Lattice Flavour Physics: Lecture 2

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LEVERHULME TRUST

UNIVERSITY OF Southampton

- SM incomplete. Two complementary approaches to searching for new physics:
  1. Search for new particles at LHC and elsewhere;
  2. Precision flavour physics.
- The charged-current interactions are of the form

$$J_\mu^+ = (\bar{u},\bar{c},\bar{t})_L \gamma_\mu V_{\rm CKM} \begin{pmatrix} d \\ s \\ b \end{pmatrix}_L \equiv (\bar{u},\bar{c},\bar{t})_L \gamma_\mu \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}_L$$

  - $V_{\rm CKM}$ has four parameters $\lambda, A, \bar{\rho}, \bar{\eta}$.
  - Huge number of processes to over-determine these four parameters and search for inconsistencies.
- Precision flavour physics requires good quantitative control of non-perturbative QCD effects $\Rightarrow$ Lattice QCD.

$$A_{if} = \frac{G_F}{\sqrt{2}} V_{CKM} \sum_i C_i(\mu) \langle f|\,O_i(\mu)\,|i\rangle \ .$$

- I am not going to speak to all the slides on renormalisation posted at lecture 1.

  The details will be provided by R.Sommer

- Consider

$$A_{if} = \frac{G_F}{\sqrt{2}} V_{CKM} \sum_i C_i(\mu) \langle f | O_i(\mu) | i \rangle \ .$$

  - The Wilson coefficient functions C_i must correspond to the same renormalisation scheme (definition) as the O_i.
  - The standard renormalisation schemes for perturbative calculations are based on *dimensional regularisation* ($\overline{\text{MS}}\ \cdots$) and these are purely perturbative; we cannot perform simulations in $4+2\varepsilon$ dimensions.
  - In principle, providing both a^{-1} and μ are sufficiently large, the $O_i^{\overline{\text{MS}}}(\mu)$ can be obtained from bare lattice operators $O_i^{\text{Latt}}(a)$ using perturbation theory.
  - However, lattice perturbation theory frequently converges slowly (e.g. partly because of tadpole diagrams) and is technically complicated, e.g. for a scalar propagator,

$$\frac{1}{k^2+m^2} \to \frac{1}{\sum_\mu \{\frac{4}{a^2}\sin^2\frac{k_\mu a}{2}\} + m^2} \ .$$

$\Rightarrow$ *Non-perturbative renormalisation*

A General Method for Nonperturbative Renormalisation of Lattice Operators

G.Martinelli, C.Pittori, CTS, M.Testa and A.Vladikas, Nucl. Phys. B445 (1995) 81

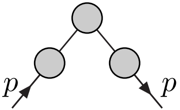
More details on NPR in Rainer Sommer's lectures.

- Consider an operator O, which depends on the scale a, but which does not mix under renormalization with other operators:

$$O_R(\mu) = Z_O(\mu a)\, O_{\rm Latt}(a)\,.$$

  The task is to determine Z_O .

- In the Rome-Southampton RI-Mom scheme, we impose that the matrix element of the operator between parton states, in the Landau gauge say, is equal to the tree level value for some specified external momenta.
  - These external momenta correspond to the renormalisation scale.
- I will illustrate the idea by considering the scalar density $S = \bar{q}q$.
  - Since $m_q(\bar{q}q)$ does not need renormalization, $Z_m Z_S = 1$, so from the determination of Z_S we obtain Z_m.



(i) Fix the gauge (to the Landau gauge say).

(ii) Evaluate the unamputated Green function:

$$G(x,y) = \langle 0 \,|\, u(x)\, [\bar{u}(0) d(0)]\, \bar{d}(y) \,|\, 0 \rangle$$

and Fourier transform to momentum space, at momentum p as in the diagram, $\Rightarrow\; G(p)$.

(iii) Amputate the Green function:

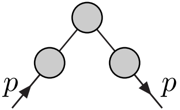
$$\Pi^{ij}_{S,\alpha\beta}(p) = S^{-1}(p)\, G(p)\, S^{-1}(p)\,,$$

where α,β (i,j) are spinor (colour) indices.

At tree level $\Pi^{ij}_{\alpha\beta}(p) = \delta_{\alpha\beta}\delta^{ij}$ and it is convenient to define

$$\Lambda_S(p) = \frac{1}{12}\,\mathrm{Tr}\,[\Pi_S(p)\, I]\,,$$

so that at tree-level $\Lambda_S = 1$.



- So far we have calculated the amputated Green function, in diagrammatic language, we have calculated the one-particle irreducible vertex diagrams.
- In order to determine the renormalization constant we need to multiply by $\sqrt{Z_q}$ for each external quark (i.e. there are two such factors).

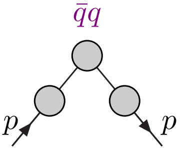
(iv) We now evaluate Z_q. There are a number of ways of doing this, perhaps the best is to use the non-renormalization of the conserved vector current:

$$Z_q \Lambda_{V_C} = 1 \quad \text{where} \quad \Lambda_{V_C} = \frac{1}{48} \mathrm{Tr}\left[\Pi_{V_C^\mu}(p)\, \gamma^\mu\right].$$

- This is equivalent to the definition

$$Z_q = -\frac{i}{48} \mathrm{tr}\left(\gamma_\rho \frac{\partial S_{\mathrm{latt}}^{-1}}{\partial p_\rho}\right)$$

at $p^2 = \mu^2$.



- We now have all the ingredients necessary to impose the renormalization condition. We define the renormalized scalar density S_R by $S_R(\mu) = Z_S(\mu a) S_{\text{Latt}}(a)$ where
$$Z_S \frac{\Lambda_S(p)}{\Lambda_{V_C}(p)} = 1\,,$$
for $p^2 = \mu^2$.
- The scalar density has a non-zero anomalous dimension and therefore Z_S depends on the scale μ.
- The renormalization scheme here is a MOM scheme. We called it the RI-MOM scheme, where the *RI* stands for *Regularization Independent* to underline the fact that the renormalized operators do not depend on the bare theory (i.e. the lattice theory).

- Since we cannot perform simulations with lattice spacings $< 1/M_W$ or $1/m_t$ we exploit the standard technique of the Operator Product Expansion and write schematically:
$$\text{Physics} = \sum_i C_i(\mu) \times \langle f|O_i(\mu)|i\rangle\,.$$
- Until relatively recently, the Wilson coefficients $C_i(\mu)$ were typically calculated with much greater precision than our knowledge of the matrix elements.
  - The C_i are typically calculated in schemes based on dimensional regularisation (such as $\overline{\text{MS}}$) which are intrinsically perturbative.
  - We have seen that we can compute the matrix elements non-perturbatively, with the operators renormalised in schemes which have a non-perturbative definition (such as RI-MOM schemes) but not in purely perturbative schemes based on dim.reg.
- Thus the determination of the C_i in $\overline{\text{MS}}$-like schemes is not the complete perturbative calculation. Matching between $\overline{\text{MS}}$ and non-perturbatively defined schemes must also be performed.
  - This is beginning to be done.
  - We are now careful to present tables of matrix elements of operators renormalized in RI-MOM schemes, which can be used to gain better precision once improved perturbative calculations are performed.

- Most of the compilations in this talk are taken from the results of the 2019 FLAG collaboration: “Review of lattice results concerning low energy particle physics,”
  S. Aoki + 34 authors arXiv:1902.08191 (537 pages!)

- This fourth edition is an extension and continuation of the work started by Flavianet Lattice Averaging Group:
  G. Colangelo, S. Dürr, A. Juttner, L. Lellouch, H. Leutwyler, V. Lubicz, S. Necco, C. T. Sachrajda, S. Simula, A. Vladikas, U. Wenger, H. Wittig arXiv:1011.4408

- Motivation - to present to the wider community an *average* of lattice results for important quantities obtained after a critical expert review.
- Danger - It is important that original papers (particularly those which pioneer new techniques) get recognised and cited appropriately by the community.
- The closing date for arXiv:1902.08191.00299 was Sep 30th 2018.

Quantity	■	$N_f=2+1+1$	■	$N_f=2+1$	■	$N_f=2$
m_s(MeV)	4	93.44(68)	6	92.03(88)	2	101(3)
m_{ud}(MeV)	2	3.410(43)	5	3.364(41)	1	3.6(2)
m_s/m_{ud}	3	27.23(10)	4	27.42(12)	1	27.3(9)
m_d(MeV)	1	4.88(20)	1	4.67(9)	1	4.8(23)
m_u(MeV)	1	2.50(17)	1	2.27(9)	1	2.40(23)
m_u/m_d	1	0.513(31)	1	0.485(19)	1	0.50(4)
$f_+^{K\pi}(0)$	2	0.9706(27)	2	0.9677(27)	1	0.9560(57)(62)
f_{K^+}/f_{π^+}	3	1.1932(19)	6	1.1917(37)	1	1.205(18)
f_K(MeV)	3	155.7(3)	3	155.7(7)	1	157.5(2.4)
f_π(MeV)			3	130.2(8)		
$\hat{B}_K$	1	0.717(18)(16)	4	0.7625(97)	1	0.727(22)(12)

Quantity	■	$N_f=2+1+1$	■	$N_f=2+1$	■	$N_f=2$
$\bar{m}_c(3\,\mathrm{GeV})$ (GeV)	5	$0.988(7)$	3	$0.992(6)$		
m_c/m_s	3	$11.768(33)$	2	$11.82(16)$		
$\bar{m}_b(\bar{m}_b)(GeV)$	5	$4.198(12)$	1	$4.164(23)$		
f_D(MeV)	2	$212.0(7)$	2	$209.0(2.4)$	1	$208(7)$
f_{D_s}(MeV)	2	$249.9(5)$	4	$248.0(1.6)$	2	$242.5(5.8)$
f_{D_s}/f_D	2	$1.1783(16)$	3	$1.174(7)$	1	$1.20(2)$
$f_+^{D\pi}(0)$	1	$0.612(35)$	1	$0.666(29)$		
$f_+^{DK}(0)$	1	$0.765(31)$	1	$0.747(19)$		
f_B(MeV)	4	$190.0(1.3)$	5	$192.0(4.3)$	2	$188(7)$
f_{B_s}(MeV)	4	$230.3(1.3)$	5	$228.4(3.7)$	2	$227(7)$
f_{B_s}/f_B	4	$1.209(5)$	5	$1.201(16)$	2	$1.206(23)$
$f_{B_d}\sqrt{\hat{B}_{B_d}}$(MeV)			3	$225(9)$	1	$216(10)$
$f_{B_s}\sqrt{\hat{B}_{B_s}}$(MeV)			3	$274(8)$	1	$262(10)$
$\hat{B}_{B_d}$			3	$1.30(10)$	1	$1.30(6)$
$\hat{B}_{B_s}$			3	$1.35(6)$	1	$1.32(5)$
ξ			2	$1.206(17)$	1	$1.225(31)$
$\hat{B}_{B_s}/\hat{B}_{B_d}$			2	$1.032(38)$	1	$1.007(21)$

$$\alpha^{(5)}_{\overline{\rm MS}}(M_Z) = 0.11823(81) \quad \text{from 7 papers}$$

$$\Lambda^{(5)}_{\overline{\rm MS}} = 211(10)\,\text{MeV} \quad \text{from 7 papers}$$

Mostly kaon physics

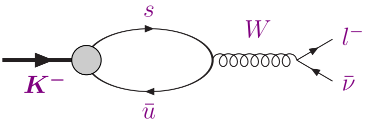
1. “Standard" physical quantities in kaon physics.
   - 1a) Leptonic and Semileptonic Kaon Decays
   - 1b) ε_K and Neutral Kaon Mixing
2. Long-distance contributions to weak processes
   - 2a) Long-distance contributions - Introduction to Theoretical Issues
   - 2b) Physics motivation
   - 2c) Generic issues in computing long-distance contributions
   - 2d) Status of RBC-UKQCD Calculations
3. Directly computing $K \to \pi\pi$ decays amplitudes

- By *standard quantities* I mean those for which the non-perturbative QCD effects are contained in matrix elements of the form:

$$\langle 0|\,O(0)\,|H\rangle \qquad \text{or} \qquad \langle H_2|\,O(0)\,|H_1\rangle$$

where $O(0)$ is a local composite operator and H, H_1, H_2 are single hadron states.

- We have already been introduced to the leptonic decays constants, e.g. for the kaon:



- Non-perturbative QCD effects are contained in the matrix element $\langle 0|\,\bar u\gamma^\mu(1-\gamma^5)s\,|K^-(p)\rangle$.
- Lorentz Inv. + Parity $\Rightarrow$ $\langle 0|\,\bar u\gamma^\mu s\,|K^-(p)\rangle = 0.$ Similarly $\langle 0|\,\bar u\gamma^\mu\gamma^5 s\,|K^-(p)\rangle = if_K p^\mu$.
- All QCD effects are contained in a single constant, f_K, the K-meson's *(leptonic) decay constant.* ($f_\pi \simeq 132$ MeV)
- Calculations such as these enable the determination of CKM matrix elements, e.g.

$$\Gamma(K^-\to\ell^-\bar\nu_\ell) = \frac{G_F^2\,|V_{us}|^2 f_K^2}{8\pi}\, m_K\, m_\ell^2\left(1-\frac{m_\ell^2}{m_K^2}\right)^2 .$$

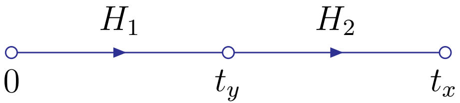
- From experimental measurements we have $\frac{|V_{us}f_K^\pm|}{|V_{ud}f_\pi^\pm|} = 0.2760(4)$

M.Moulson, arXiv:1704.04104; PDG 2016

Consider now a three-point correlation function of the form:

$$C_3(t_x,t_y) = \int d^3x\,d^3y\; e^{i\vec p\cdot\vec x}\; e^{i\vec q\cdot\vec y}\; \langle 0|\, J_2(\vec x,t_x)\, O(\vec y,t_y)\, J_1^\dagger(\vec 0,0)\,|0\rangle\ ,$$

where $J_{1,2}$ may be interpolating operators for different particles and we assume that $t_x > t_y > 0$.

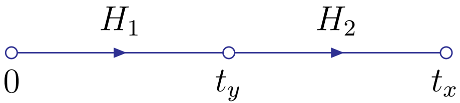


For sufficiently large times t_y and $t_x - t_y$

$$\begin{aligned} C_3(t_x,t_y) \quad &\simeq \quad \frac{e^{-E_1 t_y}}{2E_1}\;\frac{e^{-E_2(t_x-t_y)}}{2E_2}\;\langle 0|J_2(0)|H_2(\vec p)\rangle \\ &\times\langle H_2(\vec p)|O(0)|H_1(\vec p+\vec q)\rangle\;\langle H_1(\vec p+\vec q)|J_1^\dagger(0)|0\rangle\ , \end{aligned}$$

where $E_1^2 = m_1^2 + (\vec p+\vec q)^2$ and $E_2^2 = m_1^2 + \vec p^{\,2}$.

Three-Point Correlation Functions (Cont.)

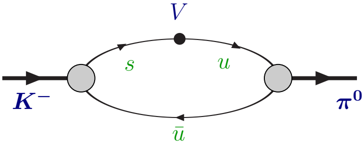


- From the evaluation of two-point functions we have the masses and the matrix elements of the form $|\langle 0|J|H(\vec{p})\rangle|$. Thus, from the evaluation of three-point functions we obtain matrix elements of the form $|\langle H_2|O|H_1\rangle|$.
- Important examples include:
  - $K^0-\bar{K}^0$ ($B^0-\bar{B}^0$) mixing. In this case

$$O=\bar{s}\gamma^\mu(1-\gamma^5)d\;\bar{s}\gamma_\mu(1-\gamma^5)d\,.$$

  - Semileptonic and rare radiative decays of hadrons of the form $B\to\pi,\rho+$leptons or $B\to K^*\gamma$. Now O is a quark bilinear operator such as $\bar{b}\gamma^\mu(1-\gamma^5)u$ or an *electroweak penguin* operator.

- V_{us} can be determined $K\to\pi\ell\bar\nu$ ($K_{\ell 3}$) decays.



- Space-Time symmetries allow us to parametrise the non-perturbative strong interaction effects in terms of invariant form-factors.
  For example, for $K\to\pi$ semileptonic decays ($K_{\ell 3}$ decays):

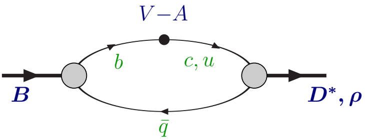
$$\langle\pi(p)|V^\mu|K(k)\rangle \;=\; f^+(q^2)\left[(p+k)^\mu-\frac{m_B^2-m_P^2}{q^2}\,q^\mu\right]+f^0(q^2)\,\frac{m_K^2-m_\pi^2}{q^2}\,q^\mu\,,$$

  where $q=k-p$.
- From experimental measurements we have

$$|V_{us}|f^+(0)=0.2165(4)\,.$$

M.Moulson, arXiv:1704.04104

- Heavier mesons can also decay semileptonically into vector mesons with four independent form-factors.



- For decays into a vector V $(=\rho, D^*$ for example), a conventional decomposition is

$$\langle V(k,\varepsilon)|V^\mu|B(p)\rangle = \frac{2V(q^2)}{m_B+m_V}\varepsilon^{\mu\gamma\delta\beta}\varepsilon^*_\beta p_\gamma k_\delta$$

$$\langle V(k,\varepsilon)|A^\mu|B(p)\rangle = i(m_B+m_V)A_1(q^2)\varepsilon^{*\mu} - i\frac{A_2(q^2)}{m_B+m_V}\varepsilon^*\cdot p\,(p+k)^\mu + i\frac{A(q^2)}{q^2}2m_V\varepsilon^*\cdot p\,q^\mu\ ,$$

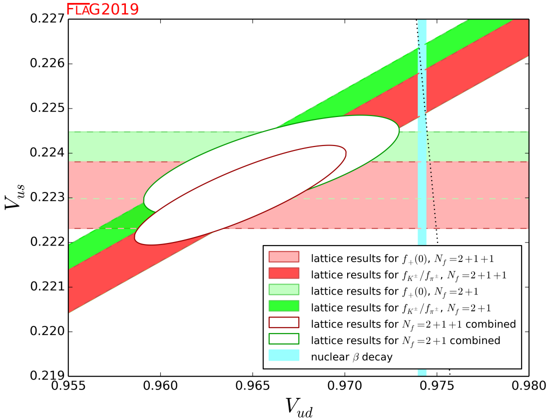
where ε is the polarisation vector of the final-state meson, and $q=p-k$.

$$\{A_3 = \frac{m_B+m_V}{2m_V}A_1 - \frac{m_B-m_V}{2m_V}A_2\ .\}$$

- FLAG2019 Averages:

N_F	V_{us}	V_{ud}	$f^+(0)$	f_K/f_π
2+1+1	0.2249(7)	0.97437(16)	0.9627(35)	1.196(3)
2+1	0.2249(5)	0.97438(12)	0.9627(28)	1.196(3)
2	0.2256(19)	0.97423(44)	0.9597(83)	1.192(9)

- The latest value for V_{ud} from superallowed nuclear β-decays is $|V_{ud}| = 0.97420(21)$. J. Hardy and I. S. Towner, arXiv:1807.01146
- **At this level of precision, isospin-breaking contributions, including electromagnetic effects, must be taken into account.**
  G.Martinelli's lectures at this school.
  - Parametrically, these are of $O(\alpha_{\text{em}})$ or $O((m_u - m_d)/\Lambda_{\text{QCD}})$, i.e. of $O(1\%)$ or so.
- The $N_F = 2+1+1$ results $\Rightarrow$ $|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 0.9797(74)$; a 2.7σ deviation from unitarity.
  - Taking instead V_{ud} from nuclear β-decays and $f^+(0)$ or f_K/f_π from the lattice results give $0.99884(53)$ and $0.99986(46)$ respectively.


FLAG2019
0.227
0.226
0.225
0.224
0.223
0.222
0.221
0.220
0.219
V_{us}
0.955
0.960
0.965
0.970
0.975
0.980
V_{ud}
lattice results for $f_+(0)$, $N_f = 2+1+1$
lattice results for $f_{K^\pm}/f_{\pi^\pm}$, $N_f = 2+1+1$
lattice results for $f_+(0)$, $N_f = 2+1$
lattice results for $f_{K^\pm}/f_{\pi^\pm}$, $N_f = 2+1$
lattice results for $N_f = 2+1+1$ combined
lattice results for $N_f = 2+1$ combined
nuclear β decay

$$\varepsilon_K = \frac{A(K_L \to (\pi\pi)_{I=0})}{A(K_S \to (\pi\pi)_{I=0})} = e^{i\phi_\varepsilon}\sin\phi_\varepsilon \left[\frac{\mathrm{Im}\langle \bar K^0\,|H_W^{\Delta S=2}\,|K^0\rangle}{\Delta m_K} + \text{L.D. effects}\right]$$

where

Buras, Guadagnoli, arXiv:0805.3887

$$\begin{aligned} |\varepsilon_K| &= 2.228(11)\times 10^{-3} \\ \phi_\varepsilon &= \arctan\frac{\Delta m_K}{\Delta\Gamma_K/2} = 43.52(5)^\circ \\ \Delta m_K &= m_{K_L} - m_{K_S} = 3.4839(59)\times 10^{-12}\,\text{MeV} \\ \Delta\Gamma_K &= \Gamma_S - \Gamma_L = 7.3382(33)\times 10^{-12}\,\text{MeV}. \end{aligned}$$

- It is conventional to present the short-distance contribution in terms of the B_K parameter:

$$\langle \bar K^0\,|H_W^{\Delta S=2}\,|K^0\rangle \propto \langle \bar K^0\,|(\bar s\gamma^\mu(1-\gamma^5)d)\,(\bar s\gamma_\mu(1-\gamma^5)d)|K^0\rangle \equiv \frac{8}{3} f_K^2 m_K^2 B_K(\mu)\,.$$

- Lattice calculations of B_K have been performed since the mid 1980s. The precision is now such that the $O(5\%)$ long-distance (LD) effects have to be considered.

Buras, Guadagnoli, Isidori arXiv:1002.3612

Quantity	■	$N_f = 2+1+1$	■	$N_f = 2+1$	■	$N_f = 2$
m_s(MeV)	4	93.44(68)	6	92.03(88)	2	101(3)
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- An important element of the motivation of the RBC/UKQCD collaborations study of the long-distance (distances $> 1/m_c$) is to avoid perturbation theory at the scale of m_c.
- The dominant contribution to $\varepsilon_K \propto |V_{cb}|^4$ and PDG(2019) quote $|V_{cb}| = (40.9 \pm 1.1) \times 10^{-3}$ error on B_K is no longer the dominant one.

- Beyond the standard model there are in general 5 independent operators which contribute neutral Kaon mixing:

$$\mathscr{H}^{\Delta S=2} = \sum_{i=1}^{5} C_i(\mu)\, Q_i^{\Delta S=2}(\mu)\,.$$

- The five operators are:

$$\begin{aligned}
Q_1^{\Delta S=2} &= [\bar{s}^i \gamma_\mu (1-\gamma_5) d^i]\,[\bar{s}^j \gamma_\mu (1-\gamma_5) d^j] \\
Q_2^{\Delta S=2} &= [\bar{s}^i (1-\gamma_5) d^i]\,[\bar{s}^j (1-\gamma_5) d^j] \\
Q_3^{\Delta S=2} &= [\bar{s}^i (1-\gamma_5) d^j]\,[\bar{s}^j (1-\gamma_5) d^i] \\
Q_4^{\Delta S=2} &= [\bar{s}^i (1-\gamma_5) d^i]\,[\bar{s}^j (1+\gamma_5) d^j] \\
Q_5^{\Delta S=2} &= [\bar{s}^i (1-\gamma_5) d^j]\,[\bar{s}^j (1+\gamma_5) d^i]
\end{aligned}$$

  - i,j are colour indices.
- The matrix elements can be calculated in a similar way to B_K and a review of the current results can be found in section 6.3 in FLAG2019.
- $Q_1^{\Delta S=2}$ transforms as (27,1) under SU(3)$_L\times$SU(3)$_R$, $Q_2^{\Delta S=2}$ and $Q_3^{\Delta S=2}$ as $(6,\bar{6})$ and $Q_4^{\Delta S=2}$ and $Q_5^{\Delta S=2}$ as (8,8) $\Rightarrow$ Renormalization matrix is block diagonal.

- In this section I briefly review the RBC-UKQCD collaborations' development of computations of long-distance contributions, i.e. matrix elements of the form

$$\int d^4x \, \langle f \,|\, T[O_1(x)\, O_2(0)] \,|\, i \rangle \,,$$

where $O_{1,2}$ are local composite operators.

- The FCNC applications we have been pursuing are:

  (a) Δm_K and ε_K;
  (b) Rare kaon decays $K \to \pi \ell^+ \ell^-$ and $K^+ \to \pi^+ \nu \bar{\nu}$.

I will briefly provide a physics motivation for each of these applications.

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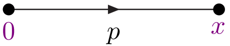
- Most of the discussion in this section also applies to the calculation of electromagnetic effects. G.Martinelli's lectures at this school
- More details of the presentation below can be found in C.J.D. Lin et al, hep-lat/0208007

- Consider the propagator of the free scalar particle of mass m:

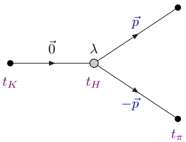


For $t_x > 0$ we have

$$\int d^3x \int \frac{dk_4}{2\pi}\,\frac{1}{L^3}\sum_{\vec k}\frac{e^{ik_4t_x+i\vec k\cdot\vec x}\,e^{-i\vec p\cdot\vec x}}{k_4^2+\vec k^2+m^2}=\frac{e^{-\omega_p t_x}}{2\omega_p}\,,$$

where $\omega_p=\sqrt{\vec p^{\,2}+m^2}\,.$

- Next consider a trivial $\frac{\lambda}{2}\phi_K\phi_\pi^2$ model for the $K\pi\pi$ Hamiltonian:



$$=\lambda\,\frac{e^{-m_K(t_H-t_K)}}{2m_K}\,\frac{e^{-2\omega_p(t_\pi-t_H)}}{(2\omega_p)^2}\,.$$

  - After computing the correlation function and amputating the external lines we would correctly obtain the amplitude λ.

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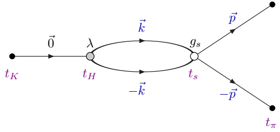
- Less trivial is what happens when we allow the pions to re-scatter through a $g_s \frac{\phi_\pi^4}{4!}$ interaction:

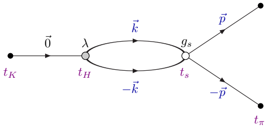


- We have to perform the t_s integration over the whole lattice.
- Consider first the time ordering as indicated in the diagram: $t_K < t_H < t_s < t_\pi$.
- Amputating the lowest order external propagators we have:

$$-\lambda g_s\, e^{-2\omega_p t_H} \frac{1}{L^3} \sum_{\vec{k}} \int_{t_H}^{t_\pi} dt_s\; \frac{e^{-2\omega_k(t_s-t_H)}}{(2\omega_k)^2}\; e^{2\omega_p t_s}$$

$$= -\lambda g_s \frac{1}{L^3} \sum_{\vec{k}} \frac{1}{(2\omega_k)^2}\; \int_{t_H}^{t_\pi} dt_s\; e^{-2(\omega_k-\omega_p)(t_s-t_H)}\,.$$

- We see that if there are states with $\omega_k < \omega_p$ then there are exponentially growing terms in time. This is not a surprise of course.
- The dominant contribution to the correlation function comes from the vertex $K \to \pi(\vec{0})\pi(\vec{0})$ and not $K \to \pi(\vec{p})\pi(-\vec{p})$. L.Maiani & M.Testa, Phys.Lett.B245 (1990) 585.



$$-\lambda g_s \frac{1}{L^3}\sum_{\vec{k}}\frac{1}{(2\omega_k)^2}\int_{t_H}^{t_\pi} dt_s\, e^{-2(\omega_k-\omega_p)(t_s-t_H)}\,.$$

- Integrating over t_s we obtain

$$-\lambda g_s \frac{1}{L^3}\sum_{\vec{k}}\frac{1}{(2\omega_k)^2}\,\frac{1}{2(\omega_k-\omega_p)}\left[1-e^{-2(\omega_k-\omega_p)(t_\pi-t_H)}\right]$$

- Consider first the ν_p terms with $\omega_k=\omega_p$. These give the contribution:

$$-\nu_p\lambda g_s\,\frac{t_\pi-t_H}{L^3}\,\frac{1}{(2\omega_p)^2}\,,$$

which represents a finite-volume correction to the two-pion energy at this order of perturbation theory:

$$e^{-2\omega_p(t_\pi-t_H)}\left[1-\nu_p g_s\,\frac{t_\pi-t_H}{L^3}\,\frac{1}{(2\omega_p)^2}\right]\simeq e^{-W_p(t_\pi-t_H)}$$

where the two-pion energy is $W_p=2\omega_p+\nu_p g_s\frac{1}{L^3}\frac{1}{4\omega_p^2}$.

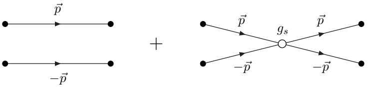
Growing Exponentials and Finite-Volume Effects (cont.)

- Combining the contributions from all the time orderings the (amputated) correlation function at this order is:

$$\frac{g_s}{L^3(4\omega_p^2)}\left(\frac{1}{4\omega_p}+(t_\pi-t_H)+\frac{1}{4\omega_p}\right)$$

$$-g_s\frac{1}{L^3}\sum_{\omega_k\neq\omega_p}\frac{1}{4\omega_k^2}\left(\frac{\omega_k e^{2(\omega_p-\omega_k)(t_\pi-t_H)}}{\omega_k^2-\omega_p^2}+\frac{\omega_k}{\omega_k^2-\omega_p^2}\right)$$

- We have already seen that the term proportional to $t_\pi - t_H$ represents a finite-volume mass shift.
- One of the terms proportional to $\frac{1}{4\omega_p}$ in the parentheses on the first line is subtracted when amputating the two-pion interpolating operator at the sink.



- The terms proportional to $e^{2(\omega_p-\omega_k)(t_\pi-t_H)}$ have either to be identified and subtracted (particularly if $\omega_k < \omega_p$) or may be negligible (if $\omega_k > \omega_p$ and $t_\pi - t_H$ is sufficiently large).

- Let $f(p^2)$ be a smooth function. For a sufficiently large L:

$$\frac{1}{L}\sum_n f(p_n^2) = \int \frac{dp}{2\pi} f(p^2)\,,$$

where $p_n = (2\pi/L)\,n$ and the relation holds "locally".

- In actual lattice calculations the spacing between momenta are $O(\text{few }100\,\text{MeV})$ so we would not expect such a local relation to be sufficiently accurate.
- However using the Poisson summation formula:

$$\sum_{n=-\infty}^{\infty} \delta(x-n) = \sum_{n=-\infty}^{\infty} \exp(2\pi i n x)$$

we obtain the powerful exact relation

$$\frac{1}{L}\sum_{n=-\infty}^{\infty} f(p_n^2) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} f(p^2) + \sum_{l\neq 0}\int_{-\infty}^{\infty} \frac{dp}{2\pi} f(p^2) e^{i(lpL)}\,,$$

which implies that

$$\boxed{\frac{1}{L}\sum_n f(p_n^2) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} f(p^2)\,,}$$

up to exponentially small corrections in L.

  - This is the starting point for all calculations of FV effects.

- Consider now $$\frac{1}{L}\sum_{k_n}\frac{f(k_n^2)}{q^2-k_n^2}\,.$$
- We assume that f has no singularities on the real axis so that

$$\frac{1}{L}\sum_{k_n}\frac{f(k_n^2)-f(q^2)}{q^2-k_n^2}=\int\frac{dk}{2\pi}\frac{f(k^2)-f(q^2)}{q^2-k^2}$$

up to exponentially small finite-volume corrections.

- We now rewrite the above as:

$$\frac{1}{L}\sum_{k_n}\frac{f(k_n^2)}{q^2-k_n^2}=\mathscr{P}\int\frac{dk}{2\pi}\frac{f(k^2)}{k^2-q^2}+f(q^2)\sum_{k_n}\frac{1}{q^2-k_n^2}-f(q^2)\mathscr{P}\int\frac{dk}{2\pi}\frac{1}{k^2-q^2}$$

where $\mathscr{P}$ represents *principal value*.

- Using $$\mathscr{P}\int\frac{dk}{2\pi}\frac{1}{k^2-q^2}=0\quad\text{and}\quad\sum_{n=-\infty}^{\infty}\frac{1}{n^2-x^2}=-\frac{\pi\cot(\pi x)}{x}$$

we obtain

$$\frac{1}{L}\sum_{n}\frac{f(k_n)}{q^2-k_n^2}=\mathscr{P}\int\frac{dk}{2\pi}\frac{f(k)}{q^2-k^2}+\frac{f(q)}{2q}\cot\frac{qL}{2}\,.$$

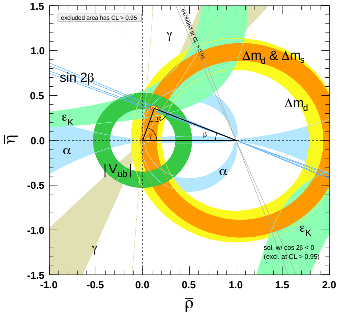
- The 1-D discussion above has been generalised to the
  - i) two-particle spectrum below the inelastic threshold:
    M.Luscher, Comm.Math.Phys. **105** (1986) 153; Nucl.Phys. **B354** (1991) 531.
  - ii) Matrix elements with two particles in the final state, e.g. $K \to \pi\pi$
    L.Lellouch and M.Luscher, hep-lat/0003023;
    C.J.D.Lin, G.Martinelli, C.T.S., and M.Testa, hep-lat/0104006;
    C. h. Kim, C.T.S., and S.R.Sharpe, hep-lat/0507006; $\cdots$
  - iii) $m_{K_L} - m_{K_S}$ and related quantities (see below):
    N.H.Christ, X.Feng, G.Martinelli and C.T.S., arXiv:1504.01170
- Luscher's work demonstrated a beautiful connection between the two-particle spectrum and the elastic scattering phase.
- There has been heroic and successful work to try to extend the formalism to three-particle states, but its implementation is very complicated.
  For a recent review see M.T.Hansen and S.R.Sharpe, arXiv:1901.00483 and S.Sharpe's lectures

- $\Delta m_K = m_{K_L} - m_{K_S} = 3.483(6)\times 10^{-12}$ MeV is tiny on the scale of $\Lambda_{\rm QCD}$.
- This FCNC quantity is therefore an excellent one in which to search for new-physics effects.
- It is frequently said that Flavour Physics can probe scales which are unreachable in colliders.
  - Here, if we could reproduce the experimental Δm_K in the SM to 10% accuracy and if we imagine an effective new-physics $\Delta S=2$ contribution $\frac{1}{\Lambda^2}(\bar{s}\cdots d)(\bar{s}\cdots d)$ then $\Lambda \gtrsim (10^3-10^4)$ TeV.
- Since $\langle \bar{K}^0|H_W|K^0\rangle \neq 0$ at second order in the weak interactions, the masses of the two eigenstates, K_L and K_S, are not equal.
- As well as computing the non-perturbative long-distance contributions from scales of $O(\Lambda_{\rm QCD})$, we aim to avoid the necessity of performing perturbation theory at the scale of m_c. For Δm_K this has proved particularly slowly convergent.

J.Brod & M.Gorbahn, arXiv:1108.2036

- ε_K is one of the standard inputs into the unitarity triangle analysis.
- SD dominance $\Rightarrow$ the leading non-perturbative QCD effects are contained in $\langle \bar{K}^0 | O_{LL}^{\Delta S=2} | K^0 \rangle$ which is given by B_K and f_K.
  - Now known to $O(2\text{-}3\%)$ precision.
- Currently a large uncertainty is due to that in V_{cb}^4. ($V_{cb} = (42.2 \pm 0.8) \times 10^{-3}$ with a significant uncertainty on the error.)

PDG 2018



PDG 2016

- LD effects are estimated to be of $O(5\text{-}10\%)$.

  A.Buras, D.Guadagnoli and G.Isidori, arXiv:1002.3612

- The aim of this work is to compute the LD effects with controlled uncertainties.

- NA62 ($K^+ \to \pi^+ \nu \bar{\nu}$) and KOTO ($K_L \to \pi^0 \nu \bar{\nu}$) are beginning their experimental programme to study these decays. These FCNC processes provide ideal probes for the observation of new physics effects.
- The dominant contributions from the top quark $\Rightarrow$ they are also very sensitive to V_{ts} and V_{td}.
- Experimental results and bounds:

$$\mathrm{Br}(K^+ \to \pi^+ \nu \bar{\nu})_{\text{exp}} = 1.73^{+1.15}_{-1.05} \times 10^{-10}$$

A.Artamonov et al. (E949), arXiv:0808.2459

$$\mathrm{Br}(K_L \to \pi^0 \nu \bar{\nu}) \leq 2.6 \times 10^{-8} \text{ at } 90\% \text{ confidence level},$$

J.Ahn et al. (E291a), arXiv:0911.4789

- Sample recent theoretical predictions:

$$\mathrm{Br}(K^+ \to \pi^+ \nu \bar{\nu})_{\text{SM}} = (9.11 \pm 0.72) \times 10^{-11}$$
$$\mathrm{Br}(K_L \to \pi^0 \nu \bar{\nu})_{\text{SM}} = (3.00 \pm 0.30) \times 10^{-11},$$

A.Buras, D.Buttazzo, J.Girrbach-Noe, R.Knejgens, arXiv:1503.02693

- To what extent can lattice calculations reduce the theoretical uncertainty?

- To what extent can lattice calculations reduce the theoretical uncertainty?
- $K \to \pi\nu\bar{\nu}$ decays are SD dominated and the hadronic effects can be determined from CC semileptonic decays such as $K^+ \to \pi^0 e^+ \nu$.
  - Lattice calculations of the $K_{\ell 3}$ form factors are well advanced,

    FLAG review, S.Aoki et al, arXiv:1607.00299
- LD contributions, i.e. contributions from distances greater than $1/m_c$ are negligible for K_L decays and are expected to be $O(5\%)$ for K^+ decays.
  - K_L decays are therefore one of the cleanest places to search for the effects of new physics.
  - The aim of our study is to compute the LD effects in K^+ decays. These provide a significant, if probably still subdominant, contribution to the theoretical uncertainty (which is dominated by the uncertainties in CKM matrix elements).
  - A phenomenological estimate of the long distance effects, estimated these to enhance the branching fraction by 6% with an uncertainty of 3%.

    G.Isidori, F.Mescia and C.Smith, hep-ph/0503107
- Lattice QCD can provide a first-principles determination of the LD contribution with controlled errors.
  - Given the NA62 experiment, it is timely to perform a lattice QCD calculation of these effects.

There are three main contributions to the amplitude:

1. Short distance contributions: F.Mescia, C,Smith, S.Trine hep-ph/0606081

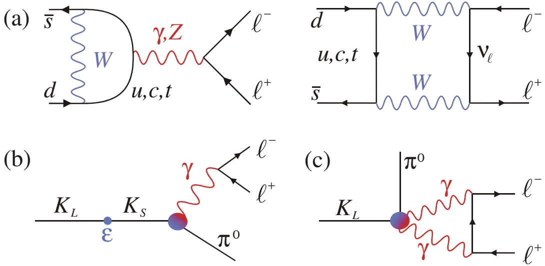
$$H_{\rm eff} = -\frac{G_F \alpha}{\sqrt{2}} V_{ts}^* V_{td} \{ y_{7V} (\bar{s}\gamma_\mu d)(\bar{\ell}\gamma^\mu \ell) + y_{7A} (\bar{s}\gamma_\mu d)(\bar{\ell}\gamma^\mu \gamma_5 \ell) \} + \text{h.c.}$$

   - Direct CP-violating contribution.
   - In BSM theories other effective interactions are possible.

2. Long-distance indirect CP-violating contribution

$$A_{ICPV}(K_L \to \pi^0 \ell^+ \ell^-) = \varepsilon A(K_1 \to \pi^0 \ell^+ \ell^-) \simeq \varepsilon A(K_S \to \pi^0 \ell^+ \ell^-).$$

3. The two-photon CP-conserving contribution $K_L \to \pi^0 (\gamma^* \gamma^* \to \ell^+ \ell^-)$.



- The current phenomenological status for the SM predictions is nicely summarised by: V.Cirigliano et al., arXiv1107.6001

$$\mathrm{Br}(K_L \to \pi^0 e^+ e^-)_{\mathrm{CPV}} = 10^{-12} \times \left\{ 15.7|a_S|^2 \pm 6.2|a_S| \left(\frac{\mathrm{Im}\,\lambda_t}{10^{-4}}\right) + 2.4 \left(\frac{\mathrm{Im}\,\lambda_t}{10^{-4}}\right)^2 \right\}$$

$$\mathrm{Br}(K_L \to \pi^0 \mu^+ \mu^-)_{\mathrm{CPV}} = 10^{-12} \times \left\{ 3.7|a_S|^2 \pm 1.6|a_S| \left(\frac{\mathrm{Im}\,\lambda_t}{10^{-4}}\right) + 1.0 \left(\frac{\mathrm{Im}\,\lambda_t}{10^{-4}}\right)^2 \right\}$$

  - $\lambda_t = V_{td} V_{ts}^*$ and $\mathrm{Im}\,\lambda_t \simeq 1.35 \times 10^{-4}$.
  - $|a_S|$, the amplitude for $K_S \to \pi^0 \ell^+ \ell^-$ at $q^2 = 0$ as defined below, is expected to be $O(1)$ but the sign of a_S is unknown. $|a_S| = 1.06^{+0.26}_{-0.21}$.
  - For $\ell = e$ the two-photon contribution is negligible.
  - Taking the positive sign (?) the prediction is

$$\begin{aligned}
\mathrm{Br}(K_L \to \pi^0 e^+ e^-)_{\mathrm{CPV}} &= (3.1 \pm 0.9) \times 10^{-11} \\
\mathrm{Br}(K_L \to \pi^0 \mu^+ \mu^-)_{\mathrm{CPV}} &= (1.4 \pm 0.5) \times 10^{-11} \\
\mathrm{Br}(K_L \to \pi^0 \mu^+ \mu^-)_{\mathrm{CPC}} &= (5.2 \pm 1.6) \times 10^{-12}.
\end{aligned}$$

  - The current experimental limits (KTeV) are:

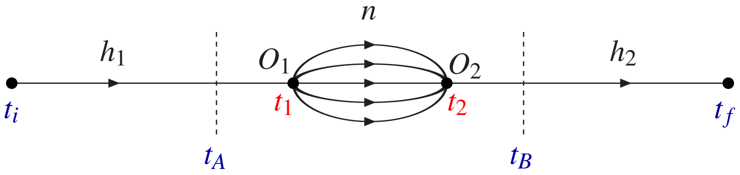
$$\mathrm{Br}(K_L \to \pi^0 e^+ e^-) < 2.8 \times 10^{-10} \quad \text{and} \quad \mathrm{Br}(K_L \to \pi^0 \mu^+ \mu^-) < 3.8 \times 10^{-10}.$$

(i) The fiducial volume

(ii) Unphysical exponentially growing contributions

(iii) Finite-volume corrections

(iv) Renormalization



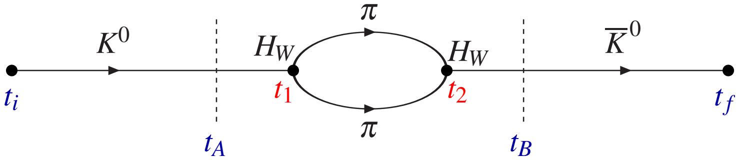
- How do you prepare the states $h_{1,2}$ in the generic integrated correlation function:

$$\int d^4x \int d^4y \, \langle h_2 \,|\, T\{O_1(x)\, O_2(y)\} \,|\, h_1\rangle \,,$$

  when the time of the operators is integrated?

- The practical solution is to integrate over a large subinterval in time $t_A \leq t_{x,y} \leq t_B$, but to create h_1 and to annihilate h_2 well outside of this region.
- This is the natural modification of standard field theory for which the asymptotic states are prepared at $t \to \pm\infty$ and then the operators are integrated over all time.
- This approach has been successfully implemented in all our projects.

(ii) Exponentially growing exponentials illustrated with $\Delta m_K^{\rm FV}$



- Δm_K is given by

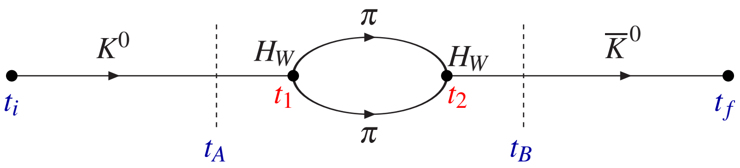
$$\Delta m_K \equiv m_{K_L} - m_{K_S} = 2\mathscr{P}\sum_\alpha \frac{\langle \bar{K}^0\,|\,\mathscr{H}_W\,|\,\alpha\rangle\,\langle\alpha\,|\,\mathscr{H}_W\,|\,K^0\rangle}{m_K - E_\alpha} = 3.483(6)\times 10^{-12}\,\text{MeV}.$$

- The above correlation function gives ($T = t_B - t_A + 1$)

$$C_4(t_A,t_B;t_i,t_f) \;=\; |Z_K|^2 e^{-m_K(t_f-t_i)} \sum_n \frac{\langle \bar{K}^0\,|\,\mathscr{H}_W\,|\,n\rangle\,\langle n\,|\,\mathscr{H}_W\,|\,K^0\rangle}{(m_K-E_n)^2} \times \left\{ e^{(m_K-E_n)T} - (m_K-E_n)T - 1 \right\}.$$

- From the coefficient of T we can therefore obtain

$$\Delta m_K^{\rm FV} \equiv 2\sum_n \frac{\langle \bar{K}^0\,|\,\mathscr{H}_W\,|\,n\rangle\,\langle n\,|\,\mathscr{H}_W\,|\,K^0\rangle}{(m_K-E_n)}\,.$$



$$C_4(t_A,t_B;t_i,t_f) = |Z_K|^2 e^{-m_K(t_f-t_i)} \sum_n \frac{\langle \bar{K}^0 \,|\, \mathscr{H}_W \,|\, n\rangle \langle n \,|\, \mathscr{H}_W \,|\, K^0\rangle}{(m_K-E_n)^2} \times \left\{ e^{(m_K-E_n)T} - (m_K-E_n)T - 1 \right\}.$$

- The presence of terms which (potentially) grow exponentially in T is a generic feature of calculations of matrix elements of bilocal operators.

- Theoretically the FV correction for the two-pion intermediate state is given by
  N.H.Christ, X.Feng, G.Martinelli and C.T.S., arXiv:1504.01170

$$\Delta m_K - \Delta m_K^{FV} = 2\mathscr{P}\int \mathrm{dE}\,\rho_V(E)\,\frac{f(E)}{m_K - E} - 2\sum_n \frac{f(E_n)}{m_K - E_n} = -2\left(f(m_K)\cot(h)\,\frac{\mathrm{dh}}{\mathrm{dE}}\right)_{E=m_K},$$

  where

$$f(m_K) = {}_V\langle \bar{K}^0|H_W|(\pi\pi)_{E=m_K}\rangle_V\;{}_V\langle(\pi\pi)_{E=m_K}|H_W|K_0\rangle_V \quad \text{and} \quad h(k) = \delta(k) + \phi(k)\,.$$

- The implementation of this formula requires the knowledge of the phase-shift $\delta(k_{m_K})$ (and its derivative) which can be calculated in principle, but which may have to be estimated.
- Our preliminary estimate of the FV correction is that they are $\ll$ statistical uncertainty.
- Further studies are needed to confirm whether this is a general feature.

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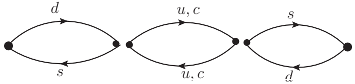
$$\int d^4x \, \langle h_2 \,|\, T\{O_1(x)\, O_2(0)\} \,|\, h_1 \rangle \,,$$

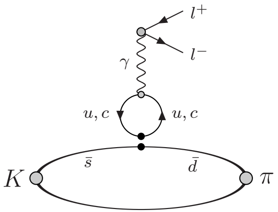
- The local operators $O_{1,2}$ are renormalised in a standard way, e.g. non-perturbatively into a RI-SMOM scheme & then perturbatively into the $\overline{\text{MS}}$ scheme if appropriate.
- However, additional ultraviolet divergences may arise as $x \to 0$.
- This does not happen in two of our cases in the four-flavour theory:

1. Δm_K



• Taking the u-quark component of the operators $\Rightarrow$ a quadratic divergence.
• GIM mechanism & $V-A$ nature of the currents $\Rightarrow$ elimination of both quadratic and logarithmic divergences.

2 $K \to \pi \ell^+ \ell^-$ decays:



- Quadratic divergence is absent by gauge invariance $\Rightarrow$ Logarithmic divergence.
  - Checked explicitly for Wilson and Clover at one-loop order.
    G.Isidori, G.Martinelli and P.Turchetti, hep-lat/0506026
  - Absence of power divergences does not require GIM.
  - Logarithmic divergence cancelled by GIM.

- Such an absence of additional divergences as $x \to 0$ is not generic and for example, is not the case for ε_K or for $K \to \pi \nu \bar{\nu}$ rare kaons decays.

- I use the decay $K^+ \to \pi^+ \nu\bar{\nu}$ for illustration, but the procedure is general.

1. Start by integrating out the W and Z bosons (as well as the t-quark). The transition amplitude takes the form

$$\langle \pi^+ \nu\bar{\nu} | \{ (C_A^{\overline{\text{MS}}} O_A^{\overline{\text{MS}}}) (C_B^{\overline{\text{MS}}} O_B^{\overline{\text{MS}}}) \}_\mu^{\overline{\text{MS}}} | K^+ \rangle + C_0^{\overline{\text{MS}}}(\mu) \langle \pi^+ \nu\bar{\nu} | O_0^{\overline{\text{MS}}}(\mu) | K^+ \rangle ,$$

   - The notation is $\{O_A^S O_B^S\}^{S'} = \int d^4x\, T\{O_A^S(x) O_B^S(0)\}^{S'}$, where $S(S')$ denotes the scheme used to renormalise the local (bilocal) operators.
   - O_0 is a local operator, necessary to include the short-distance effects correctly. In this case it is $O_0 = (\bar{s}\gamma_L^\mu d)(\bar{\nu}\gamma_L^\mu \nu)$.

2. The μ-dependence of the coefficients is determined by perturbative running.

3. In the traditional phenomenological approach, at $\mu \simeq m_c$ the charm-quark is also integrated out

$$\{ C_A^{\overline{\text{MS}}} O_A^{\overline{\text{MS}}} C_B^{\overline{\text{MS}}} O_B^{\overline{\text{MS}}} \}_\mu^{\overline{\text{MS}}} \quad \to \quad C_A^{\overline{\text{MS}}}(\mu) C_B^{\overline{\text{MS}}}(\mu) r_{AB}^{\overline{\text{MS}}}(\mu) O_0^{\overline{\text{MS}}}(\mu),$$

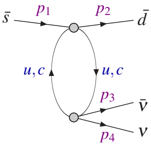
where $r_{AB}^{\overline{\text{MS}}}(\mu)$ is calculated in perturbation theory.
   - But at the scale $m_c \simeq 1$ GeV it is doubtful how well the OPE converges and how precise perturbation theory may be.
   - We therefore keep the charm quark as a dynamical degree of freedom.

Renormalization of the Bilocal Operator (Cont.)

$$\langle \pi^+ \nu\bar\nu | \{C_A^{\overline{\rm MS}} O_A^{\overline{\rm MS}}\, C_B^{\overline{\rm MS}} O_B^{\overline{\rm MS}}\}_\mu^{\overline{\rm MS}} | K^+\rangle + C_0^{\overline{\rm MS}}(\mu)\langle \pi^+ \nu\bar\nu | O_0^{\overline{\rm MS}}(\mu) | K^+\rangle,$$

- In order to calculate the perform the calculation with operators renormalised at $\mu > m_c$, it is necessary to compute the matrix element of the bilocal operator.
- Since the $\overline{\rm MS}$ scheme is purely perturbative, it is necessary to introduce an intermediate (RI-SMOM) scheme. We write

$$\{O_A^{\rm RI}\, O_B^{\rm RI}\}_{\mu_0}^{\rm RI} \equiv Z_{O_A}^{\rm lat\to RI}(a\mu_0) Z_{O_B}^{\rm lat\to RI}(a\mu_0)\{O_A^{\rm lat} O_B^{\rm lat}\}_a^{\rm lat} - X_{AB}(\mu_0, a) O_0^{\rm RI}(\mu_0).$$



By choosing, e.g. $p_1 = (\xi,\xi,0,0)$, $p_2 = (\xi,0,\xi,0)$, $p_3 = (0,\xi,0,\xi)$ and $p_4 = (0,0,-\xi,-\xi)$, with $p_i^2 = \mu_0^2$, the loop momentum is μ_0^2 and we have short-distance dominance of the off-shell Green functions.

- We choose to define (and determine) X by imposing

$$\langle \bar d \nu\bar\nu | \{O_A^{\rm RI}\, O_B^{\rm RI}\}_{\mu_0}^{\rm RI} | \bar s\rangle = 0$$

at some chosen renormalisation scale $p_i^2 = \mu_0^2$.

$$\langle\pi^+\nu\bar\nu|\{C_A^{\overline{\rm MS}}O_A^{\overline{\rm MS}}\,C_B^{\overline{\rm MS}}O_B^{\overline{\rm MS}}\}_\mu^{\overline{\rm MS}}|K^+\rangle+C_0^{\overline{\rm MS}}(\mu)\langle\pi^+\nu\bar\nu|O_0^{\overline{\rm MS}}(\mu)|K^+\rangle,$$

- Finally we need to determine the bilocal matrix element in the $\overline{\rm MS}$ scheme and this is unavoidably perturbative. We write

$$\{O_A^{\overline{\rm MS}}\,O_B^{\overline{\rm MS}}\}_\mu^{\overline{\rm MS}}=Z_{O_A}^{\rm RI\to\overline{MS}}(\mu/\mu_0)Z_{O_B}^{\rm RI\to\overline{MS}}(\mu/\mu_0)\{O_A^{\rm RI}\,O_B^{\rm RI}\}_{\mu_0}^{\rm RI}+Y_{AB}(\mu,\mu_0)O_0^{\rm RI}(\mu_0).$$

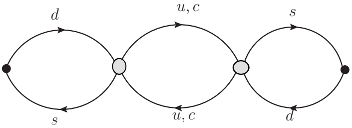
- We propose to evaluate the matching coefficient Y by calculating in perturbation theory

$$\langle\bar d\nu\bar\nu|\{O_A^{\overline{\rm MS}}\,O_B^{\overline{\rm MS}}\}_\mu^{\overline{\rm MS}}|\bar s\rangle_{p_i^2=\mu_0^2}=Y_{AB}(\mu,\mu_0)\langle\bar d\nu\bar\nu|O_0^{\rm RI}(\mu_0)|\bar s\rangle_{p_i^2=\mu_0^2}\,.$$

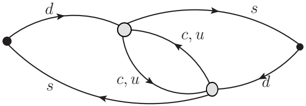
- In this way we are able to compute the $K^+\to\pi^+\nu\bar\nu$ amplitude in a lattice calculation.

RBC-UKQCD

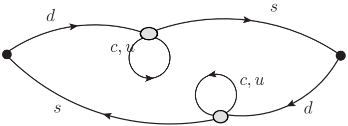
- There are four types of diagram to be evaluated:



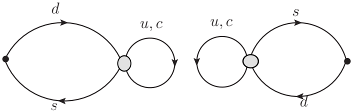
Type 1



Type 2

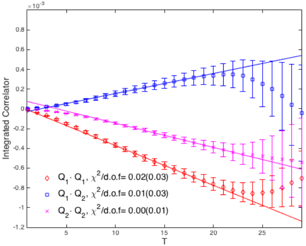


Type 3



Type 4

- Following the development of the theoretical background and exploratory numerical studies, we presented the first numerical results at physical masses at Lattice 2017 Z.Bai, N.H.Christ, CTS; EPJ Web Conf 175 (2018) 13017
  and updated them at Lattice 2018 B.Wang; arXiv:1812.05302.
- The calculation is performed on a $64^3 \times 128 \times 12$ lattice with Möbius DWF and the Iwasaki gauge action. a^{-1}=2.359(7) GeV, $m_\pi = 135.9(3)$ MeV and $m_K = 496.9(7)$ MeV. T.Blum et al., RBC-UKQCD Collabs., arXiv:1411.7017
  Charm-physics studies with this action $\Rightarrow am_c \simeq 0.32 - 0.33$. We have used $am_c \simeq 0.31$ and studied the dependence on m_c.



all diagrams

- 2018 preliminary result is
  $$\Delta m_K = 7.9(1.3)(2.1) \times 10^{-12}\,\text{MeV}\,,$$
  to be compared to the physical value
  $$(\Delta m_K)^{\text{phys}} = 3.483(6) \times 10^{-12}\,\text{MeV}\,.$$
- The dominant systematic error is due to discretisation effects because $am_c \simeq 0.31$.
- Preliminary update Bigeng Wang @ Lattice 2019
  $\Delta m_K = 7.7(0.7)(2.0) \times 10^{-12}\,\text{MeV}\,,$
- Future project planned on finer lattices at SUMMIT.

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RBC-UKQCD, **Z.Bai** et al.

$$\varepsilon_K^{\rm Exp} = 2.228(11)\times 10^{-3}$$

- There has been no journal publication on the long-distance contribution to ε_K even though the whole theoretical background has been developed.
- A number of conference papers have been presented including:

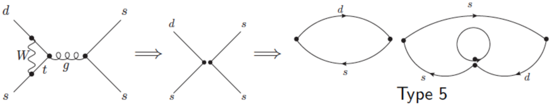
  "Long distance part of ε_K from lattice QCD"

  Z.Bai, arXiv:1611.06601
- The preliminary results below were obtained from 200 configurations on a $N_f = 2+1$ flavour ensemble using DWF and Iwasaki gauge action on a $24^3\times 64\times 16$ lattice with $a^{-1} = 1.78\,$GeV. C.Allton et al, arXiv:0804.0473
- The quark masses are unphysical, $m_\pi = 339\,$MeV, $m_K \simeq 592\,$MeV and $m_c^{\overline{\rm MS}}(2\,{\rm GeV}) = 968\,$MeV.
- Our preliminary result for the LD contribution at these unphysical masses is

$$\varepsilon_K^{\rm LD} = 0.11(0.08)\times 10^{-3}\,.$$

  - The central (unphysical) value is about 5% of the physical ε_K which is consistent with expectations of the long-distance contribution.
- It is hoped to restart the computation of the long-distance contributions to ε_K in the autumn?

- We need Im $M_{\bar{0}0} \Rightarrow t$-quark contributions not suppressed $\Rightarrow$ QCD penguin operators contribute and we have a Type 5 topology.



- The contributions to the amplitude have CKM factors of the form $\lambda_i\lambda_j$ where $\{i,j\} = \{u,c,t\}$ and $\lambda_i = V_{id}V^*_{is}$.
- In the standard approach, the unitarity relation $\lambda_u + \lambda_c + \lambda_t = 0$ is used to eliminate λ_u. We find it convenient instead to eliminate λ_c, leaving contributions proportional to
  - λ_u^2 which is real and hence does not contribute to ε_K;
  - λ_t^2 which are perturbative;
  - $\lambda_u\lambda_t$ which need to be calculated using lattice QCD.

G.Isidori, G.Martinelli and P.Turchetti, hep-lat/0506026

- We now turn to the CPC decays $K_S \to \pi^0 \ell^+ \ell^-$ and $K^+ \to \pi^+ \ell^+ \ell^-$ and consider

$$A^i_\mu = \int d^4x\, e^{-iq\cdot x} \langle \pi(p) \,|\, \mathrm{T}\{J^\mu_{\mathrm{em}}(x)\, Q_i(0)\,\} \,|\, K(k) \rangle\,,$$

  where Q_i is an operator from the $\Delta S = 1$ effective weak Hamiltonian.

- EM gauge invariance implies that

$$A^i_\mu = \frac{\omega_i(q^2)}{(4\pi)^2} \left\{ q^2 (p+k)^\mu - (m_K^2 - m_\pi^2)\, q^\mu \right\}.$$

- The theoretical framework has been developed and an exploratory numerical calculation for the K^+ decay has been performed.

N.Christ, X.Feng, A.Portelli and CTS, arXiv1507.03094

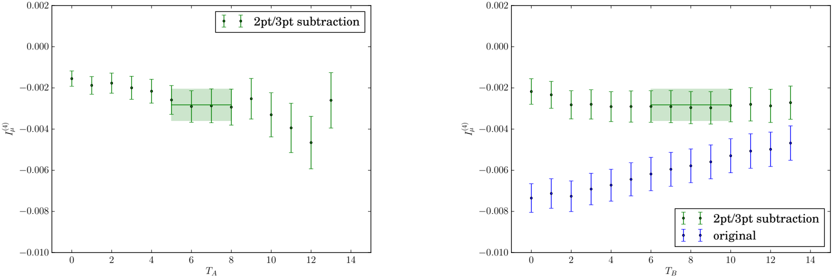
N.Christ, X.Feng, A.Jüttner, A.Lawson, A.Portelli and CTS, arXiv1608.07585

- These decays are an important focus for the UK effort on the DiRAC resources it has been allocated recently.

- The exploratory numerical study was performed on a $24^3 \times 64$ DWF+Iwasaki RBC-UKQCD ensembles with $m_\pi \simeq 420\,$MeV, $m_K \simeq 625\,$MeV), $a^{-1} \simeq 1.78\,$fm.

  N.Christ, X.Feng, A.Jüttner, A.Lawson, A.Portelli and CTS, arXiv:1608.07585

  - 128 configurations were used with $\vec{k} = \vec{0}$ and $\vec{p} =$(1,0,0), (1,1,0) and (1,1,1) in units of $2\pi/L$. With this kinematics we are in the unphysical region, $q^2 < 0$ and the charm quark is also lighter than physical $m_c^{\overline{\rm MS}}(2\,\text{GeV}) \simeq 520$ MeV.

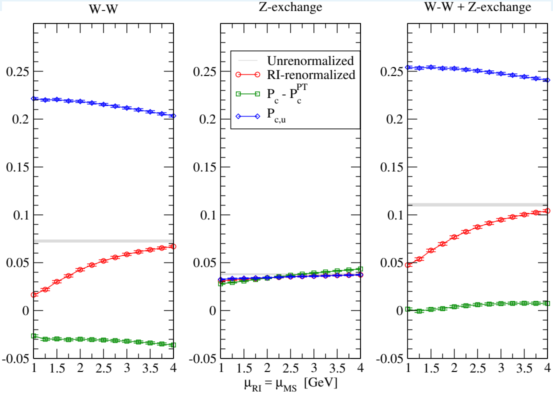


$\int_{t_J-T_A}^{t_J+8} \tilde{\Gamma}_0^{(4)}\, dt_H$ $\qquad$ $\int_{t_J-6}^{t_J+T_B} \tilde{\Gamma}_0^{(4)}\, dt_H$

$$A_0^+(q^2) = -0.0028(6).$$

- To what extent can lattice calculations reduce the theoretical uncertainty?
- $K \to \pi\nu\bar{\nu}$ decays are SD dominated and the hadronic effects can be determined from CC semileptonic decays such as $K^+ \to \pi^0 e^+ \nu$.
- LD contributions, i.e. contributions from distances greater than $1/m_c$ are negligible for K_L decays and are expected to be $O(5\%)$ for K^+ decays.
  - K_L decays are therefore one of the cleanest places to search for the effects of new physics.
  - The aim of our lattice study is to compute the LD effects in K^+ decays. (These provide a significant, if probably still subdominant, contribution to the theoretical uncertainty, which is dominated by the uncertainties in CKM matrix elements.)
- Lattice QCD can provide a first-principles determination of the LD contribution with controlled errors.
- The theoretical framework has been developed and implemented in an exploratory calculation. N.Christ, X.Feng, A.Portelli & CTS, arXiv:1605.04442

  Z.Bai, N.Christ, X.Feng, A,Lawson, A.Portelli & CTS, arXiv:1701.02858 & 1806.11520
- Ongoing work, led by X.Feng, includes a study of the momentum dependence on a 32^3 lattice at a^{-1}=1.37 GeV with $m_\pi \simeq 170$ MeV but lighter m_c as well as generating data in a physical simulation.

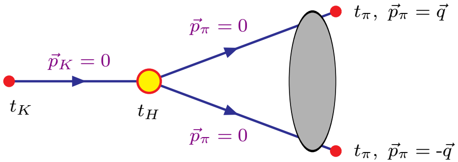


- Details of simulation: 800 configs on a $16^3 \times 32$ lattice with $N_f = 2+1$ DWF, $a^{-1} \simeq 1.73$ GeV, $m_\pi \simeq 420$ MeV, $m_K \simeq 563$ MeV and $m_c^{\overline{\rm MS}}(2\,\text{GeV}) \simeq 863$ MeV.
- For this unphysical kinematics, we find

$$P_c = 0.2529(\pm 13)(\pm 32)(-45) \quad \text{and} \quad \Delta P_c = 0.0040(\pm 13)(\pm 32)(-45)\,.$$

- Large cancellation between WW and Z-exchange contributions.

- $K \to \pi\pi$ decays are a very important class of processes for standard model phenomenology with a long and noble history.
  - It is in these decays that both indirect and direct CP-violation was discovered.
- Bose Symmetry $\Rightarrow$ the two-pion state has isospin 0 or 2.
- Among the very interesting issues are the origin of the $\Delta I = 1/2$ rule (Re A_0/Re $A_2 \simeq 22.5$) and an understanding of the experimental value of ε'/ε, the parameter which was the first experimental evidence of direct CP-violation.
- The evaluation of $K \to \pi\pi$ matrix elements requires an extension of the standard computations of $\langle 0\,|\,O(0)\,|\,h\rangle$ and $\langle h_2\,|\,O(0)\,|\,h_1\rangle$ matrix elements with a single hadron in the initial and/or final state.



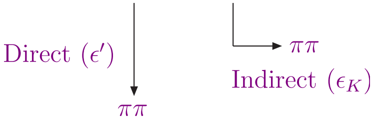
- $K \to \pi\pi$ correlation function is dominated by lightest state, i.e. the state with two-pions at rest. Maiani and Testa, PL B245 (1990) 585

$$C(t_\pi) = A + B_1 e^{-2m_\pi t_\pi} + B_2 e^{-2E_\pi t_\pi} + \cdots$$

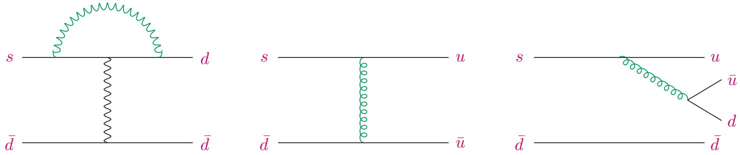
  - Solution 1: Study an excited state. Lellouch and Lüscher, hep-lat/0003023
  - Solution 2: Introduce suitable boundary conditions such that the $\pi\pi$ ground state is $|\pi(\vec{q})\pi(-\vec{q})\rangle$. RBC-UKQCD, C.h.Kim hep-lat/0311003

For B-decays, with so many intermediate states below threshold, this is the main obstacle to producing reliable calculations.

- Directly CP-violating decays are those in which a CP-even (-odd) state decays into a CP-odd (-even) one: $K_L \;\propto\; K_2 \;+\; \bar{\epsilon} K_1\,.$



- Consider the following contributions to $K \to \pi\pi$ decays:



$I = 0$, Complex (a) $I = 0$, Real (b) $I = 0$ or 2, Real (c)

Direct CP-violation in kaon decays manifests itself as a non-zero relative phase between the $I = 0$ and $I = 2$ amplitudes.

- We also have *strong phases*, δ_0 and δ_2 which are independent of the form of the weak Hamiltonian.

$$\mathcal{H}_{\rm eff}^{\Delta S=1} = \frac{G_F}{\sqrt{2}} V_{ud}V_{us}^* \sum_{i=1}^{10} [z_i(\mu) + \tau y_i(\mu)]\, Q_i\,, \text{ where } \tau = -\frac{V_{ts}^* V_{td}}{V_{us}^* V_{ud}} \text{ and}$$

Current – Current Operators

$$Q_1 = (\bar{s}d)_L(\bar{u}u)_L \qquad Q_2 = (\bar{s}^i d^j)_L(\bar{u}^j u^i)_L$$

QCD Penguin Operators

$$Q_3 = (\bar{s}d)_L \textstyle\sum_{q=u,d,s}(\bar{q}q)_L \qquad Q_4 = (\bar{s}^i d^j)_L \sum_{q=u,d,s}(\bar{q}^j q^i)_L$$

$$Q_5 = (\bar{s}d)_L \textstyle\sum_{q=u,d,s}(\bar{q}q)_R \qquad Q_6 = (\bar{s}^i d^j)_L \sum_{q=u,d,s}(\bar{q}^j q^i)_R$$

Electroweak Penguin Operators

$$Q_7 = \tfrac{3}{2}(\bar{s}d)_L \textstyle\sum_{q=u,d,s} e_q(\bar{q}q)_L \qquad Q_8 = \tfrac{3}{2}(\bar{s}^i d^j)_L \sum_{q=u,d,s} e_q(\bar{q}^j q^i)_L$$

$$Q_9 = \tfrac{3}{2}(\bar{s}d)_L \textstyle\sum_{q=u,d,s} e_q(\bar{q}q)_R \qquad Q_{10} = \tfrac{3}{2}(\bar{s}^i d^j)_L \sum_{q=u,d,s} e_q(\bar{q}^j q^i)_R$$

This 10 operator basis is very natural but over-complete:

$$\begin{aligned} Q_{10} - Q_9 &= Q_4 - Q_3 \\ Q_4 - Q_3 &= Q_2 - Q_1 \\ 2Q_9 &= 3Q_1 - Q_3\,. \end{aligned}$$

- For A_2, there is no vacuum subtraction and we can use the Wigner-Eckart theorem to write

$$\underbrace{\langle(\pi\pi)^{I=2}_{I_3=1}\,|}_{\frac{1}{\sqrt{2}}(\langle\pi^+\pi^0|+\langle\pi^0\pi^+|)}\quad Q^{\Delta I=3/2}_{\Delta I_3=1/2,i}\,|\,K^+\rangle=\frac{3}{2}\underbrace{\langle(\pi\pi)^{I=2}_{I_3=2}\,|}_{\langle\pi^+\pi^+|}Q^{\Delta I=3/2}_{\Delta I_3=3/2,i}\,|\,K^+\rangle\,,$$

  and impose anti-periodic conditions on the d-quark in one or more directions.
- If we impose the anti-periodic boundary conditions in all 3 directions then the ground state is

$$\left|\pi\left(\frac{\pi}{L},\frac{\pi}{L},\frac{\pi}{L}\right)\,\pi\left(\frac{-\pi}{L},\frac{-\pi}{L},\frac{-\pi}{L}\right)\right\rangle.$$

- With an appropriate choice of L and the number of directions, we can arrange that $E_{\pi\pi}=m_K$.
- Isospin breaking by the boundary conditions is harmless (exponentially small in the volume) here. CTS & G.Villadoro, hep-lat/0411033
  - This is not the case for $\Delta I=1/2$ transitions $\Rightarrow$ G-parity boundary conditions.

- In 2015 RBC-UKQCD published our first result for ε'/ε computed at physical quark masses and kinematics, albeit still with large relative errors:

$$\left.\frac{\varepsilon'}{\varepsilon}\right|_{\text{RBC-UKQCD}} = (1.38 \pm 5.15 \pm 4.59) \times 10^{-4}$$

to be compared with

$$\left.\frac{\varepsilon'}{\varepsilon}\right|_{\text{Exp}} = (16.6 \pm 2.3) \times 10^{-4}\,.$$

RBC-UKQCD, arXiv:1505.07863

- This is by far the most complicated project that I have ever been involved with.
- This single result hides much important (and much more precise) information which we have determined along the way.
- We are updating the results with about ≥ 6 times the statistics and much improved techniques for reducing the systematic uncertainties.
- It is planned to present updated results within the next few months.

Ongoing work

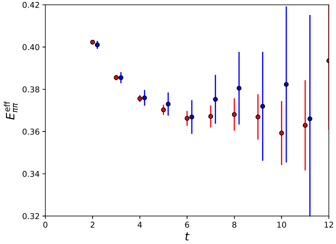
- Elastic $\pi\pi$ phase-shifts are obtained by measuring $E_{\pi\pi} - 2E_\pi$ and using Lüscher's formula.
- Isospin 0, two-pion correlators are noisy (primarily due to the vacuum subtraction) and measuring the ground-state two-pion energy is challenging.
- A puzzle from our 2015 paper was that we found $\delta_S^{I=0}(m_K^2) = (23.8 \pm 4.9 \pm 1.2)^\circ$ to be compared to $\sim 35^\circ$ from dispersive analyses.

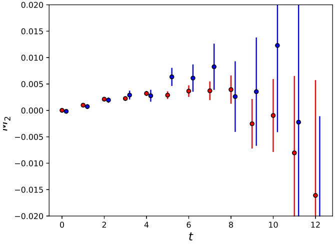
  G.Colangelo, J.Gasser, & H.Leutwyler, hep-ph/0103088
- With increased statistics, and more importantly the use of additional interpolating operators for the two-pion state, we are able to understand and reduce the contamination from excited states and now find $\delta_S^{I=0}(m_K^2) = (30.9 \pm 1.5 \pm 3.0)^\circ$.

  T.Wang and C.Kelly, PoS Lattice 2018 (2018) 276

Puzzle is resolved.

- In order to reduce possible contamination of excited states, we have now included the additional two-pion interpolating operators into the $K \to \pi\pi$ analysis and plan to present the results later this year.





Q_2

- Blue points - 216 configurations   Red points 1438 configurations
- I stress that more important in the improved analysis is the use of additional two-pion interpolating operators.

- The amplitude A_2 is considerably simpler to evaluate that A_0.
- Our first results for A_2 at physical kinematics were obtained at a single, rather coarse, value of the lattice spacing ($a \simeq 0.14$ fm). Estimated discretization errors at 15%. arXiv:1111.1699, arXiv:1206.5142
- Our latest results were obtained on two new ensembles, 48^3 with $a \simeq 0.11$ fm and 64^3 with $a \simeq 0.084$ fm so that we can make a continuum extrapolation:

$$\begin{aligned} \mathrm{Re}(A_2) &= 1.50(4)_{\mathrm{stat}}(14)_{\mathrm{syst}} \times 10^{-8}\ \mathrm{GeV}. \\ \mathrm{Im}(A_2) &= -6.99(20)_{\mathrm{stat}}(84)_{\mathrm{syst}} \times 10^{-13}\ \mathrm{GeV}. \end{aligned}$$

arXiv:1502.00263

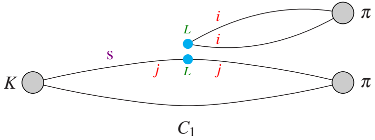
- The experimentally measured value is $\mathrm{Re}(A_2) = 1.479(4) \times 10^{-8}$ GeV.
- Although the precision can still be significantly improved (partly by perturbative calculations), the calculation of A_2 at physical kinematics can now be considered as standard.

RBC-UKQCD Collaboration, arXiv:1212.1474

- Re A_2 is dominated by a simple operator:

$$O^{3/2}_{(27,1)} = (\bar{s}^i d^i)_L \{(\bar{u}^j u^j)_L - (\bar{d}^j d^j)_L\} + (\bar{s}^i u^i)_L (\bar{u}^j d^j)_L$$

and two diagrams:



- Re A_2 is proportional to $C_1 + C_2$.
- The contribution to Re A_0 from Q_2 is proportional to $2C_1 - C_2$ and that from Q_1 is proportional to $C_1 - 2C_2$ with the same overall sign.
- Colour counting might suggest that $C_2 \simeq \frac{1}{3} C_1$.
- We find instead that $C_2 \approx -C_1$ so that A_2 is significantly suppressed!
- **We believe that the strong suppression of Re A_2 and the (less-strong) enhancement of Re A_0 is a major factor in the $\Delta I = 1/2$ rule.**

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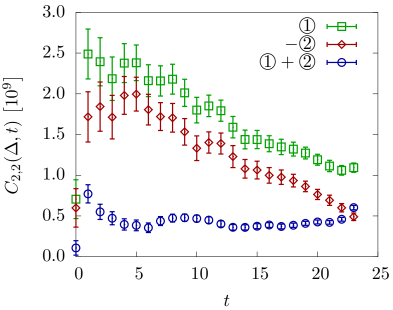

Physical Kinematics

- Notation ⓘ $\equiv C_i,\ i = 1,2$.
- Of course before claiming a quantitative understanding of the $\Delta I = 1/2$ rule we needed to compute Re A_0 at physical kinematics and found a results of $\simeq 31 \pm 12$ to be compared to the experimental value of 22.5.
- Much early phenomenology was based on the vacuum insertion approach. although the qualitative picture we find had been suggested by Bardeen, Buras and Gerard in 1987.

Many exciting lattice studies in kaon physics underway.

- For leptonic and semileptonic decays, in order to make further progress in phenomenology it is necessary to compute the isospin-breaking corrections (including radiative correction).
  - This is underway! G.Martinelli's lectures
- For the evaluation of long-distance contributions to physical quantities in kaon physics, the theoretical framework has been developed and exploratory computations have been performed.
  - Physical mass calculations are beginning (for Δm_K they are well advanced) and results will be available in $\lesssim$ 2 years.
- As a results of our work, the computation of A_2 is now almost "standard".
- Our results for ε'/ε will be updated later this summer, along with many related results.
- ε'/ε is now a quantity which is amenable to lattice computations.
  - The lattice contribution is the determination of the matrix elements $\langle\pi\pi|Q_i^{\text{RI-SMOM}}|K\rangle$. These contain all the NP QCD effects and are then processed to give ε'/ε.

- It appears that the explanation of the $\Delta I = 1/2$ rule has a number of components, of which the significant cancelation between the two dominant contributions to ReA_2 is a major one.
- We have completed the first calculation of ε'/ε with controlled errors $\Rightarrow$ motivation for further refinement (systematic improvement by collecting more statistics, working on larger volumes, $\geq$2 lattice spacings etc.)
  - I stress that our particular direct contribution is the determination of the matrix elements $\langle\pi\pi|Q_i^{\text{RI-SMOM}}|K\rangle$. These contain all the NP QCD effects and are then processed to give ε'/ε.
- Other non-standard calculations of the RBC-UKQCD collaborations include the evaluation of Δm_K, the long-distance contribution to ε_K and the study of long-distance contributions to rare kaon decays.