# Lattice Flavour Physics: Lecture 3

**Chris Sachrajda**

Department of Physics and Astronomy
University of Southampton
Southampton SO17 1BJ
UK

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Unfinished kaon physics from lecture 2

1. Final comment on long-distance contributions
2. Directly computing $K \to \pi\pi$ decay amplitudes

Mostly $B$s

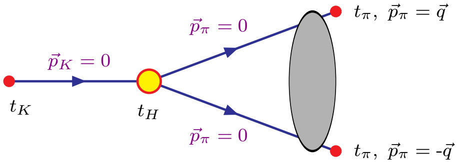
1. $B$-physics anomalies
2. Lattice $B$-physics and effective theories
3. $f_B$ and $f_{B_s}$
4. $V_{ub}$ and $V_{cb}$
5. Case Study: $B_S \to \mu^+\mu^-$
6. The Golden Mode - $B \to K_S J/\Psi$

- The word *long* needs a comparison scale.
- In the RBC-UKQCD programme of evaluating "long-distance" contributions discussed in lecture 2, by long-distance we mean distances $< 1/m_c$ so that the charm quark is treated as a propagating quark.
- The renormalisation scale $\mu > m_c$ so that perturbation theory can be avoided at the charm scale.
  - Perturbation theory at $\mu \simeq m_c$ is frequently poorly converging, as in e.g. $\Delta m_K$.

J.Brod & M.Gorbahn, arXiv:1108.2036

## 2. Directly computing $K \to \pi\pi$ decay amplitudes

- $K \to \pi\pi$ decays are a very important class of processes for standard model phenomenology with a long and noble history.
    - It is in these decays that both indirect and direct CP-violation was discovered.
- Bose Symmetry $\Rightarrow$ the two-pion state has isospin 0 or 2.
- Among the very interesting issues are the origin of the $\Delta I = 1/2$ rule (Re $A_0$/Re $A_2 \simeq 22.5$) and an understanding of the experimental value of $\varepsilon'/\varepsilon$, the parameter which was the first experimental evidence of direct CP-violation.
- The evaluation of $K \to \pi\pi$ matrix elements requires an extension of the standard computations of $\langle 0\,|\,O(0)\,|\,h\rangle$ and $\langle h_2\,|\,O(0)\,|\,h_1\rangle$ matrix elements with a single hadron in the initial and/or final state.



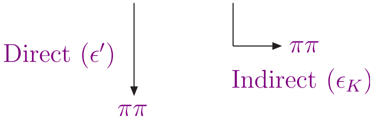

- $K \to \pi\pi$ correlation function is dominated by lightest state, i.e. the state with two-pions at rest. Maiani and Testa, PL B245 (1990) 585

$$C(t_\pi) = A + B_1 e^{-2m_\pi t_\pi} + B_2 e^{-2E_\pi t_\pi} + \cdots$$

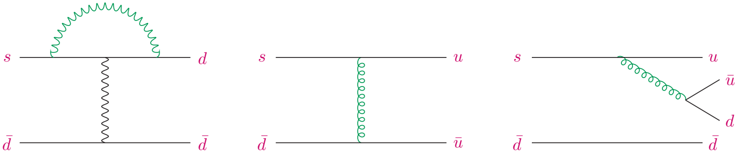
  - Solution 1: Study an excited state. Lellouch and Lüscher, hep-lat/0003023
  - Solution 2: Introduce suitable boundary conditions such that the $\pi\pi$ ground state is $|\pi(\vec{q})\pi(-\vec{q})\rangle$. RBC-UKQCD, C.h.Kim hep-lat/0311003

For $B$-decays, with so many intermediate states below threshold, this is the main obstacle to producing reliable calculations.

- Directly $CP$-violating decays are those in which a $CP$-even (-odd) state decays into a $CP$-odd (-even) one: $K_L \;\propto\; K_2 \;+\; \bar{\epsilon} K_1$ .




- Consider the following contributions to $K \to \pi\pi$ decays:




$I = 0$, Complex (a) | $I = 0$, Real (b) | $I = 0$ or $2$, Real (c)

Direct $CP$-violation in kaon decays manifests itself as a non-zero relative phase between the $I = 0$ and $I = 2$ amplitudes.

- We also have *strong phases*, $\delta_0$ and $\delta_2$ which are independent of the form of the weak Hamiltonian.

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$$\mathcal{H}_{\rm eff}^{\Delta S=1} = \frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \sum_{i=1}^{10} [z_i(\mu) + \tau y_i(\mu)] Q_i \,, \text{ where } \tau = -\frac{V_{ts}^* V_{td}}{V_{us}^* V_{ud}} \text{ and}$$

Current – Current Operators

$$Q_1 = (\bar{s}d)_L (\bar{u}u)_L \qquad Q_2 = (\bar{s}^i d^j)_L (\bar{u}^j u^i)_L$$

QCD Penguin Operators

$$Q_3 = (\bar{s}d)_L \textstyle\sum_{q=u,d,s} (\bar{q}q)_L \qquad Q_4 = (\bar{s}^i d^j)_L \textstyle\sum_{q=u,d,s} (\bar{q}^j q^i)_L$$

$$Q_5 = (\bar{s}d)_L \textstyle\sum_{q=u,d,s} (\bar{q}q)_R \qquad Q_6 = (\bar{s}^i d^j)_L \textstyle\sum_{q=u,d,s} (\bar{q}^j q^i)_R$$

Electroweak Penguin Operators

$$Q_7 = \tfrac{3}{2} (\bar{s}d)_L \textstyle\sum_{q=u,d,s} e_q (\bar{q}q)_L \qquad Q_8 = \tfrac{3}{2} (\bar{s}^i d^j)_L \textstyle\sum_{q=u,d,s} e_q (\bar{q}^j q^i)_L$$

$$Q_9 = \tfrac{3}{2} (\bar{s}d)_L \textstyle\sum_{q=u,d,s} e_q (\bar{q}q)_R \qquad Q_{10} = \tfrac{3}{2} (\bar{s}^i d^j)_L \textstyle\sum_{q=u,d,s} e_q (\bar{q}^j q^i)_R$$

This 10 operator basis is very natural but over-complete:

$$\begin{aligned} Q_{10} - Q_9 &= Q_4 - Q_3 \\ Q_4 - Q_3 &= Q_2 - Q_1 \\ 2Q_9 &= 3Q_1 - Q_3 \,. \end{aligned}$$

- For $A_2$, there is no vacuum subtraction and we can use the Wigner-Eckart theorem to write

$$\underbrace{\langle(\pi\pi)^{I=2}_{I_3=1}\,|}_{\frac{1}{\sqrt{2}}(\langle\pi^+\pi^0|+\langle\pi^0\pi^+|)}\quad Q^{\Delta I=3/2}_{\Delta I_3=1/2,i}\,|\,K^+\rangle=\frac{3}{2}\underbrace{\langle(\pi\pi)^{I=2}_{I_3=2}\,|}_{\langle\pi^+\pi^+|}Q^{\Delta I=3/2}_{\Delta I_3=3/2,i}\,|\,K^+\rangle\,,$$

  and impose anti-periodic conditions on the d-quark in one or more directions.

- If we impose the anti-periodic boundary conditions in all 3 directions then the ground state is

$$\Big|\pi\Big(\frac{\pi}{L},\frac{\pi}{L},\frac{\pi}{L}\Big)\,\pi\Big(\frac{-\pi}{L},\frac{-\pi}{L},\frac{-\pi}{L}\Big)\Big\rangle\,.$$

- With an appropriate choice of $L$ and the number of directions, we can arrange that $E_{\pi\pi}=m_K$.

- Isospin breaking by the boundary conditions is harmless (exponentially small in the volume) here. CTS & G.Villadoro, hep-lat/0411033
  - This is not the case for $\Delta I=1/2$ transitions $\Rightarrow$ $G$-parity boundary conditions.

- In 2015 RBC-UKQCD published our first result for $\varepsilon'/\varepsilon$ computed at physical quark masses and kinematics, albeit still with large relative errors:

$$\frac{\varepsilon'}{\varepsilon}\bigg|_{\text{RBC-UKQCD}} = (1.38 \pm 5.15 \pm 4.59) \times 10^{-4}$$

to be compared with

$$\frac{\varepsilon'}{\varepsilon}\bigg|_{\text{Exp}} = (16.6 \pm 2.3) \times 10^{-4}\,.$$

RBC-UKQCD, arXiv:1505.07863

- This is by far the most complicated project that I have ever been involved with.
- This single result hides much important (and much more precise) information which we have determined along the way.
- We are updating the results with about $\geq 6$ times the statistics and much improved techniques for reducing the systematic uncertainties.
- It is planned to present updated results within the next few months.

- Elastic $\pi\pi$ phase-shifts are obtained by measuring $E_{\pi\pi} - 2E_{\pi}$ and using Lüscher's formula.
- Isospin 0, two-pion correlators are noisy (primarily due to the vacuum subtraction) and measuring the ground-state two-pion energy is challenging.
- A puzzle from our 2015 paper was that we found $\delta_S^{I=0}(m_K^2) = (23.8 \pm 4.9 \pm 1.2)^\circ$ to be compared to $\sim 35^\circ$ from dispersive analyses.

  G.Colangelo, J.Gasser, & H.Leutwyler, hep-ph/0103088
- With increased statistics, and more importantly the use of additional interpolating operators for the two-pion state, we are able to understand and reduce the contamination from excited states and now find $\delta_S^{I=0}(m_K^2) = (30.9 \pm 1.5 \pm 3.0)^\circ$.

  T.Wang and C.Kelly, PoS Lattice 2018 (2018) 276

Puzzle is resolved.

- In order to reduce possible contamination of excited states, we have now included the additional two-pion interpolating operators into the $K \to \pi\pi$ analysis and plan to present the results later this year.

- The amplitude $A_2$ is considerably simpler to evaluate that $A_0$.
- Our first results for $A_2$ at physical kinematics were obtained at a single, rather coarse, value of the lattice spacing ($a \simeq 0.14$ fm). Estimated discretization errors at 15%. arXiv:1111.1699, arXiv:1206.5142
- Our latest results were obtained on two new ensembles, $48^3$ with $a \simeq 0.11$ fm and $64^3$ with $a \simeq 0.084$ fm so that we can make a continuum extrapolation:

$$\begin{aligned}\mathrm{Re}(A_2) &= 1.50(4)_{\mathrm{stat}}(14)_{\mathrm{syst}} \times 10^{-8}\ \mathrm{GeV}.\\ \mathrm{Im}(A_2) &= -6.99(20)_{\mathrm{stat}}(84)_{\mathrm{syst}} \times 10^{-13}\ \mathrm{GeV}.\end{aligned}$$

arXiv:1502.00263

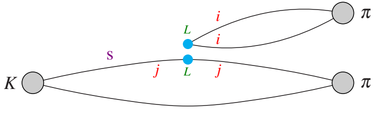
- The experimentally measured value is $\mathrm{Re}(A_2) = 1.479(4) \times 10^{-8}$ GeV.
- Although the precision can still be significantly improved (partly by perturbative calculations), the calculation of $A_2$ at physical kinematics can now be considered as standard.

RBC-UKQCD Collaboration, arXiv:1212.1474

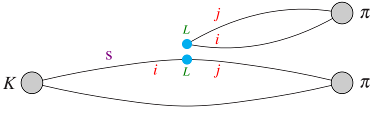
- Re $A_2$ is dominated by a simple operator:

$$O^{3/2}_{(27,1)} = (\bar{s}^i d^i)_L \{(\bar{u}^j u^j)_L - (\bar{d}^j d^j)_L\} + (\bar{s}^i u^i)_L (\bar{u}^j d^j)_L$$

and two diagrams:

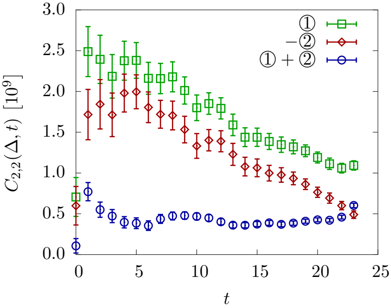



$C_1$




$C_2$

- Re $A_2$ is proportional to $C_1 + C_2$.
- The contribution to Re $A_0$ from $Q_2$ is proportional to $2C_1 - C_2$ and that from $Q_1$ is proportional to $C_1 - 2C_2$ with the same overall sign.
- Colour counting might suggest that $C_2 \simeq \frac{1}{3}C_1$.
- We find instead that $C_2 \approx -C_1$ so that $A_2$ is significantly suppressed!
- **We believe that the strong suppression of Re $A_2$ and the (less-strong) enhancement of Re $A_0$ is a major factor in the $\Delta I = 1/2$ rule.**

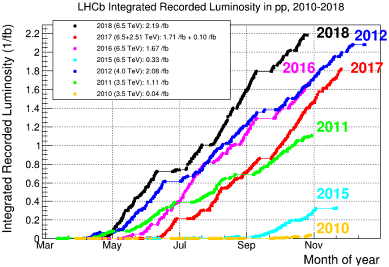



Physical Kinematics

- Notation ⓘ $\equiv C_i,\ i = 1, 2$.
- Of course before claiming a quantitative understanding of the $\Delta I = 1/2$ rule we needed to compute Re $A_0$ at physical kinematics and found a results of $\simeq 31 \pm 12$ to be compared to the experimental value of 22.5.
- Much early phenomenology was based on the vacuum insertion approach. although the qualitative picture we find had been suggested by Bardeen, Buras and Gerard in 1987.

- The $b$-quark is particularly suitable for detailed studies of the limits of the standard model and in searches for signatures of physics BSM.
  - It is sufficiently heavy ($m_B \simeq 5.3$ GeV) to have a huge number (hundreds) of decays modes.
  - It is sufficiently light that it can be produced copiously.

After the LS2 long shutdown, LHCb is preparing to take date at 5 times increased luminosity.




- The $c$-quark also leads to very interesting physics.

## 1 $R_K$ and $R_{K^*}$

- Defining $$R_H = \frac{\int dq^2\,\frac{d\Gamma(B\to H\mu^+\mu^-)}{dq^2}}{\int dq^2\,\frac{d\Gamma(B\to He^+e^-)}{dq^2}}$$

  where $H = K, K^*$ and $q^2 = (p_{\ell^+} + p_{\ell^-})^2$, the LHCb collaboration find

$$R_{K^+} = 0.846^{+0.060}_{-0.054}\,{}^{+0.16}_{-0.14} \text{ for } 1.1 < q^2 < 6\,\mathrm{GeV}^2$$ arXiv:1903.09252

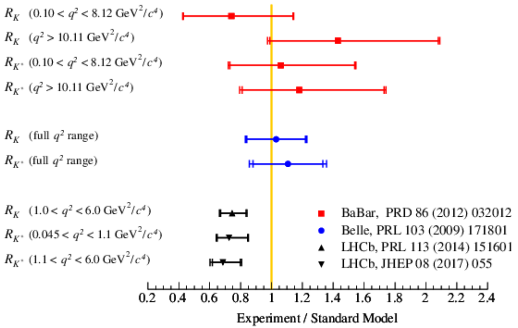
$$R_{K^{*0}} = 0.66^{+0.11}_{-0.07} \pm 0.03 \text{ for } 0.045 < q^2 < 1.1\,\mathrm{GeV}^2$$ arXiv:1705.05802

$$R_{K^{*0}} = 0.69^{+0.11}_{-0.07} \pm 0.05 \text{ for } 1.1 < q^2 < 6\,\mathrm{GeV}^2$$ arXiv:1705.05802

- For $R_K$ above and the higher $q^2$ value of $R_{K^*}$ the SM theoretical prediction is 1 to within 1% or so.
- For $R_{K^*}$ at lower $q^2$ the theoretical uncertainty is a little larger, e.g. Bordone, Isidori and Pattori find

$$R^{\mathrm{SM}}_{K^{*0}} = 0.906 \pm 0.028 \text{ for } 0.045 < q^2 < 1.1\,\mathrm{GeV}^2$$ arXiv:1705.05802

  and advocate raising the lower limit to 0.1 $\mathrm{GeV}^2$ in future analyses (which would decrease the theoretical uncertainty to 0.014).
- For $R_K$ and $R_{K^*}$ hadronic uncertainties do not play a rôle.




Review of lepton universality in $B$-decays, S.Bifani et al., arXiv:1809.06229

- This plot does not include the 2019 LHCb result for $R_K$.

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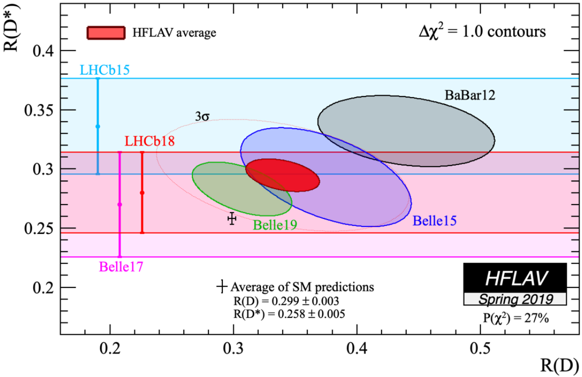
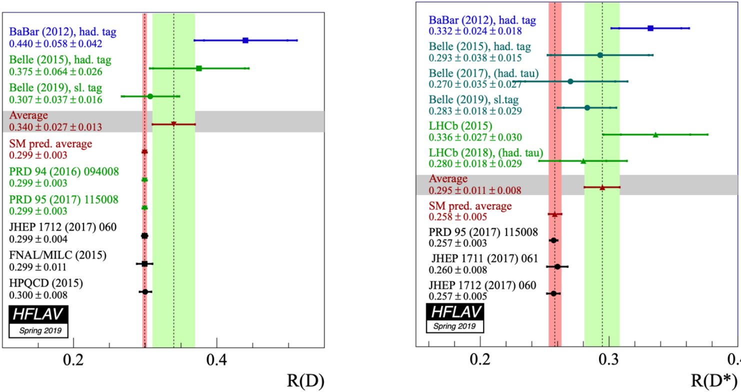
## 2 $R(D)$ and $R(D^*)$

- These are defined by: $$R_{H_c} = \frac{B(B \to H_c \tau^- \bar{\nu}_\tau)}{B(B \to H_c \ell^- \bar{\nu}_\ell)}\,,$$ where $\ell = e$ or $\mu$ at the b-factories or $\mu$ at LHCb (experimental limitations).
- "This definition cancels a large part of the theoretical ($V_{cb}$ and form factors) and experimental (branching fractions and reconstruction efficiencies) uncertainties."

  Review of lepton universality in $B$-decays, S.Bifani et al., arXiv:1809.06229
- Theoretical predictions nevertheless use lattice results for the semileptonic form-factors.


HFLAV average
$\Delta\chi^2 = 1.0$ contours
LHCb15
BaBar12
3σ
LHCb18
Belle19
Belle15
Belle17
Average of SM predictions
R(D) = 0.299 ± 0.003
R(D*) = 0.258 ± 0.005
HFLAV
Spring 2019
$P(\chi^2) = 27\%$
R(D*)
R(D)
0.2
0.25
0.3
0.35
0.4
0.5




- In the Bifani et al. review, the authors estimate that the combined pull of $R_D$ and $R_{D^*}$ from the SM is 3.6 - 3.8 $\sigma$.

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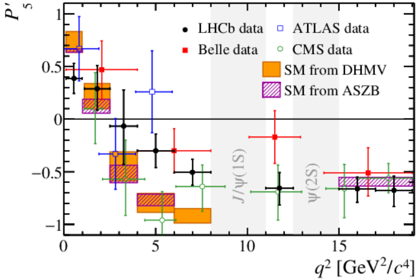
## 3 Angular Observables

- There are also deviations from SM expectations in $B \to K^* \mu^+ \mu^-$ angular observables.
- The quantities which are studied are derived from $d^4\Gamma/dq^2 d\vec{\Omega}$, where $\vec{\Omega} = (\cos\theta_\ell, \cos\theta_K, \phi)$ are three angles defined from the directions of the $B$, $\mu^+$ and the $K\pi$ decay products of the $K^*$.
  - I will not discuss this further here.

see S.Meinel's lecture



- The $b$-quark is light-enough to be produced copiously and heavy enough to have a huge number of possible decay channels.
- $m_b \sim$ 4 - 5 GeV whilst typical lattice spacings are $a^{-1}$ ~2 - 4 GeV $\Rightarrow$ we cannot yet simulate the propagation of $b$-quarks directly in QCD (however, see recent FNAL/Milc work).
- Most approaches rely on effective theories and invest a considerable effort in matching the effective theory to QCD.
  - Heavy Quark Effective Theory (expansion in $\frac{\Lambda_{\rm QCD}}{m_B}$).
  - Nonrelativistic QCD (expansion in the quark's velocity).
  - Relativistic Heavy Quarks ("Fermilab Approach" and extensions).

A. El Khadra, A. Kronfeld and P. Mackenzie, hep-lat/9604004

- Some groups also extrapolate results from the charm, or above, to the bottom region, using scaling laws where applicable and possibly using results in the static limit.
- There are still fewer, but an increasing number of calculations in heavy-quark physics, so less opportunity to check for consistency of different approaches.
- Unfortunately we do not know (yet?) how to compute non-leptonic $B$-decays ($B \to \pi\pi$, $B \to \pi K$ etc).

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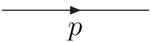
- $B$-physics is playing a central rôle in flavourdynamics and it is useful to exploit the symmetries which arise when $m_Q \gg \Lambda_{\rm QCD}$.
- The *Heavy Quark Effective Theory* (HQET) is proving invaluable in the study of heavy quark physics.
  - For scales $\ll m_Q$ the physics in HQET is the same as in QCD.
  - For scales $O(m_Q)$ and greater, the physics is different but, in principle, can be *matched* onto QCD using perturbation theory.
  - The approach of the Alpha Collaboration is to match the HQET, including the $O(1/m_b)$ corrections, nonperturbatively. Rainer Sommer's Lectures?
  - The non-perturbative physics in the same in the HQET as in QCD.

  Consider the propagator of a (free) heavy quark:

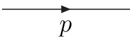



$$= i\,\frac{\not p + m}{p^2 - m_Q^2 + i\epsilon}\,.$$

- If the momentum of the quark $p$ is not far from its mass shell,

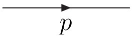
$$p_\mu = m_Q v_\mu + k_\mu\,,$$

where $|k_\mu| \ll m_Q$ and $v_\mu$ is the (relativistic) four velocity of the hadron containing the heavy quark ($v^2 = 1$), then




$$= i\,\frac{1+\not v}{2}\,\frac{1}{v\cdot k + i\epsilon} + O\left(\frac{|k_\mu|}{m_Q}\right).$$

# HQET Cont.




$$= i\,\frac{1+\not v}{2}\,\frac{1}{v\cdot k+i\epsilon} + O\left(\frac{|k_\mu|}{m_Q}\right).$$

- $(1+\not v)/2$ is a projection operator, projecting out the *large* components of the spinors.
- This propagator can be obtained from the gauge-invariant action

$$\mathscr{L}_{HQET} = \bar h(iv\cdot D)\frac{1+\not v}{2}h$$

  where $h$ is the spinor field of the heavy quark.
- $\mathscr{L}_{HQET}$ is independent of $m_Q$, which implies the existence of symmetries relating physical quantities corresponding to different heavy quarks (in practice the $b$ and $c$ quarks or Scaling Laws).
- The light degrees of freedom are also not sensitive to the spin of the heavy quark, which leads to a spin-symmetry relating physical properties of heavy hadrons of different spins.

- Consider, for example, the correlation function:

$$\int d^3x \, \langle 0 | J_H(x) \, J_H^\dagger(0) \, | 0 \rangle \, ,$$

  - $J_H^\dagger$ and $J_H$ are interpolating operators which can create or annihilate a heavy hadron $H$.
  - Here I take $H$ to be a pseudoscalar or vector meson.
  - The hadron is produced at rest, with four velocity $v = (1, \vec{0})$.
  - For example take $J_H = \bar{h}\gamma^5 q$ for the pseudoscalar meson and $J_H = \bar{h}\gamma^i q$ $(i = 1, 2, 3)$ for the vector meson. This means that the correlation function will be identical in the two cases except for the factor

$$\gamma^5 \, \frac{1+\gamma^0}{2} \, \gamma^5 = \frac{1-\gamma^0}{2}$$

    when $H$ is a pseudoscalar meson, and

$$\gamma^i \, \frac{1+\gamma^0}{2} \, \gamma^i = -3 \, \frac{1-\gamma^0}{2}$$

    when it is a vector meson.

- Correlation functions $\sim \exp(-iM_H t) \Rightarrow$ the pseudoscalar and vector mesons are degenerate (up to relative corrections of $O(\Lambda_{QCD}^2/m_Q)$):

$$M_P = M_V + O(\Lambda_{QCD}^2/m_Q)\,.$$

(or $M_V^2 - M_P^2 = \text{constant}$.)

- Heavy quark scaling laws (e.g. $f_P \sim 1/\sqrt{M_P}$) can be derived similarly.
- The difficulty is to go beyond the leading order and to determine the $O(1/m_b)$ corrections to physical quantities. There are more operators in the action, at $O(1/m_b)$:

$$\mathscr{L} = \bar{h}\,(D_4 + \delta_m)\,h + \omega_{\text{spin}}\bar{h}\,\sigma_{ij}F_{ij}\,h - \omega_{\text{kin}}\bar{h}\,\underline{D}^2\,h\,.$$

  - The coefficients $\delta_m$, $\omega_{\text{spin}}$ and $\omega_{\text{kin}}$ have to be determined by matching the lattice HQET onto QCD.
  - The higher dimensional operators $\bar{h}\,\underline{D}^2\,h$ mixes with the leading operator $\bar{h}\,D_4\,h$ with coefficients which diverge linearly with the UV cut-off.
- Higher dimensional operators also have to be added to the operators, e.g. in the evaluation of $f_B$ terms proportional to

$$\frac{1}{m_b}\bar{h}\gamma_5(\underline{\gamma}\cdot\overrightarrow{\underline{D}})q \quad \text{and} \quad \frac{1}{m_b}\bar{h}\gamma_5(\underline{\gamma}\cdot\overleftarrow{\underline{D}})q$$

have to be added to the axial current.

The aim is to construct lattice actions for simulations which will remove $O((m_Q a)^n)$ $(\forall n)$ and $O(\Lambda_{\rm QCD}a)$ discretization errors.

- Start with lattice QCD and imagine adding all possible “irrelevant" terms necessary to *improve* the action a la Symanzik.
- Use the equations of motion to reduce the number of terms to the minimum required to achieve the required precision.

$$S=\sum_{n,n'}\bar\psi_{n'}\left(\gamma_4D_4+\zeta\vec\gamma\cdot\vec D+m_0-\frac{r_t}{2}D_4^2-\frac{r_s}{2}\vec D^2+c_B\sum_{i,j}\frac{i}{4}\sigma_{ij}F_{ij}+c_E\sigma_{4i}F_{4i}\right)_{n'n}\psi_n.$$

- All higher dimensional operators which might be added to the action can be reduced to those above, up to terms of $O((\Lambda_{\rm QCD}a)^2)$.

This idea was first proposed by the Fermilab Group,

A.X.El-Khadra, A.S.Kronfeld & P.B.Mackenzie [hep-lat/9604004]

$$S=\sum_{n,n'}\bar\psi_{n'}\left(\gamma_4D_4+\zeta\vec\gamma\cdot\vec D+m_0-\frac{r_t}{2}D_4^2-\frac{r_s}{2}\vec D^2+c_B\sum_{i,j}\frac{i}{4}\sigma_{ij}F_{ij}+c_E\sigma_{4i}F_{4i}\right)_{n'n}\psi_n.$$

- In fact only 3 of the above 6 parameters ($\zeta$, $m_0$, $r_{t,s}$, $c_{E,B}$) need to be determined in order to ensure on-shell improvement.
- Example of the reduction of the number of parameters:

$$\begin{aligned}\bar\psi\sigma_{4i}F_{4i}\psi &= \bar\psi\gamma_4\gamma_i\left[D_4D_i-D_iD_4\right]\psi\\ &= -2m_0\,\bar\psi\,\vec\gamma\cdot\vec D\,\psi-4\bar\psi\vec D^2\,\psi\end{aligned}$$

  Thus a change in the coefficient $c_E$ can be compensated by a change in the coefficients $\zeta$ and $r_s$ .
- The Columbia group pointed out that a further parameter can be eliminated by making changes of variables in the fermion functional integral such as:

$$\psi\to\left(1+\chi\sigma_{4i}[D_i,D_4]\right)\psi$$

  They propose to set $r_s=r_t=1$ and then tune the 3 parameters, $m_0,\zeta$ and $c_P\equiv c_E=c_B$ .

  N.Christ, M.Li & H-W.Lin [heo-lat/0608006] .
- This has now been done non-perturbatively for both $c$ and $b$ quarks by using $m_D$ ($m_B$), $m_D^*-m_D$, ($m_B^*-m_B$) and the dispersion relations to fix the parameters.

  RBC-UKQCD arXiv:1206.2554, and updates

The Fermilab approach and its generalizations are very interesting.

- It allows for the calculation of a variety of important quantities in heavy-quark physics (e.g. leptonic decay constants, form-factors for semileptonic decays, rare decay amplitudes).
- We need to add all possible improvement terms to the operators whose matrix elements we are computing with matching coefficients which have to be determined.
- A non-perturbative procedure for evaluating the matching coefficients still has to be developed (existing results were obtained using perturbatively determined coefficients at fixed $a$).
- The coefficients of the neglected operators are functions of $m_Q a$ and one might worry that they become large, particularly for $b$-physics. However, the theory does have the correct static limit.

- In the physics of heavy quarkonia the appropriate expansion parameter is the velocity ($p \sim v$ and $K \sim v^2$) and NRQCD is designed to facilitate this expansion.

NRQCD

- $E = m(1 + O(v^2))$
- $p = O(m * v)$
- Expand in $v$.

Heavy Quark Expansion

- $E = m + O(\Lambda_{\mathrm{QCD}})$
- $p = O(\Lambda_{\mathrm{QCD}})$
- Expand in $\Lambda_{\mathrm{QCD}}/m$.

- NRQCD is also used in computations of quantities in Heavy-Light physics, where the HQET counting is relevant.

- In practice the groups doing the calculations do their book-keeping and systematics differently:
- HQET
  - The kinetic term $\vec{D}^2/2m$ is treated as a perturbation.
  - The coefficients are being determined nonperturbatively (see above).
  - The continuum limit is taken.
- NRQCD
  - The kinetic term $\vec{D}^2/2m$ is treated as a full part of the action (and included in the propagator).
  - The coefficients are being determined perturbatively (estimates of higher order corrections are included in the errors).
  - The calculations are performed at fixed $a$ ($ma$ is simply treated as a number).

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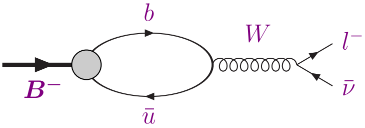
A slide from Lecture 1

- The difficulty in making predictions for weak decays of hadrons is in controlling the non-perturbative strong interaction effects.
- As a particularly simple example consider the leptonic decays of pseudoscalar mesons in general and of the $B$-meson in particular.




- Non-perturbative QCD effects are contained in the matrix element

$$\langle 0|\,\bar{b}\gamma^\mu(1-\gamma^5)u\,|B(p)\rangle\ .$$

- Lorentz Inv. + Parity $\Rightarrow$ $\langle 0|\,\bar{b}\gamma^\mu u\,|B(p)\rangle = 0$.
- Similarly $\langle 0|\,\bar{b}\gamma^\mu\gamma^5 u\,|B(p)\rangle = if_B p^\mu\ .$

All QCD effects are contained in a single constant, $f_B$, the $B$-meson's *(leptonic) decay constant.* ($f_\pi \simeq 132$ MeV)

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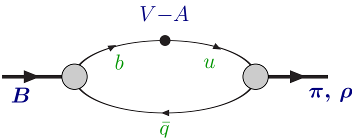
- There is no such decay of $B_s$, but it is nevertheless useful to calculate the matrix element $\langle 0|\,\bar{b}\gamma^\mu\gamma^5 s\,|B_s(p)\rangle = if_{B_s}\,p^\mu$. It enters in the prediction for $B_s \to \mu^+\mu^-$.
- In practice it is convenient to calculate $f_{B_s}$ for which the chiral extrapolation depends only on the light quarks in the sea and the ratio $f_{B_s}/f_B$ for which there is a partial cancellation of statistical and discretisation errors.
- For illustration, I list the four papers which FLAG chose for their compilations with $2+1+1$ active flavours (apologies to other authors):
  1. A.Bazavov et al., Fermilab Lattice and Milc Collaborations, arXiv:1712.09262
     HISQ action directly at the $b$-mass.
  2. A.Busone et al., ETM Collaboration, arXiv:1603.04306
     Interpolation between twisted-mass fermions and static $b$-quark.
  3. R.J.Dowdall et al., HPQCD Collaboration, arXiv:1302.2644
     Improved NRQCD.
  4. C.Hughes et al., HPQCD Collaboration, arXiv:1711.09981
     NRQCD.

## FLAG summary in heavy-quark physics

| Quantity | ■ | $N_f = 2+1+1$ | ■ | $N_f = 2+1$ | ■ | $N_f = 2$ |
|---|---|---|---|---|---|---|
| $\bar{m}_c(3\,\mathrm{GeV})$ (GeV) | 5 | $0.988(7)$ | 3 | $0.992(6)$ | | |
| $m_c/m_s$ | 3 | $11.768(33)$ | 2 | $11.82(16)$ | | |
| $\bar{m}_b(\bar{m}_b)(GeV)$ | 5 | $4.198(12)$ | 1 | $4.164(23)$ | | |
| $f_D$(MeV) | 2 | $212.0(7)$ | 2 | $209.0(2.4)$ | 1 | $208(7)$ |
| $f_{D_s}$(MeV) | 2 | $249.9(5)$ | 4 | $248.0(1.6)$ | 2 | $242.5(5.8)$ |
| $f_{D_s}/f_D$ | 2 | $1.1783(16)$ | 3 | $1.174(7)$ | 1 | $1.20(2)$ |
| $f_+^{D\pi}(0)$ | 1 | $0.612(35)$ | 1 | $0.666(29)$ | | |
| $f_+^{DK}(0)$ | 1 | $0.765(31)$ | 1 | $0.747(19)$ | | |
| $f_B$(MeV) | 4 | $190.0(1.3)$ | 5 | $192.0(4.3)$ | 2 | $188(7)$ |
| $f_{B_s}$(MeV) | 4 | $230.3(1.3)$ | 5 | $228.4(3.7)$ | 2 | $227(7)$ |
| $f_{B_s}/f_B$ | 4 | $1.209(5)$ | 5 | $1.201(16)$ | 2 | $1.206(23)$ |
| $f_{B_d}\sqrt{\hat{B}_{B_d}}$(MeV) | | | 3 | $225(9)$ | 1 | $216(10)$ |
| $f_{B_s}\sqrt{\hat{B}_{B_s}}$(MeV) | | | 3 | $274(8)$ | 1 | $262(10)$ |
| $\hat{B}_{B_d}$ | | | 3 | $1.30(10)$ | 1 | $1.30(6)$ |
| $\hat{B}_{B_s}$ | | | 3 | $1.35(6)$ | 1 | $1.32(5)$ |
| $\xi$ | | | 2 | $1.206(17)$ | 1 | $1.225(31)$ |
| $\hat{B}_{B_s}/\hat{B}_{B_d}$ | | | 2 | $1.032(38)$ | 1 | $1.007(21)$ |

- It is possible to extract $V_{ub}$ and $V_{cb}$ from either exclusive or inclusive semileptonic decays ($B \to H_c(H_u)\ell\bar{\nu}_\ell$, where $H_{c,u}$ are specific hadrons, or $B \to X_c(X_u)\ell\bar{\nu}_\ell$, where $X_{c,u}$ refer to a sum over states containing the specified quarks).
- The inclusive decays rely on the fact that the $b$-quark is heavy and therefore that perturbation theory can be used.
- In practice however, cuts have to be imposed to separate $b \to c$ decays from $b \to u$ ones which introduces technical complications. Nevertheless this is a standard method.
- Lattice simulations can be used to calculate the semileptonic decays $B \to (\pi, \rho, D, D^*)\ell\nu$ and a comparison with the experimental partial widths.




QCD effects are contained in form factors e.g. for $B \to \pi$ decays:

$$\langle \pi(p_\pi) | \bar{b}\gamma_\mu u | B(p_B) \rangle = f_0(q^2)\, \frac{m_B^2 - m_\pi^2}{q^2} q_\mu$$

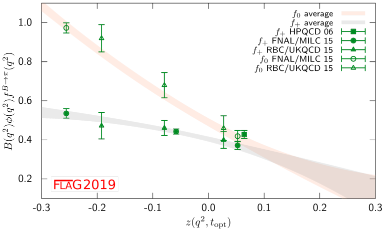
$$+ f_+(q^2) \left[ (p_\pi + p_B)_\mu - \frac{m_B^2 - m_\pi^2}{q^2} q_\mu \right]$$

where $q \equiv p_B - p_\pi$.

- For $B$-decays, in order to avoid lattice artefacts, the momentum of the $\pi$ or $\rho$ is limited $\Rightarrow$ get results only at large values of $q^2$.
- Thus $V_{ub}$ can only be obtained directly by combining the lattice results with a subset of the experimental data:

$$\Delta\zeta(q_1^2, q_2^2) = \frac{1}{|V_{ub}|^2} \int_{q_1^2}^{q_2^2} dq^2\, \frac{d\Gamma}{dq^2}\,.$$

- The lattice results can be combined with theoretically motivated parametrisations for the form factors, including constraints from analyticity and other general properties of field theory, to extend the range of the predictions.




- 5 parameter BCL fit.

- The form factors can be written in the form

$$f(q^2) = \frac{1}{B(q^2)\phi(q^2,t_0)} \sum_{n=0}^{\infty} a_n(t_0)\, z^n(q^2,t_0)$$

and letting $t^+ = (m_B + m_\pi)^2$

$$z(q^2,t_0) = \frac{\sqrt{t_+ - q^2} - \sqrt{t_+ - t_0}}{\sqrt{t_+ - q^2} + \sqrt{t_+ - t_0}}$$

the Blaschke factor $B(q^2) = z(q^2, m_{B^*}^2)$ in this case and $\phi(q^2,t_0)$ is a suitably chosen function which does not introduce any further singularities.

- The aim in choosing $\phi$ is to obtain a useful bound with the correct $q^2$ behaviour at high $q^2$ and at thresholds. see Appendix A.5 in FLAG2019, arXiv:1902.08191
- The two most popular choices are BGL and BCL:

  (BGL) Boyd, Grinstein & Lebed, hep-ph/9412324; (BCL) Bourelly, Caprini & Lellouch, arXiv:0807.2722

- PDG quote:

$$|V_{ub}|_{\rm excl} = (3.70 \pm 0.10 \pm 0.12) \times 10^{-3}\,,$$

where $\Delta\zeta(16\,{\rm GeV}^2, q^2_{\rm max})$ and $\Delta\zeta(18\,{\rm GeV}^2, q^2_{\rm max})$ are taken from

J.A.Bailey et al. (FNAL/MILC), arXiv:1503.07839; E.Dalgic et al. (HPQCD), hep-lat/0601021

- FLAG2019 quote

$$|V_{ub}|_{\rm excl} = (3.73 \pm 0.14) \times 10^{-3}\,,$$

from a BCL fit to $N_f = 2+1$ lattice and experimental data (the shapes agree well). The lattice data are taken from:

J.A.Bailey et al. (FNAL/MILC), arXiv:1503.07839; J.M.Flynn et al. (RBC/UKQCD), arXiv:1501.05373; E.Dalgic et al. (HPQCD), hep-lat/0601021

- A long-standing puzzle has been the discrepancy with the value obtained from inclusive $B \to X_u \ell \bar{\nu}$ decays, which have very different systematics.

- The PDG 2018 Review quotes an inclusive average

$$|V_{ub}|_{\rm incl} = (4.49 \pm 0.16^{+0.16}_{-0.17} \pm 0.17)\, 10^{-3}\,,$$

$$\Gamma(B^- \to \ell^- \bar{\nu}_\ell) = \frac{G_F^2\,|V_{ub}|^2 f_B^2}{8\pi}\, m_B\, m_\ell^2 \left(1 - \frac{m_\ell^2}{m_B^2}\right)^2 .$$

- It is also possible to determine $V_{ub}$ from the measured branching ratio for the decay $B \to \tau\bar{\nu}$; PDG2019 quote

$$\mathrm{Br}(B \to \tau\bar{\nu}) = (1.09 \pm 0.24) \times 10^{-4}\,.$$

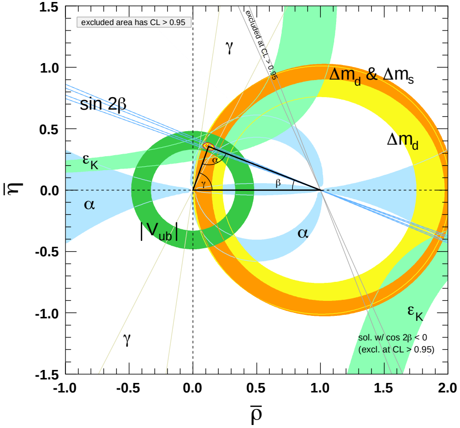
- FLAG2019 take a different value for this branching fraction, $(1.06 \pm 0.33) \times 10^{-4}$, and use the lattice values of $f_B$ for $N_f = 2+1+1$ obtained as described above, to determine

$$V_{ub} = (4.05 \pm 0.03 \pm 0.64) \times 10^{-3}\,,$$

  where the first error is from the lattice determination of $f_B$ and the second from experiment.

- $|V_{ub}|^2 = \bar{\rho}^2 + \bar{\eta}^2$ and so a precise determination of $|V_{ub}|$ determines a circle on which the vertex $A$ of the UT must lie.

# PDG2018 Unitarity Triangle



- The CKM matrix element $V_{cb}$ is a very important parameter of the SM, serving as an input into many quantities contributing to the Unitarity Triangle Analysis.
  - In particular it appears as $|V_{cb}|^4$ in the dominant term in $\varepsilon_K$.
- The two decay channels which are used to extract $|V_{cb}|_{\rm excl}$ are

$$B\to D^*\ell\nu \quad \text{and} \quad B\to D\ell\nu\,,$$

with the vector $D^*$ channel better measured.

- It is conventional with heavy mesons to use $\omega = v_B\cdot v_{D^{(*)}}$ as the argument of the form-factors, instead of $q^2$.
  - Here $v_P=\frac{p_P}{m_P}$ is the four velocity of meson $P$.
  - In the rest-frame of the $B$-meson, the zero recoil point, $\omega=1$, corresponds to the $D$ also being at rest.

$$\begin{aligned}\langle D^*|V^\mu|B\rangle &= \sqrt{m_Bm_{D^*}}\,h_V(\omega)\,\varepsilon_{\mu\nu\alpha\beta}\,\varepsilon^{*\nu}v_{D^*}^\alpha v_B^\beta\\ \langle D^*|A^\mu|B\rangle &= i\sqrt{m_Bm_{D^*}}\left[h_{A_1}(\omega)(1+\omega)\varepsilon^{*\mu}-h_{A_2}(\omega)(\varepsilon^*\cdot v_B)v_B^\mu-h_{A_3}(\omega)(\varepsilon^*\cdot v_B)v_{D^*}^\mu\right]\end{aligned}$$

- The form-factors are combined into the functions $\mathscr{G}(\omega)$ (for $B^- \to D^0\ell^-\bar{\nu}$) and $\mathscr{F}(\omega)$ (for $B^- \to D^{*0}\ell^-\bar{\nu}$) which are the quantities determined from lattice simulations.
- There are simplifications in the calculation of $\mathscr{G}(1)$ and $\mathscr{F}(1)$, which can be obtained from double ratios, in which the renormalisation of the currents cancels.
  - Also, for $B \to D^*\ell\bar{\nu}$ decays, there are no $\frac{\Lambda_{\text{QCD}}}{m_Q}$ corrections to $\mathscr{F}(1)$.

    Luke's Theorem
  - The computation of $\mathscr{F}(1)$ and the subsequent evaluation of the differential rate at $\omega = 1$ is therefore a standard approach to determining $V_{cb}$.
- Recent work is beginning to extend such calculations to a wider range of $\omega$.

| | | |
|---|---|---|
| FLAG Average for $N_f = 2+1$ (BGL) | $B \to D^* \ell \nu$ | $V_{cb} = 42.55(57)(71) \times 10^{-3}$ |
| FLAG Average for $N_f = 2+1$ (CLN) | $B \to D^* \ell \nu$ | $V_{cb} = 39.12(52)(47) \times 10^{-3}$ |
| FLAG Average for $N_f = 2+1$ | $B \to D \ell \nu$ | $V_{cb} = 40.1(1.0) \times 10^{-3}$ |
| FLAG Average for $N_f = 2+1$ (BGL) | $B \to (D, D^*) \ell \nu$ | $V_{cb} = 41.4(1.2) \times 10^{-3}$ |
| FLAG Average for $N_f = 2+1$ (CLN) | $B \to (D, D^*) \ell \nu$ | $V_{cb} = 39.44(59) \times 10^{-3}$ |
| HFLAV inclusive average | $B \to X_c \ell \nu$ | $V_{cb} = 42.46(88) \times 10^{-3}$ |

- In order to use the lattice value of $\mathscr{F}(1)$, the experimental data needs to be extrapolated using an ansatz.

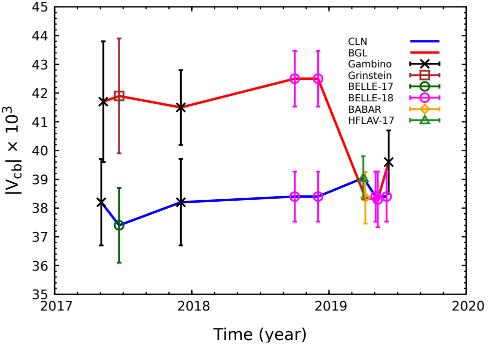
  Caprini, Lellouch, Neubert (CLN), hep-ph/9712417, Boyd, Grinstein, Lebed (BGL), hep-ph/9705252

- For $B \to D$ decays enough of the kinematic range is covered to do a direct comparison between lattice and experimental results.
- FLAG "ascribe the tension in the above determinations to a bias introduced in the fit by the CLN parameterization", however:
  - this was strongly refuted by L.Lellouch at Lattice 2019 who explains that the errors & correlations were not properly taken into account and

- As we heard yesterday, the latest Belle analysis gives a consistent result between the two parametrisations at the lower value: W.J.Lee's Colloquium

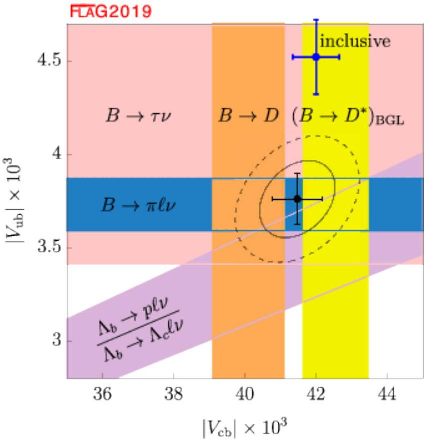


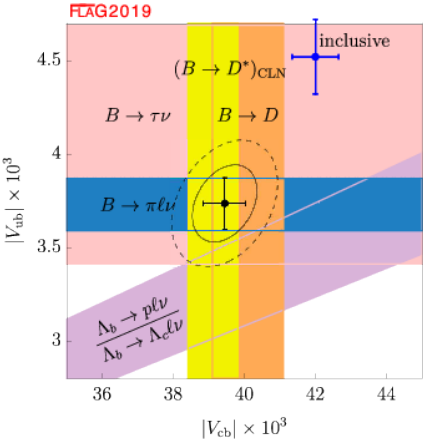
## CLN vs. BGL in $B \to D^* \ell \bar{\nu}$ decays




- At present, we find that there is no difference in exclusive $|V_{cb}|$ between CLN and BGL. $\Longrightarrow$ Resolved ???

# $(\mathbf{V_{ub}}, \mathbf{V_{cb}})$ summary



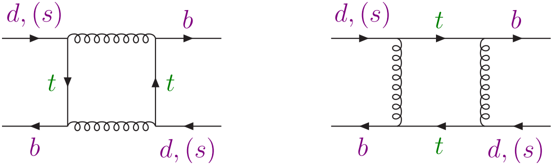



- The FLAG average for $R(D)$ is

$$R(D) \equiv \frac{B \to D\tau\nu_\tau}{B \to D\ell\nu_\ell} = 0.300(8) \qquad (\ell = e, \mu)$$

J. A. Bailey et al., (FNAL/MILC), arXiv:1503.07237; H. Na et al., (HPQCD), arXiv:1505.03925

  - FNAL/MILC: $b, c$ quarks using FNAL RHQ action.
    HPQCD: $b$ quark using NRQCD and $c$ quark using HISQ.
  - Recall that the experimental value from Babar and Belle is $R(D) = 0.340 \pm 0.027 \pm 0.013$ and this is one of the anomalies.




- In $B^0 - \bar{B}^0$ mixing, the top quark dominates and hence from the measured mass differences $\Rightarrow V_{td}$ and $V_{ts}$.
- The non-perturbative QCD effects are contained in the matrix element of the $\Delta B = 2$ operator:

$$\langle \bar{B} | O^{\Delta B=2} | B \rangle = \langle \bar{B} | \bar{b}\gamma^\mu (1-\gamma^5) d \, \bar{b}\gamma_\mu (1-\gamma^5) d | B \rangle \equiv \frac{8}{3} m_B^2 f_B^2 B_B(\mu) .$$

- PDG2018 use $\Delta m_d = (3.334 \pm 0.013) \times 10^{-10}$ MeV, $\Delta m_s = (1.1688 \pm 0.0014) \times 10^{-8}$ MeV and take the "lattice values" $f_{B_d}\sqrt{\hat{B}_{B_d}} = (219 \pm 14)$ MeV and $f_{B_s}\sqrt{\hat{B}_{B_s}} = (270 \pm 16)$ MeV FLAG2016, arXiv:1607.00299
  to obtain: $|V_{td}| = (8.1 \pm 0.5) \times 10^{-3}$ and $|V_{ts}| = (39.4 \pm 2.3) \times 10^{-3}$.

"The uncertainties are dominated by lattice QCD." PDG2018

- The uncertainties are reduced in the lattice calculation of the ratio

$$\xi = \frac{f_{B_s}\sqrt{B_{B_s}}}{f_{B_d}\sqrt{B_{B_d}}} = 1.237 \pm 0.032 \quad \Rightarrow \left|\frac{V_{td}}{V_{ts}}\right| = 0.210 \pm 0.001 \pm 0.008\,,$$

where the numerical values have been taken from PDG2018.

- For generic BSM theories, there are 5 $\Delta B = 2$ operators (and 5 $\Delta S = 2$ operators for neutral kaon mixing) whose matrix elements can be computed in a similar way.

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| Quantity | ■ | $N_f=2+1+1$ | ■ | $N_f=2+1$ | ■ | $N_f=2$ |
|---|---|---|---|---|---|---|
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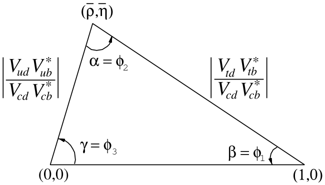
| $\hat{B}_{B_s}/\hat{B}_{B_d}$ | | | 2 | $1.032(38)$ | 1 | $1.007(21)$ |

# Back to the Unitarity Triangle

$$V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0\ .$$

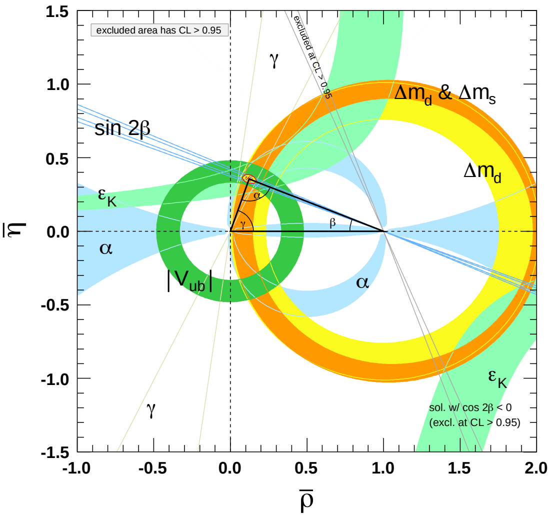



In terms of the Wolfenstein parameters, the components on the left-hand side are given by:

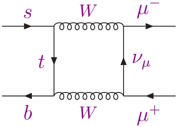
$$\begin{aligned} V_{ud}V_{ub}^* &= A\lambda^3[\bar\rho + i\bar\eta] + O(\lambda^7) \\ V_{cd}V_{cb}^* &= -A\lambda^3 + O(\lambda^7) \\ V_{td}V_{tb}^* &= A\lambda^3[1-(\bar\rho + i\bar\eta)] + O(\lambda^7)\ . \end{aligned}$$

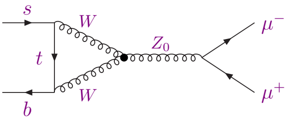
- $V_{td} \propto 1 - \bar\rho - i\bar\eta$ and so knowledge of $|V_{td}|$ fixes a circle centred on (1,0) on which the vertex $A$ must lie.

# PDG2018 Unitarity Triangle

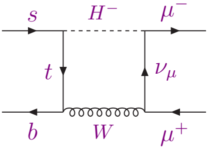


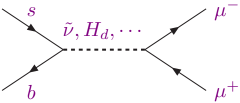
- Within the Standard Model there are box and penguin diagrams leading to the decay $B_S \to \mu^+\mu^-$:




- We shall see that the SM branching ratio is tiny and BSM there are many other potential contributions e.g.:




- This decay therefore constrains models of new physics and their parameter space.

# Case Study: $B_S \to \mu^+\mu^-$ (cont.)

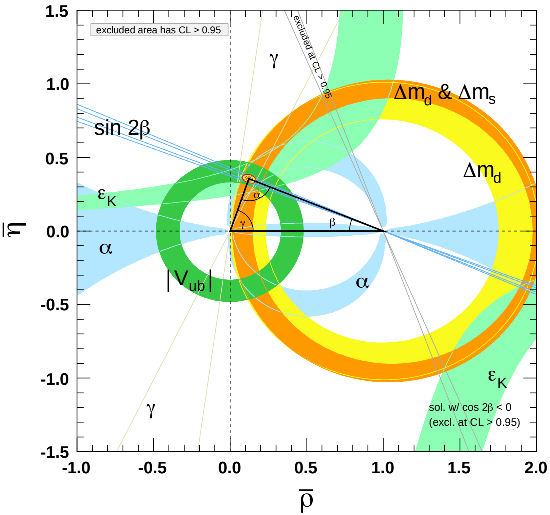
- Standard Model prediction for $B(B_s \to \mu^+\mu^-) \propto f_{B_s}^2 \times \frac{m_\mu^2}{m_{B_s}^2}$:

$$\mathrm{Br}(B_s \to \mu^+\mu^-) = (3.65 \pm 0.23) \times 10^{-9} \quad \text{Bobeth et al., arXiv : 1311.0903}$$
$$\mathrm{Br}(B_s \to \mu^+\mu^-) = (3.23 \pm 0.27) \times 10^{-9} \quad \text{Buras et al., arXiv : 1208.0934}$$

- Particle Data Group results for the branching ratios over the past decade:

| Year | PDG $B(B_s \to \mu^+\mu^-)$ |
|---|---|
| 2006 | $< 1.5 \times 10^{-7}$ |
| 2008,2010 | $< 4.7 \times 10^{-8}$ |
| 2012 | $< 6.4 \times 10^{-9}$ |
| 2014 | $(3.1 \pm 0.7) \times 10^{-9}$ |
| 2016,2017 | $(2.4^{+0.9}_{-0.7}) \times 10^{-9}$ |
| 2018 | $(2.7^{+0.6}_{-0.5}) \times 10^{-9}$ |
| 2019 | $(3.0 \pm 0.4) \times 10^{-9}$ |

- For the corresponding branching fraction of $B_d$ decays, the combined result is $(1.4^{+1.6}_{-1.4}) \times 10^{-10}$ compared to the theoretical prediction of $(1.06 \pm 0.09) \times 10^{-10}$.


excluded area has CL > 0.95
excluded at CL > 0.95
γ
Δm_d & Δm_s
sin 2β
Δm_d
ε_K
α
β
γ
α
|V_ub|
α
ε_K
γ
sol. w/ cos 2β < 0
(excl. at CL > 0.95)
η̄
ρ̄
1.5
1.0
0.5
0.0
-0.5
-1.0
-1.5
-1.0
-0.5
0.0
0.5
1.0
1.5
2.0

- We start by introducing the framework for Mixing Induced CP-Violating Decays.
- In order to study CP-violation we need to be sensitive to the weak phase $\Rightarrow$ *interference*.
- The strong interactions also generate phases, so, in general, we need to be able to control the hadronic effects.
- For the golden-mode $B \to J/\Psi K_S$ this is possible to a great degree of accuracy $\Rightarrow$ precise determination of $\sin(2\beta)$. I will now review the theoretical background behind this statement.
- The two neutral mass-eigenstates are given by

$$|B_L\rangle = \frac{1}{\sqrt{p^2+q^2}}\left(p\,|B^0\rangle + q\,|\bar{B}^0\rangle\right)$$

and

$$|B_H\rangle = \frac{1}{\sqrt{p^2+q^2}}\left(p\,|B^0\rangle - q\,|\bar{B}^0\rangle\right).$$

where $p$ and $q$ are complex parameters.

- The $2 \times 2$ mass-matrix takes the form

$$M - \frac{i\Gamma}{2} = \begin{pmatrix} A & p^2 \\ q^2 & A \end{pmatrix} .$$

where $A, p$ and $q$ are complex parameters.

- Starting with a $B^0$ meson at time $t = 0$, its subsequent evolution is governed by the Schrödinger equation:

$$|B^0_{\text{phys}}(t)\rangle = g_+(t)\,|B^0\rangle + \left(\frac{q}{p}\right) g_-(t)|\bar{B}^0\rangle \ , \quad \text{where}$$

$$\begin{aligned} g_+(t) &= \exp\left[-\frac{\Gamma t}{2}\right] \exp[-iMt] \cos\left(\frac{\Delta M\, t}{2}\right), \\ g_-(t) &= \exp\left[-\frac{\Gamma t}{2}\right] \exp[-iMt]\, i \sin\left(\frac{\Delta M\, t}{2}\right) \text{ and } M = (M_H + M_L)/2. \end{aligned}$$

- Starting with a $\bar{B}^0$ meson at $t = 0$, the time evolution is

$$|\bar{B}^0_{\text{phys}}(t)\rangle = (p/q)\, g_-(t)|B^0\rangle + g_+(t)\,|\bar{B}^0\rangle.$$

- Let $f_{CP}$ be a $CP$-eigenstate and $A,\bar{A}$ be the amplitudes

$$A \equiv \langle f_{CP}|\mathscr{H}|B^0\rangle \quad \text{and} \quad \bar{A} \equiv \langle f_{CP}|\mathscr{H}|\bar{B}^0\rangle .$$

- Defining

$$\lambda \equiv \frac{q}{p}\,\frac{\bar{A}}{A}$$

we have

$$\langle f_{CP}\,|\,\mathscr{H}\,|\,B^0_{\text{phys}}\rangle = A\,[g_+(t) + \lambda\, g_-(t)] \quad \text{and} \quad \langle f_{CP}\,|\,\mathscr{H}\,|\,\bar{B}^0_{\text{phys}}\rangle = A\,\frac{p}{q}\,[g_-(t) + \lambda\, g_+(t)].$$

- The time-dependent rates for initially pure $B^0$ or $\bar{B}^0$ states to decay into the $CP$-eigenstate $f_{CP}$ at time $t$ are given by:

$$\Gamma(B^0_{\text{phys}}(t) \to f_{CP}) = |A|^2\, e^{-\Gamma t} \times \left[\frac{1+|\lambda|^2}{2} + \frac{1-|\lambda|^2}{2}\cos(\Delta M\, t) - \text{Im}\,\lambda\,\sin(\Delta M\, t)\right]$$

$$\Gamma(\bar{B}^0_{\text{phys}}(t) \to f_{CP}) = |A|^2\, e^{-\Gamma t} \times \left[\frac{1+|\lambda|^2}{2} - \frac{1-|\lambda|^2}{2}\cos(\Delta M\, t) + \text{Im}\,\lambda\,\sin(\Delta M\, t)\right].$$

- The time-dependent asymmetry is defined as:

$$\mathscr{A}_{f_{CP}}(t) \equiv \frac{\Gamma(B^0_{\rm phys}(t)\to f_{CP})-\Gamma(\bar B^0_{\rm phys}(t)\to f_{CP})}{\Gamma(B^0_{\rm phys}(t)\to f_{CP})+\Gamma(\bar B^0_{\rm phys}(t)\to f_{CP})} = \frac{(1-|\lambda|^2)\cos(\Delta M\,t)-2{\rm Im}\,\lambda\,\sin(\Delta M\,t)}{1+|\lambda|^2}.$$

- If $|q/p|=1$ (which is the case if $\Delta\Gamma \ll \Delta M$) and $|\bar A/A|=1$ (examples of this will be presented below), then $|\lambda|=1$ and the first term on the right-hand side above vanishes.
- The form of the amplitudes $A$ and $\bar A$ is:

$$A=\sum_i A_i\,e^{i\delta_i}\,e^{i\phi_i} \quad \text{and} \quad \bar A=\sum_i A_i\,e^{i\delta_i}\,e^{-i\phi_i}$$

  - Sum is over all the contributions to the process;
  - the $A_i$ are real;
  - the $\delta_i$ are the strong phases;
  - the $\phi_i$ are the phases from the CKM matrix.

$$A=\sum_i A_i\,e^{i\delta_i}\,e^{i\phi_i}\quad\text{and}\quad \bar{A}=\sum_i A_i\,e^{i\delta_i}\,e^{-i\phi_i}$$

- In the most favourable situation, all the contributions have a single CKM phase ($\phi_D$ say) and

$$\frac{\bar{A}}{A}=\exp(-2i\phi_D).$$

- Since $\Gamma_{12}\ll M_{12}$, $q/p=\sqrt{M_{12}^*/M_{12}}\equiv\exp(-2i\phi_M)$, and

$$\lambda=\exp(-2i(\phi_D+\phi_M)).$$

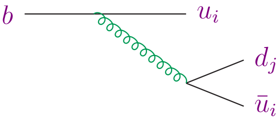
  Thus

$$\text{Im}\,\lambda=-\sin(2(\phi_D+\phi_M)).$$

- From the box diagrams:

$$\left(\frac{q}{p}\right)_{B_d}=\frac{V_{td}V_{tb}^*}{V_{td}^*V_{tb}}\quad\text{and}\quad\left(\frac{q}{p}\right)_{B_s}=\frac{V_{ts}V_{tb}^*}{V_{ts}^*V_{tb}}.$$

- Consider processes in which the $b$-quark decays through the subprocess $b \to d_j u_i \bar{u}_i$. The corresponding tree-level diagram is




for which

$$\frac{\bar{A}}{A} = \frac{V_{ib} V_{ij}^*}{V_{ib}^* V_{ij}} \,.$$

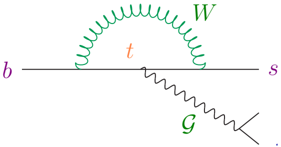
- $B_d \to J/\Psi K_S$ – In this case

$$\lambda(B \to J/\Psi K_S) = \frac{V_{td} V_{tb}^*}{V_{td}^* V_{tb}} \, \frac{V_{cs} V_{cd}^*}{V_{cs}^* V_{cd}} \, \frac{V_{cb} V_{cs}^*}{V_{cb}^* V_{cs}} \Rightarrow \mathrm{Im}\lambda = -\sin(2\beta)$$

  - The first factor is $(q/p)_{B_d}$;
  - the second factor is the analogous one for the final state kaon;
  - the third factor is $\bar{A}/A$, with $u_i = c$ and $d_j = s$.
  - Recall that

$$\beta = \arg\left(-\frac{V_{cd} V_{cb}^*}{V_{td} V_{tb}^*}\right) .$$

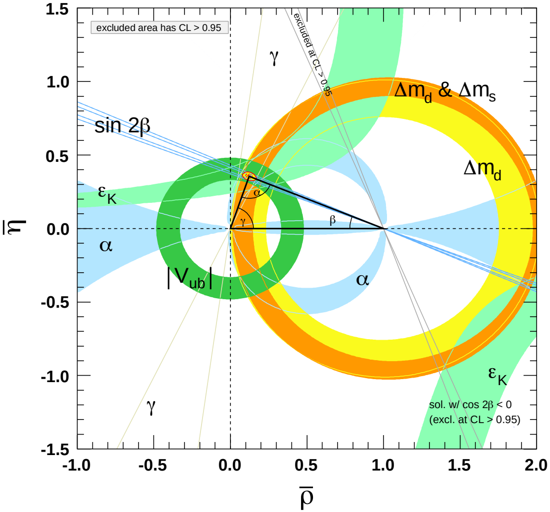
- There is also a small penguin contribution to this process:




- Phase is that of $V_{tb}V_{ts}^*$, which is is equal (to an excellent approximation) to that of $V_{cb}V_{cs}^*$.
- Thus we have a single weak phase and hence hadronic uncertainties are negligible in the determination of the $\sin(2\beta)$ from this process (*golden mode*).
- This is an (almost) ideal situation but one which is very rare.
- PDG 2018 average the results from BaBar, Belle and LHCb and obtain

$$\sin(2\beta) = 0.691 \pm 0.017\,.$$

# PDG2018 Unitarity Triangle



In these lectures, I have tried to explain:

i) that the SM is an incomplete description of fundamental physics;

ii) the importance of precision flavour physics in searches for new physics (FCNC as an important tool in this quest);
   - Even if new particles are discovered at the LHC, precision flavour physics will be central to unravelling the structure of the underlying theory;

iii) the importance of controlling the non-perturbative QCD effects and the role of lattice QCD;

iv) with a few examples, what can be computed using lattice simulations;
   - the limitations in precision in today's computations, but which are expected to be improved in (near) future simulations;
   - limitations which require new theoretical ideas to be overcome (e.g. $B \to \pi\pi$ decay amplitudes).

- The Standard Model is proving to be remarkably robust, but perhaps a few cracks are beginning to appear (e.g. $B$-physics anomalies).