Lattice Flavour Physics: Lecture 1

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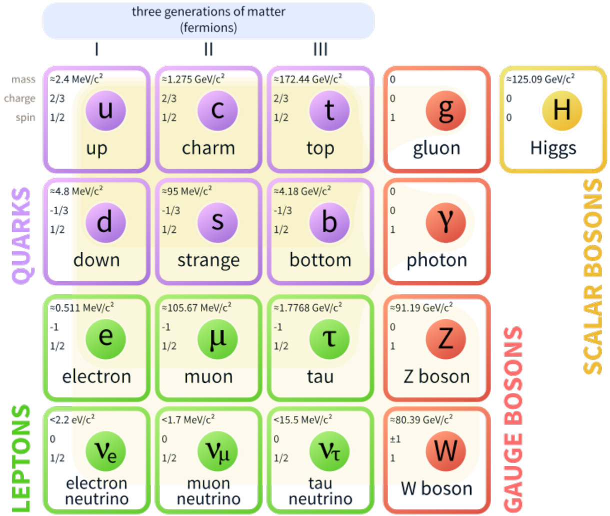
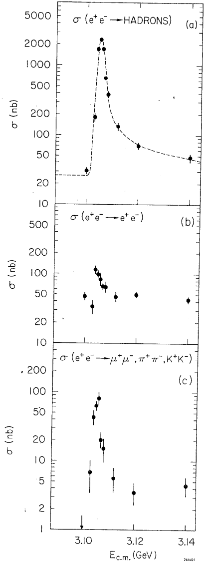
Standard Model of Elementary Particles





- Left - Discovery of J/Ψ in e^+e^- collisions at SLAC, November 1974.

  J.E.Augustin et al. PRL **33** (1974) 1406

- "Simultaneous" discovery in $pp \to e^+e^-X$ at Brookhaven. J.Aubert et al. PRL **33** (1974) 1404

- I was assigned to the working group asked to investigate whether the resonance might be non-hadronic.

  T.Neff, CTS, D.Sivers, J.Townsend, PRD **12** (1975) 1488

  1 citation

- It was soon understood that this resonance was a $c\bar{c}$ (charmonium) vector particle.

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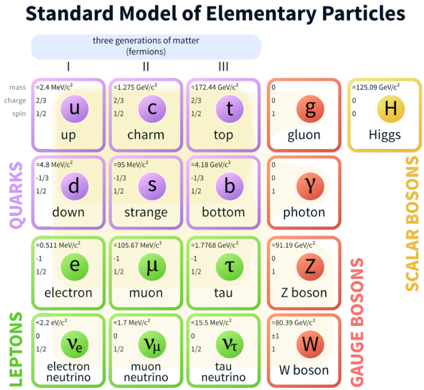
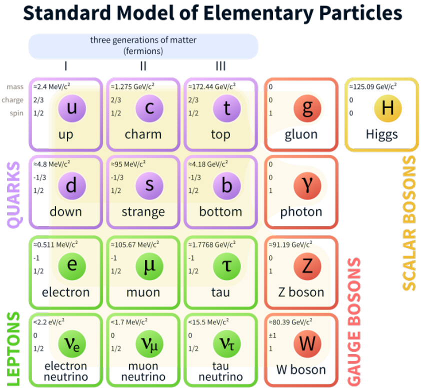


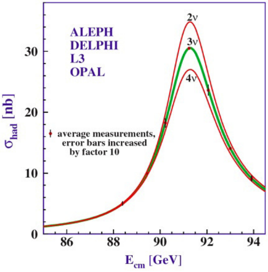
- Who ordered that?

I.I.Rabi, 1936

Discovery of the muon



- Z_0 width



$$N_\nu = 2.9840 \pm 0.0082$$

PDG 2016

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There are many reasons to believe that the Standard Model is incomplete:

- Why are the charges of the proton and electron equal and opposite:

$$\frac{Q_p + Q_e}{e} < 1 \times 10^{-21}\,.$$

  - Unification of forces?
  - Cancellation of anomalies?

The SM is incomplete

There are many reasons to believe that the Standard Model is incomplete:

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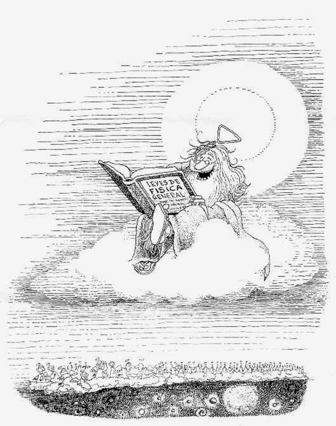
$$\frac{Q_p+Q_e}{e} < 1\times 10^{-21}\,.$$

  - Unification of forces?
  - Cancellation of anomalies?

- nature of dark matter and dark energy;
- naturalness and mass hierarchies;
- matter-antimatter asymmetry;
- strong CP-problem;
- origin of neutrino masses;
- gravity, $\cdots$



1. General introduction
2. Brief introduction to flavour physics
3. The Operator Product Expansion (OPE) in Weak Processes
4. Flavour Physics Experiments
5. Outline of lattice computation of f_P
6. Renormalisation
7. Selected Results from the Flavour Physics Lattice Averaging Group (FLAG)

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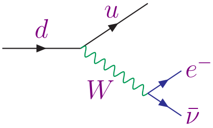
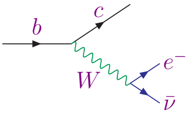
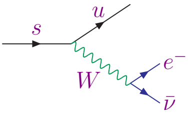
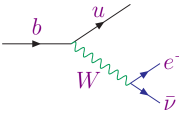
- At the level of quarks we understand nuclear β decay in terms of the fundamental process:



- With the 3 generations of quarks and leptons in the standard model this is generalized to other *charged current* processes, e.g.:



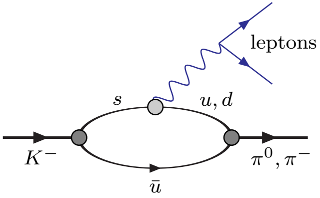




- Weak interaction eigenstates $\neq$ mass eigenstates.

Two Experimental Numbers:

$$B(K^- \to \pi^0 e^- \bar{\nu}_e) \simeq 5\% \; (K_{e3} \text{ Decay}) \quad \text{and} \quad B(K^- \to \pi^- e^+ e^-) = (3.00 \pm 0.09) \times 10^{-7}\,.$$



- Measurements like this show that $s \to u$ (charged-current) transitions are not very rare, but that *Flavour Changing Neutral Current* (FCNC) transitions, such as $s \to d$ are.
  - Since FCNC processes are *rare* in the SM, they provide an excellent laboratory for searches for new physics.
- The existence of decays such as $K^- \to \pi^0 e^- \bar{\nu}_e$ implies that we need to have a mechanism for transitions between quarks of different generations.
- The picture which has emerged is the Cabibbo-Kobayashi-Maskawa (CKM) theory of quark mixing.

Weak interaction eigenstates $\neq$ mass eigenstates:

$$U_W = \begin{pmatrix} u_W \\ c_W \\ t_W \end{pmatrix} = U_u \begin{pmatrix} u \\ c \\ t \end{pmatrix} = U_u\, U \quad \text{and} \quad D_W = \begin{pmatrix} d_W \\ s_W \\ b_W \end{pmatrix} = U_d \begin{pmatrix} d \\ s \\ b \end{pmatrix} = U_d D$$

where U_u and U_d are unitary matrices.

- For neutral currents:

$$\bar{U}_W \cdots U_W = \bar{U} \cdots U \quad \text{and} \quad \bar{D}_W \cdots D_W = \bar{D} \cdots D$$

  and no FCNC are induced. The $\cdots$ represent Dirac Matrices, but the identity in flavour.

- For charged currents:

$$J_W^{\mu\,+} = \frac{1}{\sqrt{2}}\,\bar{U}_W \gamma_L^\mu D_W = \frac{1}{\sqrt{2}}\,\bar{U}_L \gamma^\mu (U_u^\dagger U_d) D_L \equiv \frac{1}{\sqrt{2}}\,\bar{U} \gamma_L^\mu V_{\text{CKM}} D$$

- The charged-current interactions are of the form

$$J_\mu^+ = (\bar{u},\bar{c},\bar{t})_L \gamma_\mu V_{\rm CKM} \begin{pmatrix} d \\ s \\ b \end{pmatrix}_L \equiv (\bar{u},\bar{c},\bar{t})_L \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}_L$$

- 2018 Particle Data Group summary for the magnitudes of the entries (assuming unitarity):

$$\begin{pmatrix} 0.97446 \pm 0.00010 & 0.22452 \pm 0.00044 & 0.00365 \pm 0.00012 \\ 0.22438 \pm 0.00044 & 0.97359^{+0.00010}_{-0.00011} & 0.04214 \pm 0.00076 \\ 0.00896^{+0.00024}_{-0.00023} & 0.04133 \pm 0.00074 & 0.999105 \pm 0.000032 \end{pmatrix}$$

- How many parameters are there?
  - Let N_g be the number of generations.
  - $N_g \times N_g$ unitary matrix has N_g^2 real parameters.
  - $(2N_g - 1)$ of them can be absorbed into unphysical phases of the quark fields.
  - $(N_g - 1)^2$ physical parameters to be determined.

- For $N_g = 2$ there is only one parameter, which is conventionally chosen to be the Cabibbo angle:

$$V_{\rm CKM} = \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix}.$$

- For $N_g = 3$, there are 4 real parameters. Three of these can be interpreted as angles of rotation in three dimensions (e.g. the three Euler angles) and the fourth is a phase. The general parametrisation recommended by the PDG is

$$\begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{pmatrix}$$

  where c_{ij} and s_{ij} represent the cosines and sines respectively of the three angles θ_{ij}, $ij = 12$, 13 and 23. δ_{13} is the phase parameter.

- It is conventional to use approximate parametrizations, based on the hierarchy of values in $V_{\rm CKM}$ $(s_{12} \gg s_{23} \gg s_{13})$.

Wolfenstein parametrisation:

$$V_{\rm CKM} \simeq \begin{pmatrix} 1-\frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho-i\eta) \\ -\lambda & 1-\frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} + O(\lambda^4)\,.$$

- A, ρ and η are real numbers that a priori were intended to be of order unity.
- It is conventional to introduce

$$\bar\rho + i\bar\eta = -\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*} = \left(1-\frac{\lambda^2}{2}\right)(\rho - i\eta) + O(\lambda^4)\,.$$

Unitarity of the CKM-matrix we have a set of relations between the entries. A particularly useful one is:

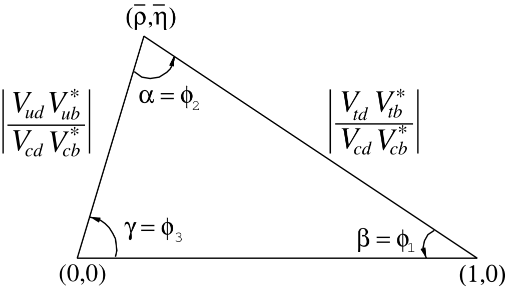
$$V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0\ .$$

In terms of the Wolfenstein parameters, the components on the left-hand side are given by:

$$\begin{aligned} V_{ud}V_{ub}^* &= A\lambda^3[\bar\rho + i\bar\eta] + O(\lambda^7) \\ V_{cd}V_{cb}^* &= -A\lambda^3 + O(\lambda^7) \\ V_{td}V_{tb}^* &= A\lambda^3[1-(\bar\rho + i\bar\eta)] + O(\lambda^7)\ . \end{aligned}$$

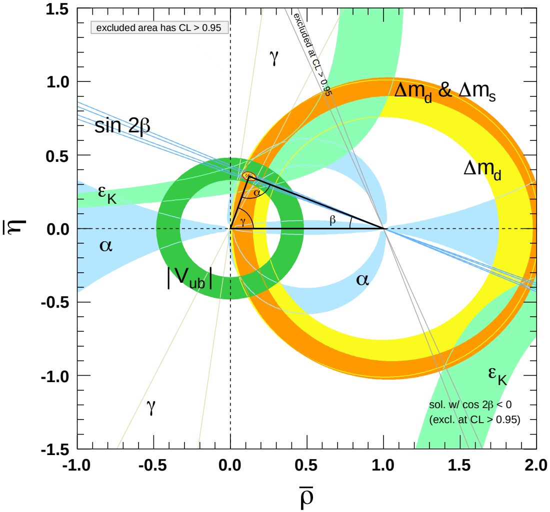
The unitarity relation can be represented schematically by the famous “unitarity triangle" (obtained after scaling out a factor of $A\lambda^3$).

$$V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0\ .$$

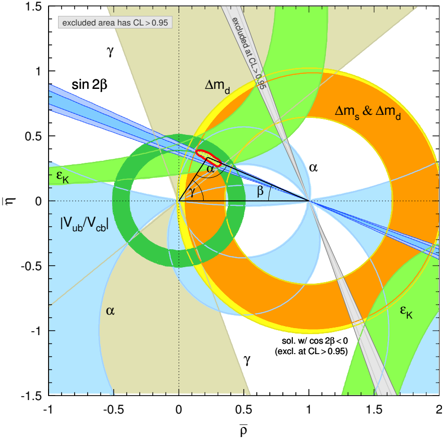


- A particularly important approach to testing the *Limits of the SM* is to over-determine the position of the vertex A to check for consistency.

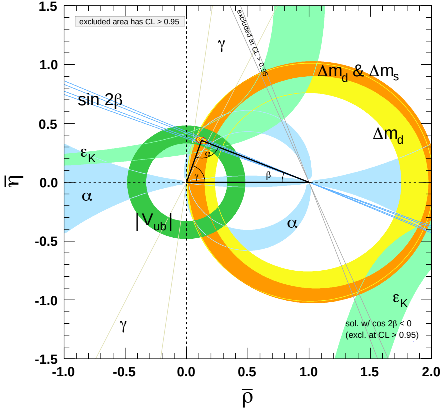
PDG2018 Unitarity Triangle



PDG2006 & 2018 Unitarity Triangle Comparison



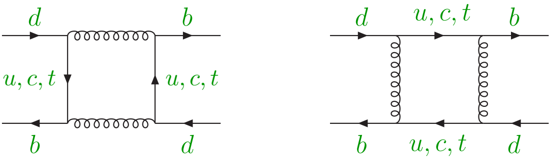
2006



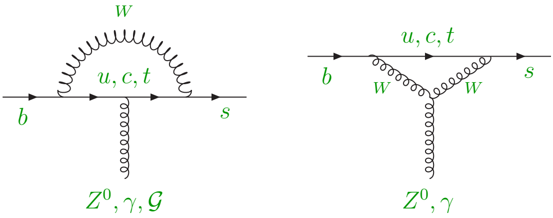
2018

We have seen that in the SM, unitarity implies that there are no FCNC reactions at tree level, i.e. there are no vertices of the type:

Quantum loops, however, can generate FCNC reactions, through *box* diagrams or *penguin* diagrams.
Example relevant for $\bar{B}^0 - B^0$ mixing:



Examples of penguin diagrams relevant for $b \to s$ transitions:



- We will discuss several of the physical processes induced by these loop-effects.
- The Glashow-Illiopoulos-Maiani (GIM) mechanism $\Rightarrow$ FCNC effects vanish for degenerate quarks ($m_u = m_c = m_t$). For example unitarity implies

$$V_{ub}V_{us}^* + V_{cb}V_{cs}^* + V_{tb}V_{ts}^* = 0$$

$\Rightarrow$ each of the above penguin vertices vanish.

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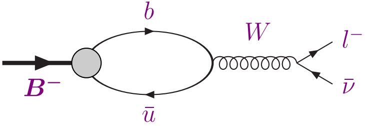
- The difficulty in making predictions for weak decays of hadrons is in controlling the non-perturbative strong interaction effects.
- As a particularly simple example consider the leptonic decays of pseudoscalar mesons in general and of the B-meson in particular.



- Non-perturbative QCD effects are contained in the matrix element

$$\langle 0|\,\bar b\gamma^\mu(1-\gamma^5)u\,|B(p)\rangle\ .$$

- Lorentz Inv. + Parity $\Rightarrow$ $\langle 0|\,\bar b\gamma^\mu u\,|B(p)\rangle = 0.$ Similarly $\langle 0|\,\bar b\gamma^\mu\gamma^5 u\,|B(p)\rangle = if_B p^\mu$.
- All QCD effects are contained in a single constant, f_B, the B-meson's *(leptonic) decay constant.* ($f_\pi \simeq 132$ MeV)
- Calculations such as these enable the determination of CKM matrix elements, e.g.

$$\Gamma(B^-\to\ell^-\bar\nu_\ell) = \frac{G_F^2\,|V_{ub}|^2 f_B^2}{8\pi}\,m_B\,m_\ell^2\left(1-\frac{m_\ell^2}{m_B^2}\right)^2 .$$

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- The property of asymptotic freedom $\Rightarrow$ quark and gluon interactions become weak at short distances, i.e. distances $\ll 1$ fm.

  2004 Nobel prize in physics to Gross, Politzer and Wilczek.

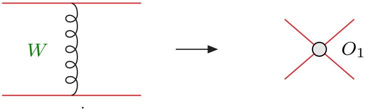
- Thus at short distances we can use perturbation theory.
- Schematically weak decay amplitudes are frequently organised as follows:

$$\mathscr{A}_{i\to f} = \sum_j C_j(\mu)\,\langle f\,|\,O_j(0)\,|\,i\rangle_\mu$$

  where

  - The C_j contain the short-distance effects and are calculable in perturbation theory;
  - the long-distance *non-perturbative* effects are contained in the matrix elements of composite local operators $\{O_i(0)\}$ which are the quantities which are computed in lattice QCD simulations;
  - the renormalization scale μ can be viewed as the scale at which we separate the short-distances from long-distances.

- Quarks interact strongly ⇒ we have to consider QCD effects even in weak processes.
- Our inability to control (non-perturbative) QCD effects is generally the largest systematic error in attempts to obtain fundamental information from experimental studies of weak processes!
- Tree-Level:

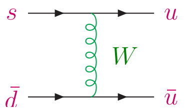


- Since $M_W \simeq 80\,$GeV, at low energies the momentum in the W-boson is much smaller than its mass ⇒ the four quark interaction can be approximated by the local Fermi β-decay vertex with coupling

$$\frac{G_F}{\sqrt{2}} = \frac{g_2^2}{8M_W^2}\ .$$

- *Asymptotic Freedom* $\Rightarrow$ we can treat QCD effects at short distances, $|x| \ll \Lambda_{QCD}^{-1}$ ($|x| < 0.1$ fm say) or corresponding momenta $|p| \gg \Lambda_{QCD}$ ($|p| > 2$ GeV say), using perturbation theory.
- The natural scale of strong interaction physics is of $O(1\ \text{fm})$ however, and so in general, and for most of the processes discussed here, non-perturbative techniques must be used.
- For illustration consider $K \to \pi\pi$ decays, for which the tree-level amplitude is proportional to

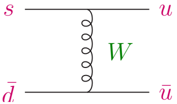
$$\frac{G_F}{\sqrt{2}}\, V_{ud}^* V_{us}\, \langle \pi\pi | (\bar{d}\gamma^\mu (1-\gamma^5)u)\, (\bar{u}\gamma_\mu (1-\gamma^5)s) | K\rangle \ .$$

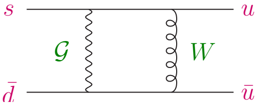


- We therefore need to determine the matrix element of the operator

$$O_1 = (\bar{d}\gamma^\mu (1-\gamma^5)u)\, (\bar{u}\gamma_\mu (1-\gamma^5)s) \ .$$

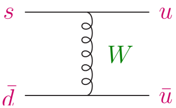
OPEs and Effective Hamiltonians (Cont.)

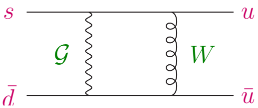




- Gluonic corrections generate a second operator $(\bar{d}T^a\gamma^\mu(1-\gamma^5)u)\,(\bar{u}T^a\gamma_\mu(1-\gamma^5)s)$, which by using Fierz Identities can be written as a linear combination of O_1 and O_2 where
$$O_2 = (\bar{d}\gamma^\mu(1-\gamma^5)s)\,(\bar{u}\gamma_\mu(1-\gamma^5)u)\,.$$
- OPE $\Rightarrow$ the amplitude for a weak decay process can be written as
$$A_{if} = \frac{G_F}{\sqrt{2}}\,V_{CKM}\sum_i C_i(\mu)\langle f|\,O_i(\mu)\,|i\rangle\ .$$
- μ is the renormalization scale at which the operators O_i are defined.
- Non-perturbative QCD effects are contained in the matrix elements of the O_i, which are independent of the large momentum scale, in this case of M_W.
- The Wilson coefficient functions $C_i(\mu)$ are independent of the states i and f and are calculated in perturbation theory.
- Since physical amplitudes manifestly do not depend on μ, the μ-dependence in the operators $O_i(\mu)$ is cancelled by that in the coefficient functions $C_i(\mu)$.

Towards more insight into the structure of the OPE.



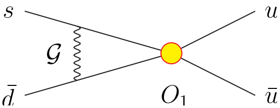


- For large loop-momenta k the right-hand graph is ultra-violet convergent:

$$\int_{k\ \text{large}} \frac{1}{k}\ \frac{1}{k}\ \frac{1}{k^2}\ \frac{1}{k^2 - M_W^2}\ d^4k\ ,$$

  ($1/k$ for each quark propagator and $1/k^2$ for the gluon propagator.)
  We see that there is a term $\sim \log(M_W^2/p^2)$, where p is some infra-red scale.

- In the OPE we do not have the W-propagator.



$$\text{Power Counting :} \quad \int_{k\ \text{large}} \frac{1}{k}\ \frac{1}{k}\ \frac{1}{k^2}\ d^4k \quad \Rightarrow \quad \textbf{divergence} \quad \Rightarrow \quad \mu\text{−dependence.}$$

- Infra-dependence is the same as in the full field-theory.

$$\log\left(\frac{M_W^2}{p^2}\right) = \log\left(\frac{M_W^2}{\mu^2}\right) + \log\left(\frac{\mu^2}{p^2}\right)$$

- The ir physics is contained in the matrix elements of the operators and the uv physics in the coefficient functions:

$$\log\left(\frac{M_W^2}{\mu^2}\right) \quad \to \quad C_i(\mu)$$

$$\log\left(\frac{\mu^2}{p^2}\right) \quad \to \quad \text{matrix element of } O_i$$

- In practice, the matrix elements are computed in lattice simulations with an ultraviolet cut-off of 2 – 4 GeV. Thus we have to resum *large logarithms* of the form $\alpha_s^n \log^n(M_W^2/\mu^2)$ in the coefficient functions $\Rightarrow$ factors of the type

$$\left[\frac{\alpha_s(M_W)}{\alpha_s(\mu)}\right]^{\gamma_0/2\beta_0} .$$

$$\left[\frac{\alpha_s(M_W)}{\alpha_s(\mu)}\right]^{\gamma_0/2\beta_0}$$

- γ_0 is the one-loop contribution to the *anomalous dimension* of the operator (proportional to the coefficient of $\log(\mu^2/p^2)$ in the evaluation of the one-loop graph above) and β_0 is the first term in the β-function, $(\beta \equiv \partial g/\partial \ln(\mu) = -\beta_0 g^3/16\pi^2)$.
- In general when there is more than one operator contributing to the right hand side of the OPE, the mixing of the operators $\Rightarrow$ matrix equations.
- The factor above represents the sum of the *leading logarithms*, i.e. the sum of the terms $\alpha_s^n \log^n(M_W^2/\mu^2)$. For almost all the important processes, the first (or even higher) corrections have also been evaluated.
- These days, for most processes of interest, the perturbative calculations have been performed to several loops (2,3,4), $\mathrm{N}^n\mathrm{LO}$ calculations.

- The *effective Hamiltonian* for weak decays takes the form

$$\mathscr{H}_{\rm eff} \equiv \frac{G_F}{\sqrt{2}} V_{CKM} \sum_i C_i(\mu)\, O_i(\mu)\ .$$

- We shall see below that for some important physical quantities (e.g. ε'/ε), there may be as many as ten operators, whose matrix elements have to be estimated.
- Lattice simulations enable us to evaluate the matrix elements non-perturbatively.
- In weak decays the large scale, M_W, is of course fixed. For other processes, most notably for deep-inelastic lepton-hadron scattering, the OPE is useful in computing the behaviour of the amplitudes with the large scale (e.g. with the momentum transfer).

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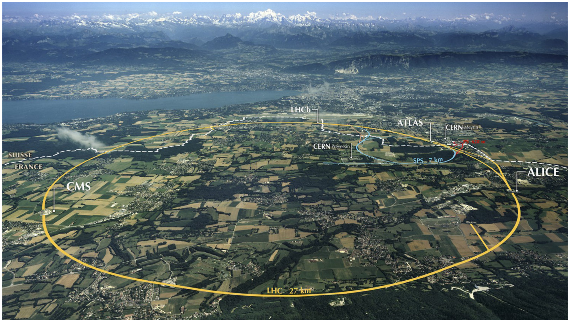
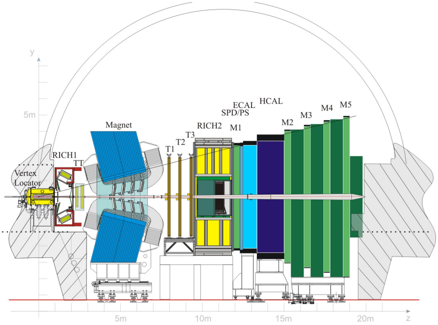
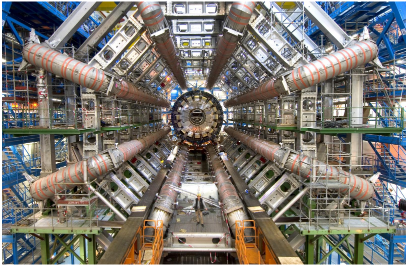


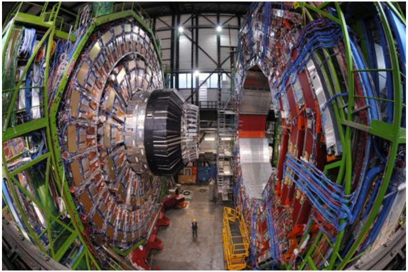
- The LHC@CERN is the world's highest energy accelerator, with proton-proton collisions @ 13 TeV.
- The LHC and experiments are now being upgraded in *Long Shutdown 2* and will start running again in 2021.

- There are 4 main detectors at the LHC, two *general purpose detectors* ATLAS and CMS (in which the Higgs Boson was discovered in 2012), as well as LHCb (focussed on flavour physics in general and B-physics in particular) and ALICE (focussed on heavy-ion physics).



- LHCb is a forward detector, designed to observe the decays of b-quarks and increasingly of the decay products the charm (and strange) quarks.
  - The b-quarks are produced and decay largely in a direction close to the beam.
- Since the LHC is a high-energy pp collider, the final-state particles include all B-hadrons (B and B_s mesons, Λ_b-baryon, etc.)

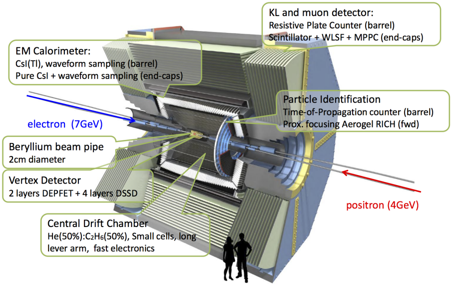




- Although CMS and ATLAS were not optimised for heavy-flavour physics, they are now also producing important results.
- There was an interesting CMS workshop in London in May, entitled “Finding new physics with 10 billion b-hadrons".

UNIVERSITY OF Southampton

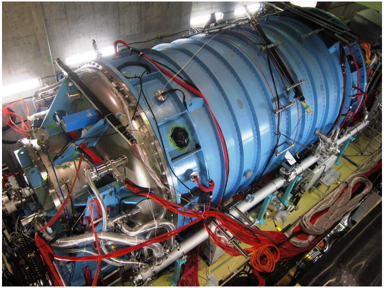
Belle II Detector



- The asymmetric e^+e^- B-factories BaBar and Belle made many fundamental discoveries including that of CP-violation in the B-system.
- Such studies will continue with the BELLE II detector at SuperKEKB which has just began taking data.
- Most of the data will be taken at the $\Upsilon(4s)$ resonance which is just above the B-$\bar{B}$ threshold.

$$m_{\Upsilon(4s)} = 10.5794(12)\,\text{GeV}\,, \quad 2m_{B^0} = 10.55928(26)\,\text{GeV}\,, \quad m_{\Upsilon(3s)} = 10.3552(5)\,\text{GeV}\,.$$

- The beams are asymmetric in energy (providing a boost to the centre-of mass frame) to allow for time-dependent CP-asymmetries to be studied.
- The Belle II Physics Book provides a good explanation of many aspects of B-physics. E.Kou et al., arXiv:1808.10567 (690 Pages!)



- The primary aim of the new NA62 (CERN) and KOTO (J-Parc) experiments is to measure rates for the rare-kaon decays $K^+ \to \pi^+ \nu\bar{\nu}$ and $K_L \to \pi^0 \nu\bar{\nu}$ respectively. Specifically:
  - NA62 hoped to have 100 events before LS2, but will now have to wait until 2021.
  - It would be wonderful if KOTO were to see any events

$$B(K_L \to \pi^0 \nu\bar{\nu})_{\rm th} \simeq 3.0 \pm 0.3 \times 10^{-11}, \quad B(K_L \to \pi^0 \nu\bar{\nu})_{\rm exp} \leq 2.6 \times 10^{-8}.$$

10^{-11} sensitivity expected in the late 2020s.

BES III

- BEPC2 is a symmetric e^+e^- collider with energies in the charm region.
- The BES III detector has a wide-ranging programme including in charm and τ physics and in tests of QCD.

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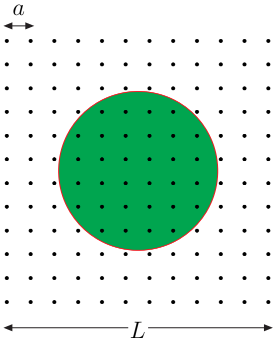
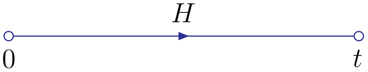


- Lattice phenomenology starts with the evaluation of correlation functions of the form:

$$\langle 0|\,O(x_1,x_2,\cdots,x_n)\,|0\rangle \;=\; \frac{1}{Z}\int [dA_\mu]\,[d\psi]\,[d\bar\psi]\,e^{-S}\,O(x_1,x_2,\cdots,x_n)\;,$$

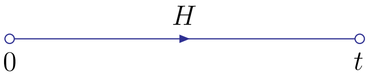
where $O(x_1,x_2,\cdots,x_n)$ is a multilocal operator composed of quark and gluon fields and Z is the partition function.

- The physics which can be studied depends on the choice of the multilocal operator O. For example:



- The functional integral is performed by discretising Euclidean space-time and using Monte-Carlo Integration.

Two-Point Correlation Functions



$$C_2(t) = \int d^3x\, e^{i\vec{p}\cdot\vec{x}}\, \langle 0|\, \phi(\vec{x},t)\, \phi^\dagger(\vec{0},0)\, |0\rangle$$

$$= \sum_n \int d^3x\, e^{i\vec{p}\cdot\vec{x}}\, \langle 0|\, \phi(\vec{x},t)\, |n\rangle\, \langle n|\phi^\dagger(\vec{0},0)\, |0\rangle$$

$$= \int d^3x e^{i\vec{p}\cdot\vec{x}}\, \langle 0|\, \phi(\vec{x},t)\, |H\rangle\, \langle H|\phi^\dagger(\vec{0},0)\, |0\rangle + \cdots$$

$$= \frac{1}{2E}\, e^{-iEt}\, \left|\langle 0|\phi(\vec{0},0)|H(p)\rangle\right|^2 + \cdots \Rightarrow \frac{1}{2E}\, e^{-Et}\, \left|\langle 0|\phi(\vec{0},0)|H(p)\rangle\right|^2 + \cdots \quad \text{(Euclidean)}$$

where $E = \sqrt{m_H^2 + \vec{p}^{\,2}}$ and we have taken H to be the lightest state created by $\phi^\dagger$. The $\cdots$ represent contributions from heavier states.

- By fitting $C(t)$ to the form above, both the energy (or, if $\vec{p} = 0$, the mass) and the modulus of the matrix element $\left|\langle 0|\phi(\vec{0},0)|H(p)\rangle\right|$ can be evaluated.
- Example: if $\phi = \bar{b}\,\gamma^\mu\gamma^5 u$ then the decay constant of the B-meson can be evaluated, $\left|\langle 0|\,\bar{b}\,\gamma^\mu\gamma^5 u\,|B^+(p)\rangle\right| = f_B\, p^\mu$.

- In Lattice QCD, while it is natural to think in terms of the lattice spacing a, the input parameter is $\beta = 6/g^2(a)$. *Dimensional Transmutation.*
- For illustration, imagine performing a simulation with $N_f = 2+1$ with $m_{ud} = m_u = m_d$ around their “physical" values.
- At each β, take two dimensionless quantities, e.g. m_π/m_Ω and m_K/m_Ω, and find the bare quark masses m_{ud} and m_s which give the corresponding physical values.These are then defined to be the physical (bare) quark masses at that β.
- Now consider a dimensionful quantity, e.g. m_Ω. The value of the lattice spacing is defined by

$$a^{-1} = \frac{1.672\,\text{GeV}}{am_\Omega(\beta, m_{ud}, m_s)}$$

  where $am_\Omega(\beta, m_{ud}, m_s)$ is the computed value in lattice units.
- Other physical quantities computed at the physical bare-quark masses will now differ from their physical values by artefacts of $O(a^2)$.
- Repeating this procedure at different β defines a scaling trajectory. Other choices for the 3 physical quantities used to define the trajectory are clearly possible.
- If the simulations are performed with m_c and/or $m_u \neq m_d$ then the procedure has to be extended accordingly.

- For each parameter of QCD (quark mass and lattice spacing) we need to sacrifice a prediction of a physical quantity.
  - In the above example we had m_{ud}, m_s and the lattice spacing a and we used m_π, m_K and m_Ω for *calibration* of the lattice.
- Note that different groups use different physical quantities for calibration, and this needs to be taken into account when making comparisons of results.

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- The quark masses $m_q(a)$ and QCD coupling constant $g(a)$ obtained as above are bare parameters with a^{-1} as the ultraviolet cut-off and with some formulation of discretised QCD as the bare theory.
- In perturbative calculations it is particularly convenient, and therefore conventional, to use the $\overline{\mathrm{MS}}$ renormalisation scheme.
  - Note that the $\overline{\mathrm{MS}}$ scheme is purely perturbative; we cannot perform simulations in $4+2\varepsilon$ dimensions.
  - Originally, providing both a^{-1} and μ are sufficiently large, renormalised quantities in the $\overline{\mathrm{MS}}$ scheme were obtained from the bare lattice ones using perturbation theory, e.g.

$$m^{\overline{\mathrm{MS}}}(\mu) = Z_m(a\mu)\, m^{\mathrm{latt}}(a)\,.$$

  - However, lattice perturbation theory frequently converges slowly (e.g. partly because of tadpole diagrams) and is technically complicated, e.g. for a scalar propagator,

$$\frac{1}{k^2+m^2} \to \frac{1}{\sum_\mu \{\frac{4}{a^2}\sin^2\frac{k_\mu a}{2}\} + m^2}\,.$$

$\Rightarrow$ *Non-perturbative renormalisation*

A General Method for Nonperturbative Renormalisation of Lattice Operators

G.Martinelli, C.Pittori, CTS, M.Testa and A.Vladikas, Nucl. Phys. B445 (1995) 81

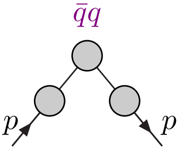
More details on NPR in Rainer Sommer's lectures.

- There are finite operators, such as V_μ and A_μ whose normalisation is fixed by Ward Identities (also Z_S/Z_P).
- Consider an operator O, which depends on the scale a, but which does not mix under renormalization with other operators:

$$O_R(\mu) = Z_O(\mu a)\, O_{\text{Latt}}(a)\,.$$

  The task is to determine Z_O .

- In the Rome-Southampton RI-Mom scheme, we impose that the matrix element of the operator between parton states, in the Landau gauge say, is equal to the tree level value for some specified external momenta.
  - These external momenta correspond to the renormalisation scale.
- I will illustrate the idea by considering the scalar density $S = \bar{q}q$.
  - Since $m_q(\bar{q}q)$ does not need renormalization, $Z_m Z_S = 1$, so from the determination of Z_S we obtain Z_m.



(i) Fix the gauge (to the Landau gauge say).

(ii) Evaluate the unamputated Green function:

$$G(x,y) = \langle 0 \,|u(x)\,[\bar{u}(0)d(0)]\,\bar{d}(y)\,|\,0\rangle$$

and Fourier transform to momentum space, at momentum p as in the diagram, $\Rightarrow\; G(p)$.

(iii) Amputate the Green function:

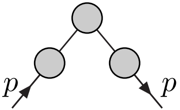
$$\Pi^{ij}_{S,\alpha\beta}(p) = S^{-1}(p)\,G(p)\,S^{-1}(p)\,,$$

where α,β (i,j) are spinor (colour) indices.

At tree level $\Pi^{ij}_{\alpha\beta}(p) = \delta_{\alpha\beta}\delta^{ij}$ and it is convenient to define

$$\Lambda_S(p) = \frac{1}{12}\,\mathrm{Tr}\,[\Pi_S(p)\,I]\,,$$

so that at tree-level $\Lambda_S = 1$.



- So far we have calculated the amputated Green function, in diagrammatic language, we have calculated the one-particle irreducible vertex diagrams.
- In order to determine the renormalization constant we need to multiply by $\sqrt{Z_q}$ for each external quark (i.e. there are two such factors).

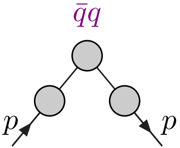
(iv) We now evaluate Z_q. There are a number of ways of doing this, perhaps the best is to use the non-renormalization of the conserved vector current:

$$Z_q \Lambda_{V_C} = 1 \quad \text{where} \quad \Lambda_{V_C} = \frac{1}{48}\text{Tr}\,[\Pi_{V_C^\mu}(p)\,\gamma^\mu]\,.$$

- This is equivalent to the definition

$$Z_q = -\frac{i}{48}\,\text{tr}\left(\gamma_\rho \frac{\partial S_{\text{latt}}^{-1}}{\partial p_\rho}\right)$$

at $p^2 = \mu^2$.



- We now have all the ingredients necessary to impose the renormalization condition. We define the renormalized scalar density S_R by $S_R(\mu) = Z_S(\mu a) S_{\rm Latt}(a)$ where

$$Z_S \, \frac{\Lambda_S(p)}{\Lambda_{V_C}(p)} = 1\,,$$

for $p^2 = \mu^2$.

- The scalar density has a non-zero anomalous dimension and therefore Z_S depends on the scale μ.
- The renormalization scheme here is a MOM scheme. We called it the RI-MOM scheme, where the *RI* stands for *Regularization Independent* to underline the fact that the renormalized operators do not depend on the bare theory (i.e. the lattice theory).

- For any renormalisation method we require a delicate window for the momenta, ideally:

$$p \gg \Lambda_{\rm QCD} \qquad \text{and} \qquad p \ll a^{-1}\,.$$

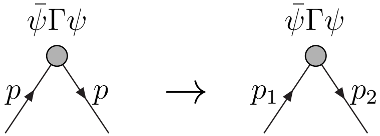
- $p \gg \Lambda_{\rm QCD}$ is required in order for perturbation theory to be applicable, so that the results can be combined with the Wilson Coefficient functions, or to translate the results into the $\overline{\rm MS}$ scheme.
- $p \ll a^{-1}$ to keep the lattice artefacts small.
- *Step Scaling*, which I won't discuss, allows us, in principle, to relax the first condition, $p \gg \Lambda_{\rm QCD}$.
  - The method is being used by a number of the large collaborations.
- One, unattractive feature however, is that we need to fix the gauge. When considering operator mixing, we have to consider non-gauge invariant (but BRST invariant) operators. See Rainer Sommer's lectures for gauge-invariant NPR.

- The RI-Mom Scheme was defined at an *exceptional* momentum, i.e. with a channel with a small (zero) momentum.
  Thus the matrix elements carry information about the physical mass spectrum which is unaccessible to perturbation theory.
- Although we showed in the RS paper that these non-perturbative effects are suppressed by powers of p^2, there is growing evidence that they present a numerical contamination which, as we strive for greater precision, should be evaluated.
- An extreme example is the pseudoscalar density where there are non-perturbative effects of the form

$$\frac{\langle 0|\bar{\psi}\psi|0\rangle}{mp^2}$$

so that it is not possible to go to the chiral limit.

Renormalization of quark bilinear operators in a MOM-scheme with a non-exceptional subtraction point, C.Sturm, Y.Aoki, N.H.Christ, T.Izubuchi, CTS, and A.Soni; arXiv:0901.2599 [hep-ph]



$$p_1^2 = p_2^2 = (p_1 - p_2)^2$$

- In this paper we develop the scheme with the non-exceptional subtraction point

$$p_1^2 = p_2^2 = (p_1 - p_2)^2 \,.$$

- We calculate the one-loop conversion factors between this scheme and the $\overline{\mathrm{MS}}$ scheme. This is entirely a continuum exercise.

- An important requirement is that the chiral Ward Identities are satisfied by the renormalized quantities.
- The wave-function renormalization is fixed by imposing the RI′-MOM condition

$$\frac{1}{12p^2}\mathrm{tr}[S_R^{-1}(p)\,\not{p}] = -1\,.$$

The definition of S differs by factors of i w.r.t. the Rome-Southampton paper.

- The renormalization conditions on S and P are

$$\frac{1}{12}\mathrm{tr}[\Lambda_{P,R}(p_1,p_2)] = 1 \quad \text{and} \quad \frac{1}{12i}\mathrm{tr}[\Lambda_{P,R}(p_1,p_2)\gamma_5] = 1\,.$$

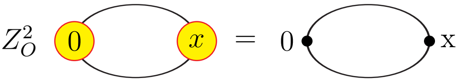
These conditions respect the chiral symmetry between S and P (e.g. the matching factors to $\overline{\mathrm{MS}}$ are the same.

- In order to preserve the WI, and in particular that $m\bar{\psi}\psi$ remains unrenormalized, we impose the mass renormalization condition

$$\frac{1}{12m_R}\mathrm{tr}[S_R^{-1}(p)] = 1 + \frac{1}{24m_R}\mathrm{tr}[q_\mu \Lambda^\mu_{A,R}(p_1,p_2)\gamma_5]\,.$$

Gauge Invariant NPR

- There are also investigations about the best way to perform NPR in a gauge invariant way.
- One possibility is to compute correlation functions at short distances in configuration space and require that the renormalised operators give the lowest order perturbative contribution.



  where the yellow circles represent the insertion of the lattice operator O_{Latt} and the right-hand diagram represents the lowest-order diagram in perturbation theory.
- The renormalization scale is now $1/|x|$, and the same constraints on the values of $1/x^2$ hold here as for the momenta in the RI-Mom scheme.
- The Alpha collaboration (and others) has been implementing a gauge invariant NPR, based on the use of the Schrödinger Functional.

One last point!

- Since we cannot perform simulations with lattice spacings $< 1/M_W$ or $1/m_t$ we exploit the standard technique of the Operator Product Expansion and write schematically:

$$\text{Physics} = \sum_i C_i(\mu) \times \langle f|O_i(\mu)|i\rangle\,.$$

- Until relatively recently, the Wilson coefficients $C_i(\mu)$ were typically calculated with much greater precision than our knowledge of the matrix elements.
  - The C_i are typically calculated in schemes based on dimensional regularisation (such as $\overline{\text{MS}}$) which are intrinsically perturbative.
  - We have seen that we can compute the matrix elements non-perturbatively, with the operators renormalised in schemes which have a non-perturbative definition (such as RI-MOM schemes) but not in purely perturbative schemes based on dim.reg.
- Thus the determination of the C_i in $\overline{\text{MS}}$-like schemes is not the complete perturbative calculation. Matching between $\overline{\text{MS}}$ and non-perturbatively defined schemes must also be performed.
  - This is beginning to be done.
  - We are now careful to present tables of matrix elements of operators renormalized in RI-MOM schemes, which can be used to gain better precision once improved perturbative calculations are performed.

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- Most of the compilations in this talk are taken from the results of the 2019 FLAG collaboration: “Review of lattice results concerning low energy particle physics,”
  S. Aoki + 34 authors arXiv:1902.08191 (537 pages!)

- This fourth edition is an extension and continuation of the work started by Flavianet Lattice Averaging Group:
  G. Colangelo, S. Dürr, A. Juttner, L. Lellouch, H. Leutwyler, V. Lubicz, S. Necco, C. T. Sachrajda, S. Simula, A. Vladikas, U. Wenger, H. Wittig arXiv:1011.4408

- Motivation - to present to the wider community an *average* of lattice results for important quantities obtained after a critical expert review.
- Danger - It is important that original papers (particularly those which pioneer new techniques) get recognised and cited appropriately by the community.
- The closing date for arXiv:1902.08191.00299 was Sep 30th 2018.

Quantity	■	$N_f = 2+1+1$	■	$N_f = 2+1$	■	$N_f = 2$
m_s(MeV)	4	$93.44(68)$	6	$92.03(88)$	2	$101(3)$
m_{ud}(MeV)	2	$3.410(43)$	5	$3.364(41)$	1	$3.6(2)$
m_s/m_{ud}	3	$27.23(10)$	4	$27.42(12)$	1	$27.3(9)$
m_d(MeV)	1	$4.88(20)$	1	$4.67(9)$	1	$4.8(23)$
m_u(MeV)	1	$2.50(17)$	1	$2.27(9)$	1	$2.40(23)$
m_u/m_d	1	$0.513(31)$	1	$0.485(19)$	1	$0.50(4)$
$f_+^{K\pi}(0)$	2	$0.9706(27)$	2	$0.9677(27)$	1	$0.9560(57)(62)$
f_{K^+}/f_{π^+}	3	$1.1932(19)$	6	$1.1917(37)$	1	$1.205(18)$
f_K(MeV)	3	$155.7(3)$	3	$155.7(7)$	1	$157.5(2.4)$
f_π(MeV)			3	$130.2(8)$		
$\hat{B}_K$	1	$0.717(18)(16)$	4	$0.7625(97)$	1	$0.727(22)(12)$

FLAG summary in heavy-quark physics

Quantity	■	$N_f=2+1+1$	■	$N_f=2+1$	■	$N_f=2$
$\bar{m}_c(3\,\mathrm{GeV})$ (GeV)	5	$0.988(7)$	3	$0.992(6)$		
m_c/m_s	3	$11.768(33)$	2	$11.82(16)$		
$\bar{m}_b(\bar{m}_b)(GeV)$	5	$4.198(12)$	1	$4.164(23)$		
f_D(MeV)	2	$212.0(7)$	2	$209.0(2.4)$	1	$208(7)$
f_{D_s}(MeV)	2	$249.9(5)$	4	$248.0(1.6)$	2	$242.5(5.8)$
f_{D_s}/f_D	2	$1.1783(16)$	3	$1.174(7)$	1	$1.20(2)$
$f_+^{D\pi}(0)$	1	$0.612(35)$	1	$0.666(29)$		
$f_+^{DK}(0)$	1	$0.765(31)$	1	$0.747(19)$		
f_B(MeV)	4	$190.0(1.3)$	5	$192.0(4.3)$	2	$188(7)$
f_{B_s}(MeV)	4	$230.3(1.3)$	5	$228.4(3.7)$	2	$227(7)$
f_{B_s}/f_B	4	$1.209(5)$	5	$1.201(16)$	2	$1.206(23)$
$f_{B_d}\sqrt{\hat{B}_{B_d}}$(MeV)			3	$225(9)$	1	$216(10)$
$f_{B_s}\sqrt{\hat{B}_{B_s}}$(MeV)			3	$274(8)$	1	$262(10)$
$\hat{B}_{B_d}$			3	$1.30(10)$	1	$1.30(6)$
$\hat{B}_{B_s}$			3	$1.35(6)$	1	$1.32(5)$
ξ			2	$1.206(17)$	1	$1.225(31)$
$\hat{B}_{B_s}/\hat{B}_{B_d}$			2	$1.032(38)$	1	$1.007(21)$

$$\alpha^{(5)}_{\overline{\rm MS}}(M_Z) = 0.11823(81) \quad \text{from 7 papers}$$
$$\Lambda^{(5)}_{\overline{\rm MS}} = 211(10)\,\text{MeV} \quad \text{from 7 papers}$$