta'} \right) = \int_{\beta_0}^{\beta(T)} d\beta' \left(\langle S_g \rangle_0 - \langle S_g \rangle_T \right)$$

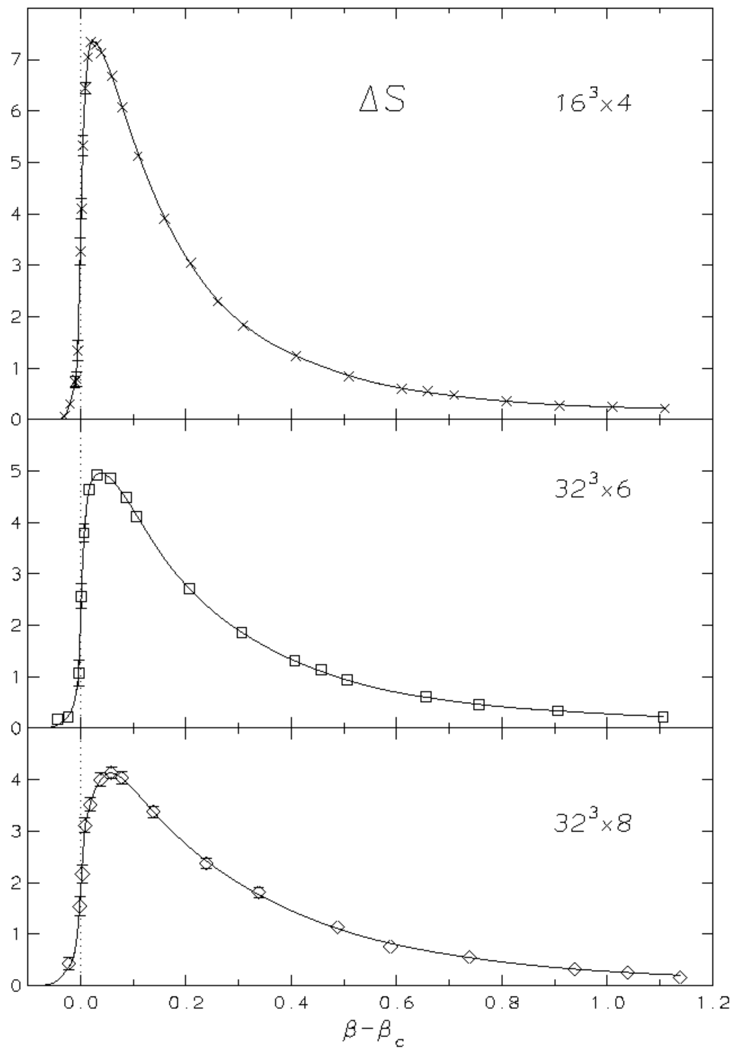
$$s = (\epsilon + p)/T = \frac{\partial p}{\partial T}$$

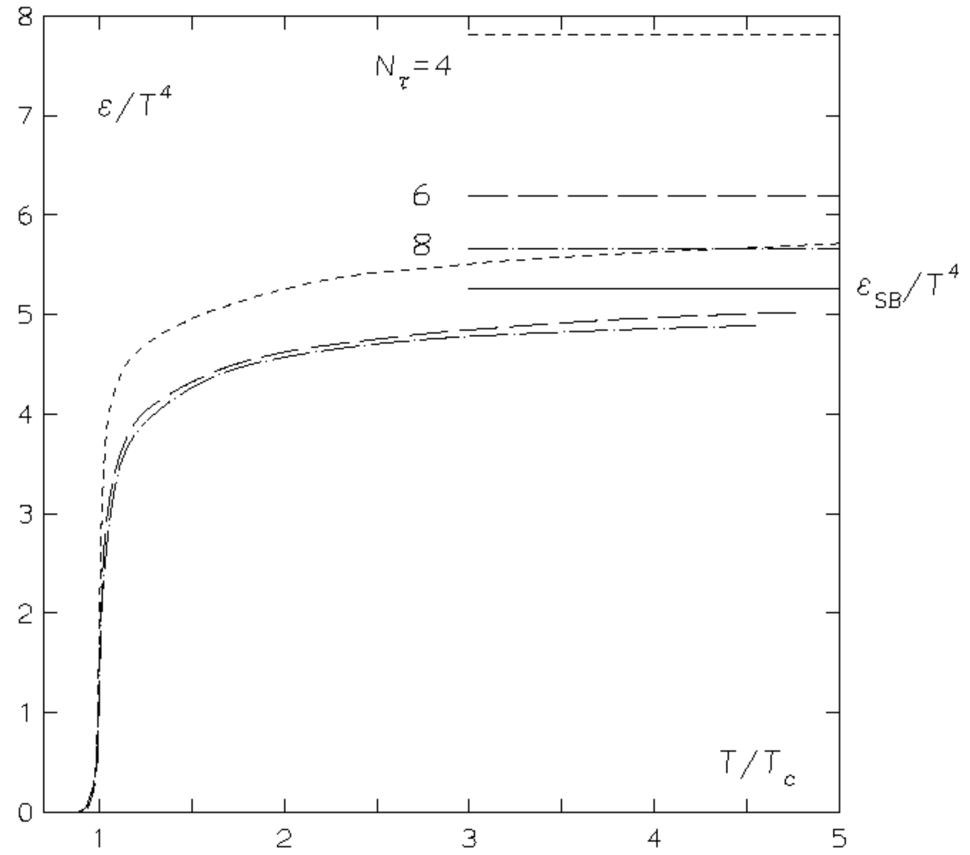
$$\frac{\epsilon - 3p}{T^4} = T \frac{d}{dT} \left(\frac{p}{T^4} \right) = T \frac{d\beta}{dT} \frac{\partial p / T^4}{\partial \beta} = - \left(a \frac{d\beta}{da} \right) \frac{\partial p / T^4}{\partial \beta}$$

⇩

computational cost go as N_τ^4 because of the vacuum subtraction large cutoff effects !

Boyd et al., NPB496 (1996) 167



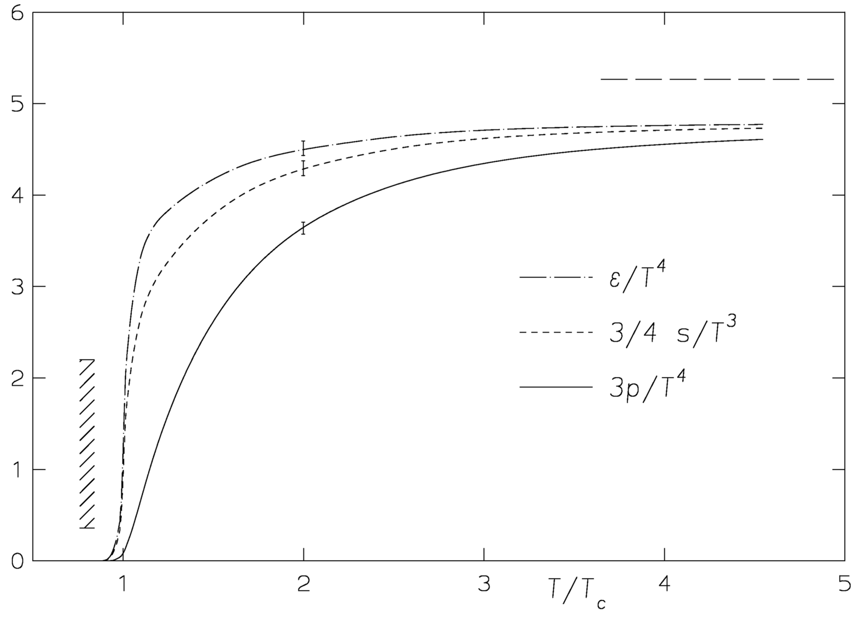


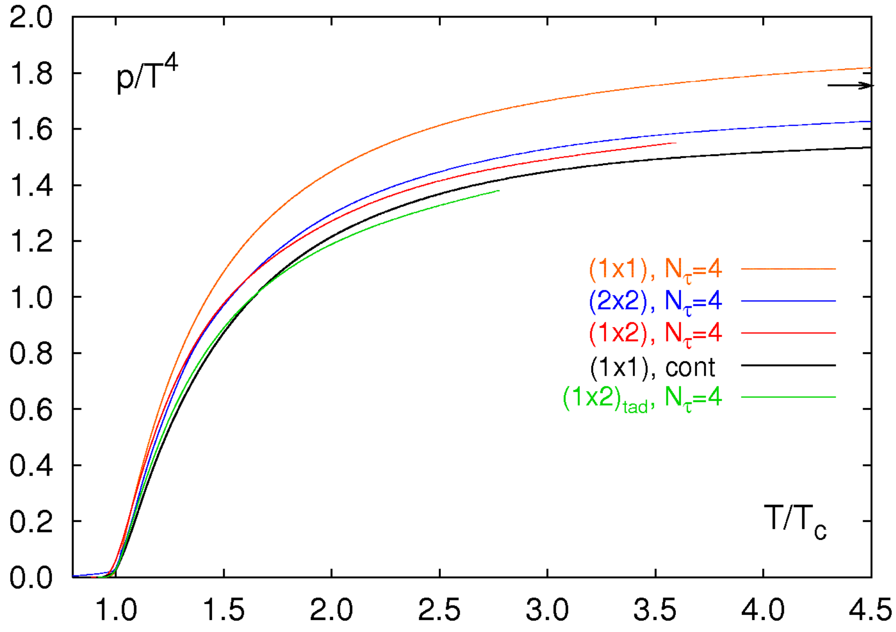
the free gas limit overestimates cutoff effects

Wilson gauge action a^2 discretization errors => $1/N_\tau^2$ corrections to the pressure

Boyd et al., NPB 496 (1996) 167
Wilson gauge action
continuum extrapolation

Karsch et al, EPJ C 6 (1999) 133
Luescher-Weisz gauge action:
large reduction of cutoff effects



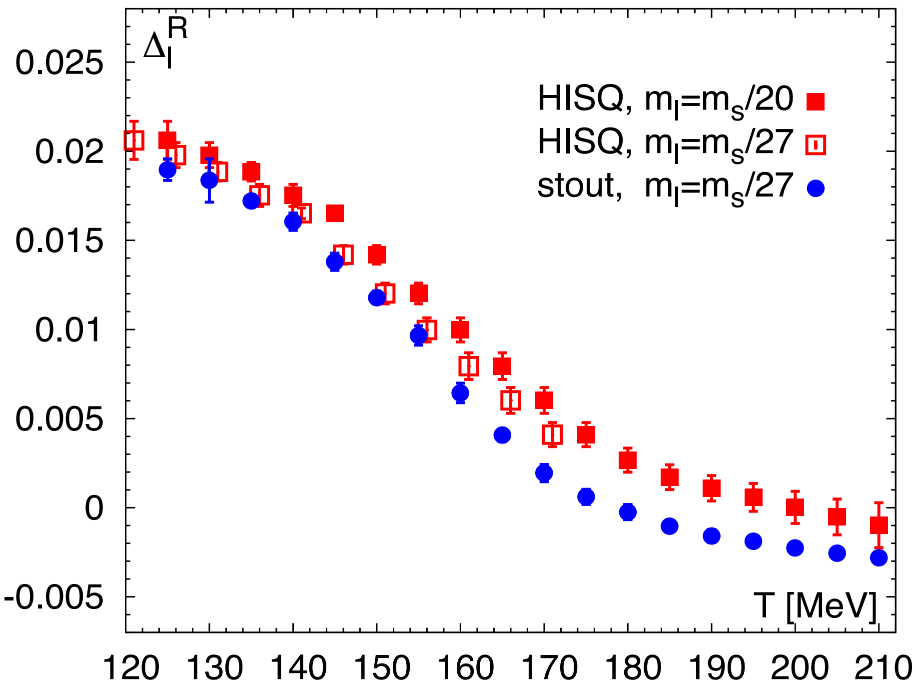


The chiral transition at non-zero temperature

Renormalized chiral condensate

$$\langle\bar\psi\psi\rangle \Rightarrow \Delta_l^R(T) =$$
$$= m_s r_1^4(\langle\bar\psi\psi\rangle_T - \langle\bar\psi\psi\rangle_{T=0}) + d,$$
$$d = m_s r_1^4 \langle\bar\psi\psi\rangle_{T=0}^{m_q=0}, \quad r_1 = 0.3106\text{fm}$$

Bazavov et al (HotQCD), PRD85 (2012) 054503;
Bazavov et al, PRD 87(2013)094505,
Borsányi et al, JHEP 1009 (2010) 073

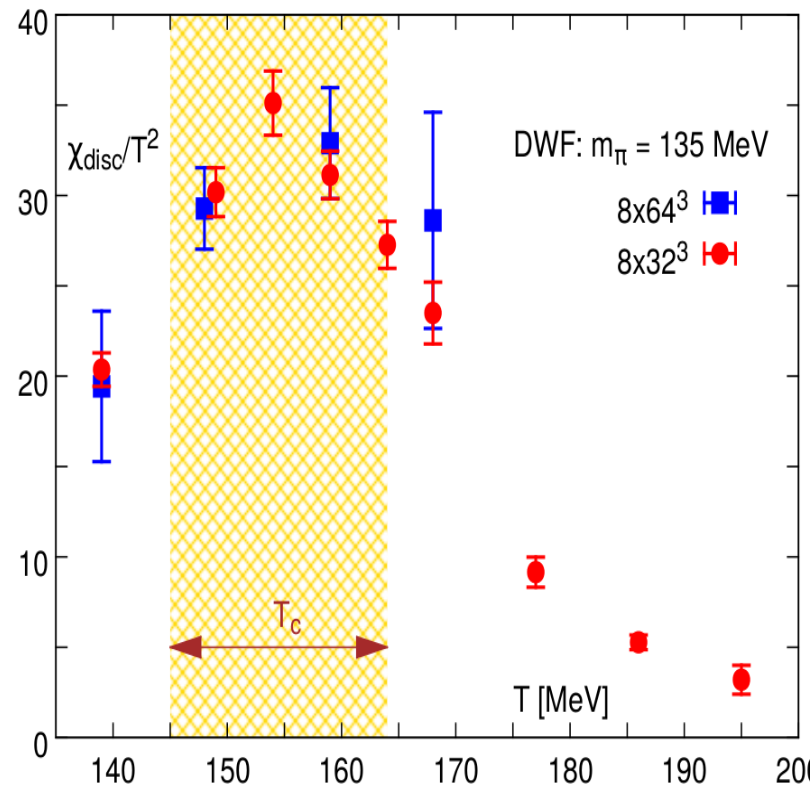


$T_c = (154 \pm 8 \pm 1(scale))$MeV

Fluctuations of the order parameter:

$$\chi_{disc} = VT^{-1}\left(\langle(\bar\psi\psi)^2\rangle - \langle\bar\psi\psi\rangle^2\right)$$

Bhattacharya et al (HotQCD), PRL 113 (2014)082001



$$T_c = (155 \pm 8 \pm 1)\text{MeV}$$

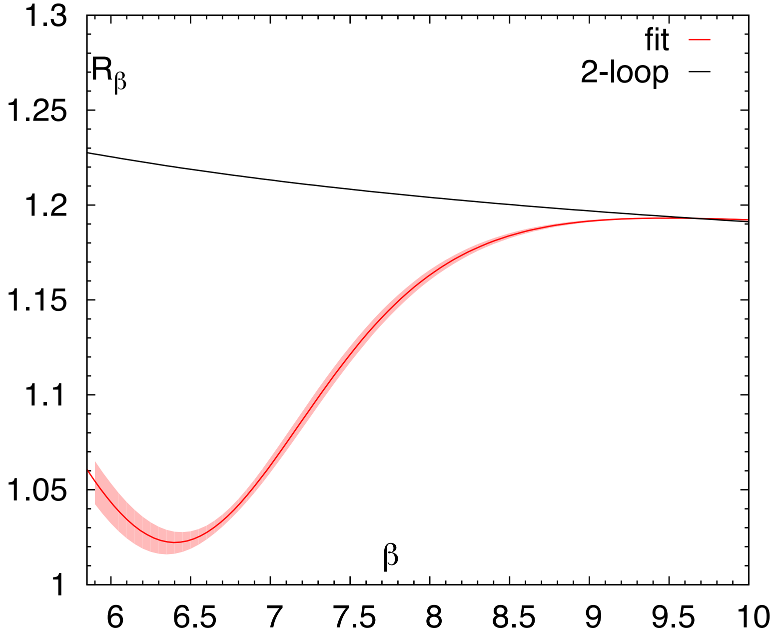
No increase with the volume
$\Rightarrow$ Crossover transition

QCD trace anomaly and the integral method

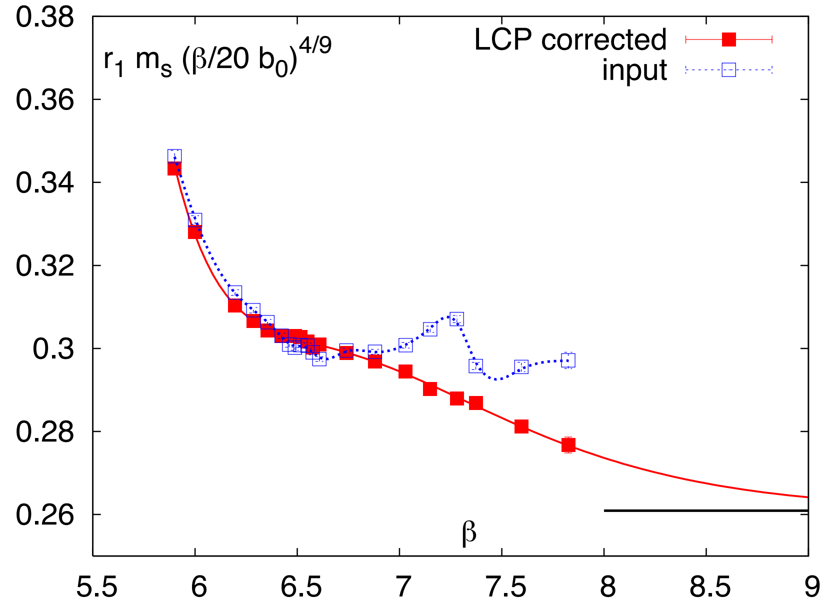
$$\frac{\Theta^{\mu\mu}(T)}{T^4} = \frac{\varepsilon - 3p}{T^4} = T\frac{d}{dT}\left(\frac{p}{T^4}\right) \Rightarrow \frac{p(T)}{T^4} - \frac{p(T_0)}{T_0^4} = \int_{T_0}^{T} dT' \frac{\Theta^{\mu\mu}(T')}{T'^5},$$

$$\frac{\Theta^{\mu\mu}(T)}{T^4} = \frac{\epsilon - 3p}{T^4} = R_\beta\{\langle S_G\rangle_0 - \langle S_G\rangle_T\} - R_\beta R_m\{2m_l(\langle\bar{q}q\rangle_0 - \langle\bar{q}q\rangle) + m_s(\langle\bar{s}s\rangle_0 - \langle\bar{s}s\rangle_T)\}$$

$$R_\beta(\beta) = -a\frac{d\beta}{da},\ R_m = \frac{1}{m_q(\beta)}\frac{dm_q(\beta)}{d\beta},\ \beta = 10/g^2$$



Bazavov, PP, Weber, PRD97 (2018) 014510

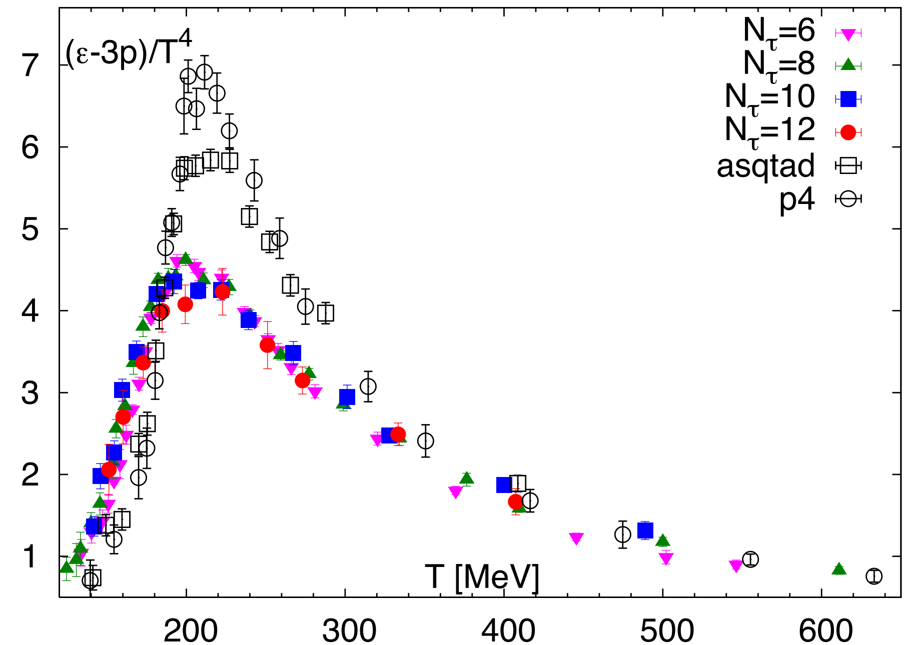


Bazavov et al, PRD 90 (2014) 094503

QCD results on the trace anomaly

2+1 flavor QCD calculations with almost physical light and strange quark masses

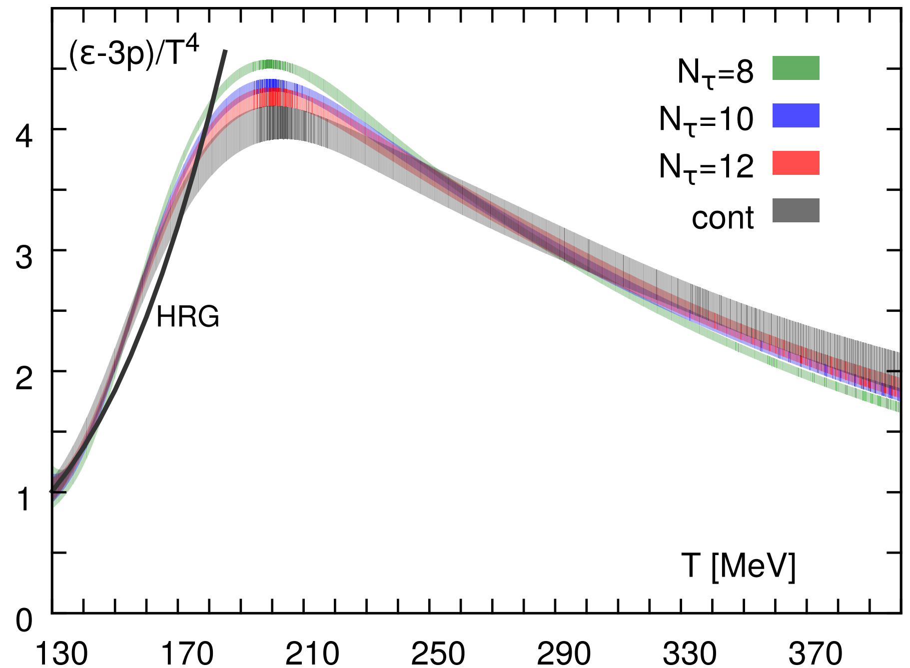
Bazavov et al, PRD 90 (2014) 094503



The peak height is much reduced compared to the asqtad and p4 $N_\tau=8$ calculations

Agreement with p4 and asqtad calculations for $T>350$ MeV

Small cutoff effects for HISQ except for $N_\tau=6$

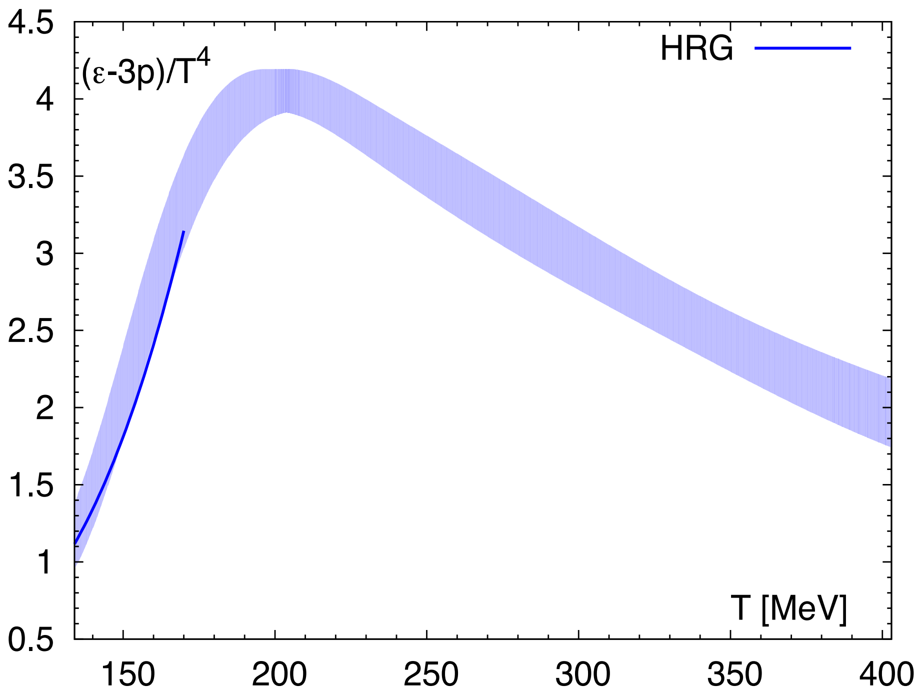


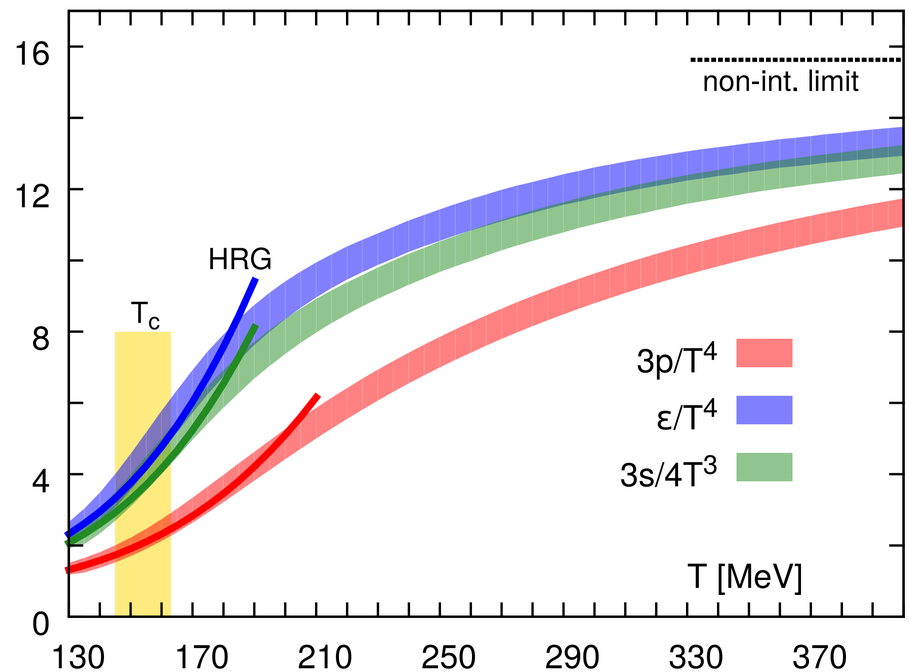
Perform spline interpolation of all the $N_\tau>6$ data with spline coefficients having $a+b/N_\tau^2$ *form* ,
stabilize the spline demanding
that ε-$3p$ is given by HRG at $T=130$ MeV

QCD thermodynamics in the continuum limit

Set the lower integration limit to $T_0 = 130$ MeV and take $p_0 = p^{HRG}(T=130 \text{ MeV})$ ➡ $p(T)$

Bazavov et al, PRD 90 (2014) 094503



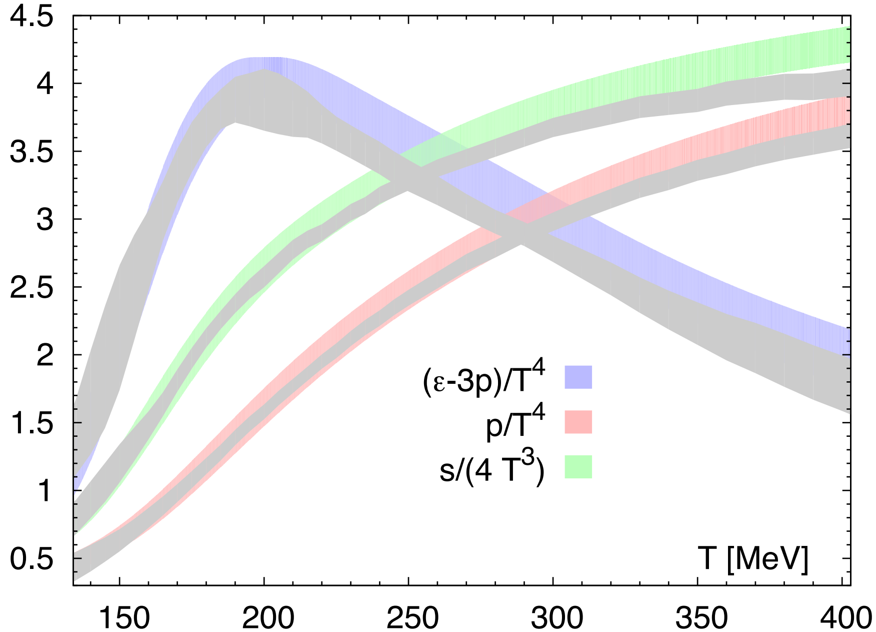


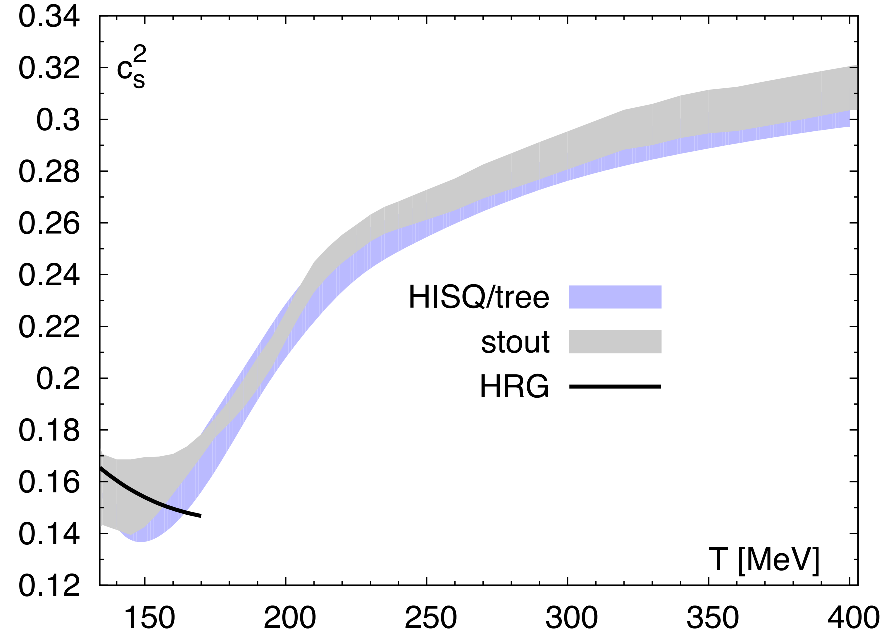
$$T_c = 156 \pm 1.5 \text{ MeV} \qquad \epsilon_{nucl} \simeq 150 \text{MeV}/\text{fm}^3$$

$$\epsilon_c = 420(60) \text{ MeV}/\text{fm}^3 \qquad \epsilon_{proton} \simeq 450 \text{MeV}/\text{fm}^3$$

HRG: all resonances from PDG treated as stable (zero width) particles in an ideal gas

Comparison of different continuum limit





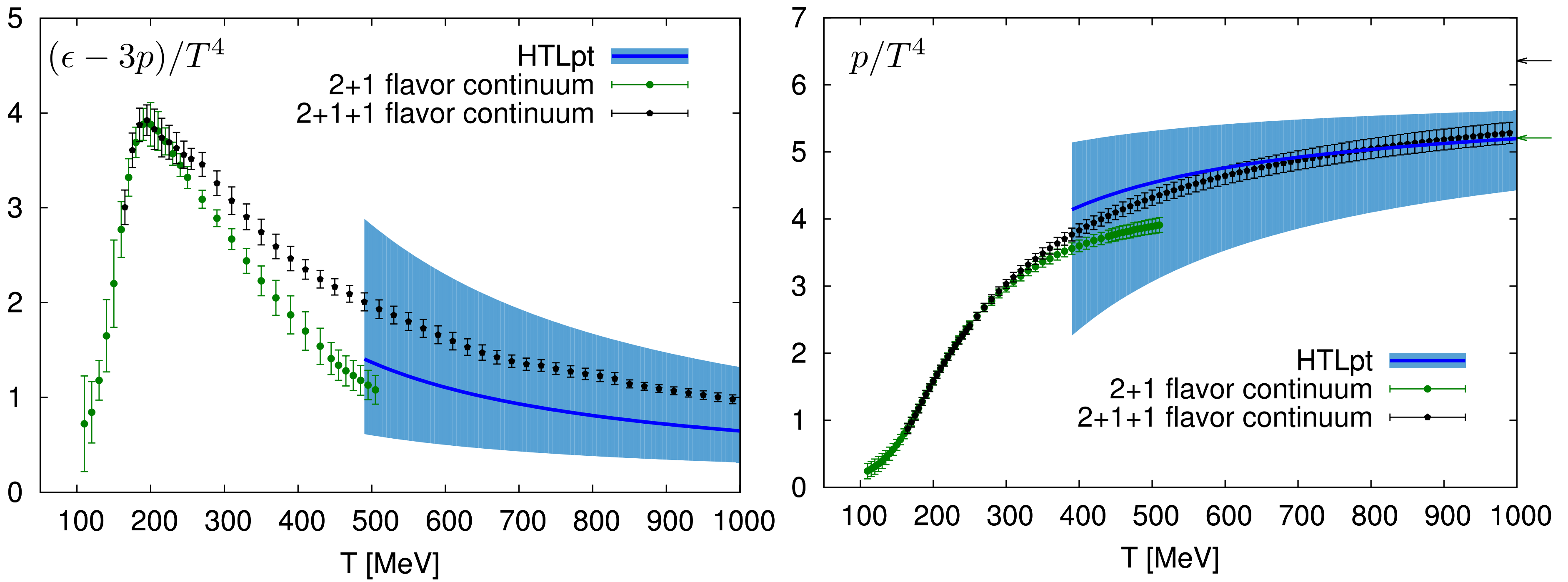
Continuum results obtained with stout and HISQ action agree reasonably well given their errors (some tension for the entropy density)

Even in the transition region the speed of sound is not much smaller than the HRG speed of sound (the EoS is never really soft)

HISQ: Bazavov et al, PRD 90 (2014) 094503
stout: Borsányi et al, PLB730 (2014) 99

The role of charm quarks in QCD EoS

Borsányi et al, Nature 539 (2016) 69



The contribution of charm quarks becomes significant for *T>400* MeV

QCD thermodynamics at non-zero chemical potential

Taylor expansion :

$$\frac{p(T,\mu_B,\mu_Q,\mu_S,\mu_C)}{T^4} = \sum_{ijkl} \frac{1}{i!j!k!l!} \chi_{ijkl}^{BQSC} \left(\frac{\mu_B}{T}\right)^i \left(\frac{\mu_Q}{T}\right)^j \left(\frac{\mu_S}{T}\right)^k \left(\frac{\mu_C}{T}\right)^l$$ hadronic

$$\frac{p(T,\mu_u,\mu_d,\mu_s,\mu_c)}{T^4} = \sum_{ijkl} \frac{1}{i!j!k!l!} \chi_{ijkl}^{udsc} \left(\frac{\mu_u}{T}\right)^i \left(\frac{\mu_d}{T}\right)^j \left(\frac{\mu_s}{T}\right)^k \left(\frac{\mu_c}{T}\right)^l$$ quark

$$\chi_{ijkl}^{abcd} = T^{i+j+k+l} \frac{\partial^i}{\partial \mu_b^i} \frac{\partial^j}{\partial \mu_b^j} \frac{\partial^k}{\partial \mu_c^i} \frac{\partial^l}{\partial \mu_d^l} \ln Z(T,V,\mu_a,\mu_b,\mu_c,\mu_d) \,|_{\mu_a=\mu_b=\mu_c=\mu_d=0}$$

Taylor expansion coefficients give the susceptibilities, i.e. the fluctuations and correlations of conserved charges, e.g.

$$\chi_2^X = \chi_X = \frac{1}{VT^3} \left(\langle X^2 \rangle - \langle X \rangle^2\right) \qquad \chi_{11}^{XY} = \frac{1}{VT^3} \left(\langle XY \rangle - \langle X \rangle \langle Y \rangle\right)$$

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information about carriers of the conserved charges (hadrons or quarks)

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probes of deconfinement