Hadron-Hadron Interactions from Lattice QCD

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HISKP, University of Bonn
Extended Twisted Mass Collaboration

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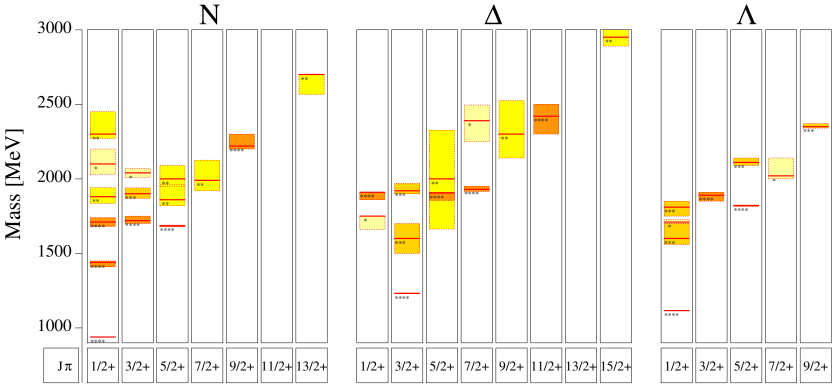
Motivation: Particle Zoo

QCD gives rise to a very rich particle spectrum, here nucleon, Δ and Λ



[PDG 2016, picture: B. Metsch]

- most states are resonances $\rightarrow$ decay to two or more particles

$\Rightarrow$ first principles theoretical computation highly valuable

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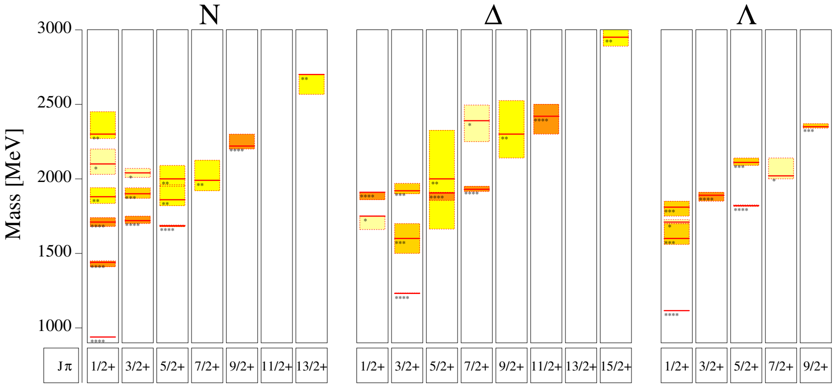

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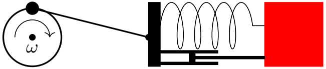
What is a resonance?

- consider a simple, driven damped harmonic oscillator



- equations of motion

$$\dot{x}^2 + 2D\omega_0\dot{x} + \omega^2 x = A\sin(\omega t)$$

- solution is known

$$x(t) = A'\sin(\omega t + \delta)$$

- for D small, a resonance occurs

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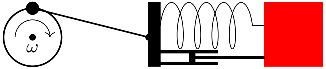

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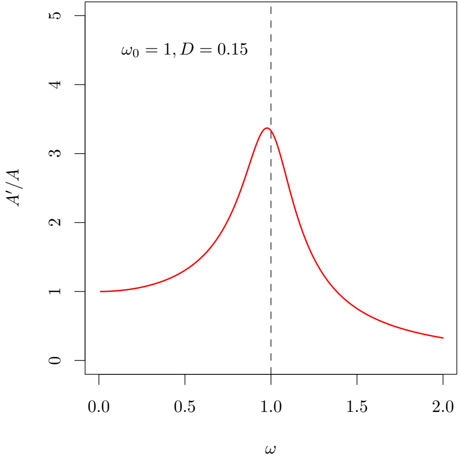

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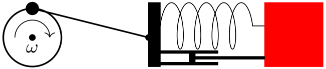

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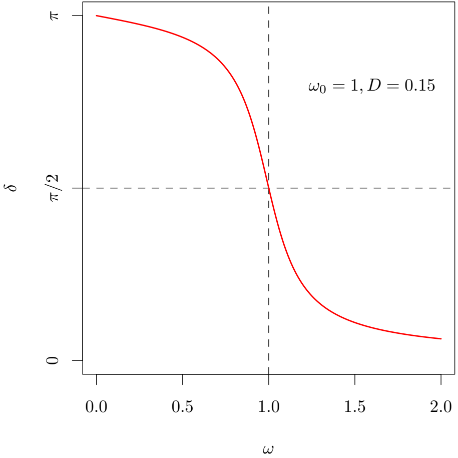

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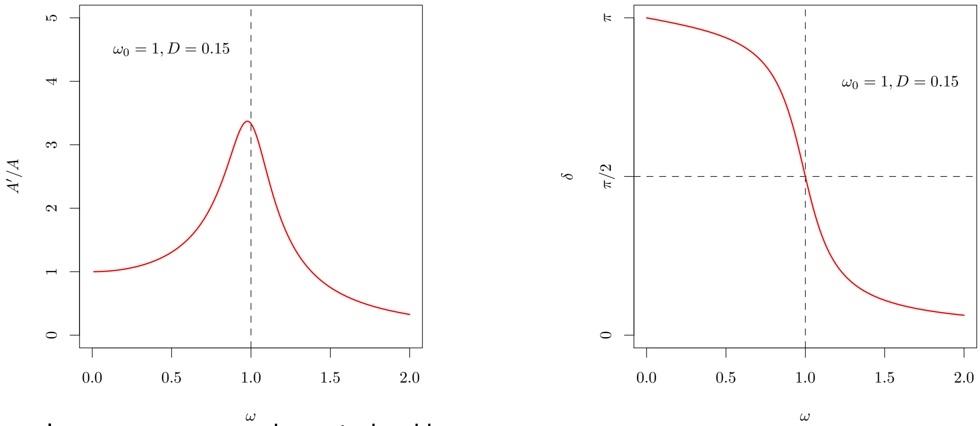
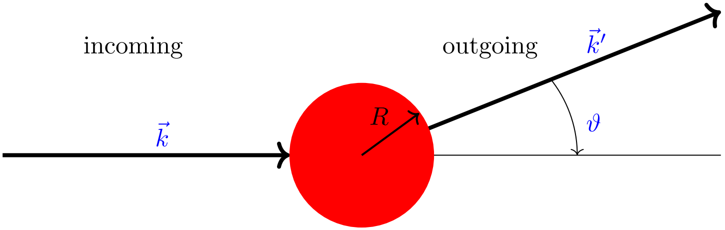


In general: resonances are characterised by

- resonance amplification in the amplitude ratio
- phase shift δ crosses $\pi/2$

Scattering of Particles

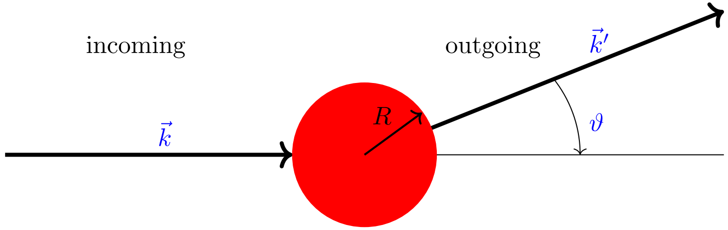
- consider interaction of finite range R, spherically symmetric



- interested in the elastic case
- in- and outgoing particles described by waves (asymptotic states)
- energy conservation: $k = |\vec{k}| = |\vec{k}'|$
- analyticity and unitarity further constrain the system

$\Rightarrow$ incoming and outgoing waves differ only by a phase shift

- consider interaction of finite range R, spherically symmetric



- analyticity and unitarity:

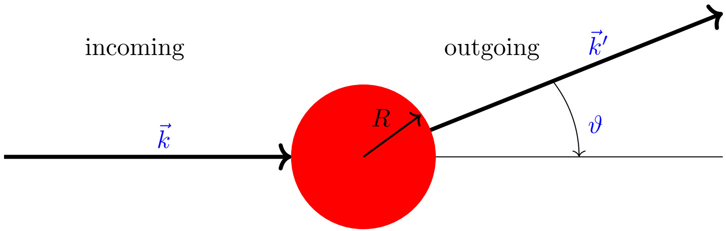
$\Rightarrow$ scattering amplitude in the partial wave expansion

$$f_k(\vartheta) = -\frac{8\pi}{M}\sum_{\ell=0}^{\infty}(2\ell+1)\ f_\ell(k)\ P_\ell(\cos\vartheta)$$

with partial wave amplitude and phase shift δ_ℓ

$$f_\ell(k) = \frac{1}{k\cot\delta_\ell(k) - ik}$$

Scattering of Particles



- it actuallly suffices to know the phase shifts $\delta_\ell(k)$
- often, even a single partial wave is enough (due to symmetries)

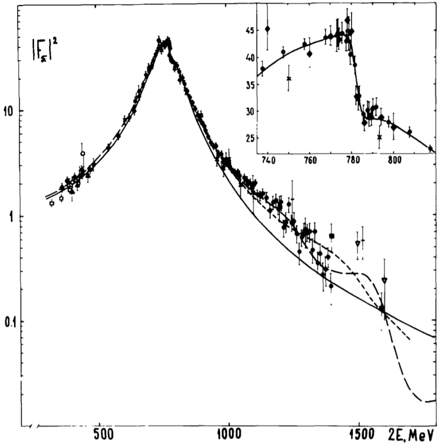
$\Rightarrow$ at small energies (small k) one can expand

$$k^{2\ell+1}\cot\delta_\ell \;=\; \frac{1}{a_\ell} + \frac{r}{2}k^2 + \ldots$$

- a_ℓ scattering length, r effective range

The ρ-Resonance

- ρ: lowest QCD resonance (together with the σ)
- in $\pi\pi$ channel with $I = 1$
- experimentally well measured
- clear signal in the amplitude $\Rightarrow$ pion formfactor



[Barkov et al., Nucl.Phys. B256 (1985)]

The ρ-Resonance

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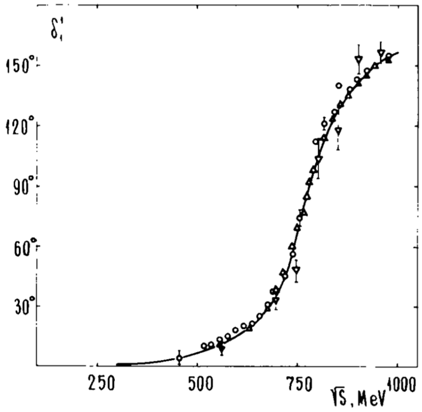
- experimentally well measured
- clear signal in the amplitude $\Rightarrow$ pion formfactor
- textbook example of a resonance phase shift



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The Lüscher Method

... or why finite volume effects can be useful

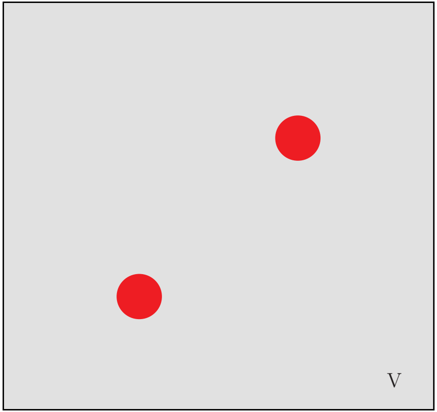
- lattice stochastic methods:
  work in finite volume / Euclidean space-time (→ Karl Jansen)

- consequence: energy levels are quantised

⇒ eigenvalues of the lattice Hamiltonian

- Maiani and Testa:
  interactions properties cannot be studied directly
  [Maiani and Testa, (1990)]

⇒ there is no one-to-one correspondence of an energy level to a resonance state instead: use finite volume as vehicle...



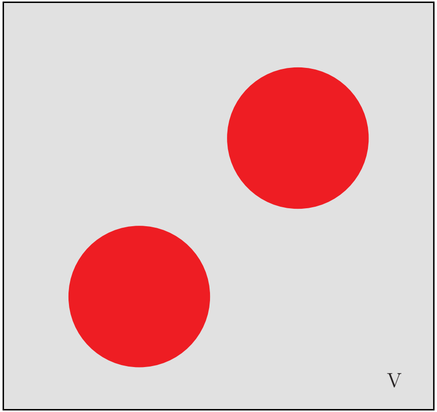
- for $V \to \infty$:

$\Rightarrow$ interaction probability very low

$\Rightarrow$ $E_{2p}(p = 0) = 2E_{1p}$

Lüscher Method

instead: use finite volume as vehicle...



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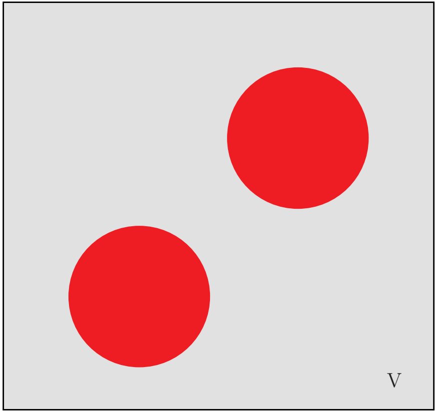

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$\Rightarrow$ $E_{2p}(p=0)$ receives corrections $\propto 1/V$

- Lüscher: correction in $1/V$ related to scattering properties!

[Lüscher, 1986]

How does this relation look like?

Consider the easiest case of two scalar particles at zero total momentum and assume small scattering momentum $|k|$:

- then, the finite volume dependence reads

[Lüscher, 1986]

$$\delta E_2 = E_2 - 2M_1 = -\frac{4\pi a_0}{M_1 L^3}\left(1 + c_1\frac{a_0}{L} + c_2\left(\frac{a_0}{L}\right)^2 + \frac{2\pi r a_0^2}{M_1^2 L^3} + ...\right)$$

- known c-numbers c_i
- finite range expansion assumed to be valid
- higher partial waves neglected

Example: Complex ϕ^4 Theory

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- Let's have a look at an easy example

  [Romero-Lopez, Rusetsky, CU, EPJC (2018)]

⇒ complex ϕ^4 theory as toy model

- lattice action

$$S = \sum_x \Big(-\kappa \sum_\mu (\varphi_x^\star \varphi_{x+\mu} + cc) + \lambda(|\varphi_x|^2 - 1)^2 + |\varphi_x|^2 \Big)$$

- big advantage: fast to simulate

⇒ can simulate basically arbitrary volumes

- and study the interaction of two scalar particles

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$\Rightarrow$ can simulate basically arbitrary volumes

- and study the interaction of two scalar particles

  (or three, four, five, ... scalar particles)

- need to compute $\Delta E = E_2 - 2M_1$
- single particle energy from

$$C_1(t) = \sum_{t'} \sum_{x,y} \langle \hat{\mathcal{O}}_\varphi(\mathbf{x}, t') \hat{\mathcal{O}}_\varphi^\dagger(\mathbf{y}, t + t') \rangle \quad \overset{t\to\infty}{\propto} \quad e^{-M_1 t}$$

- n-particle energy from

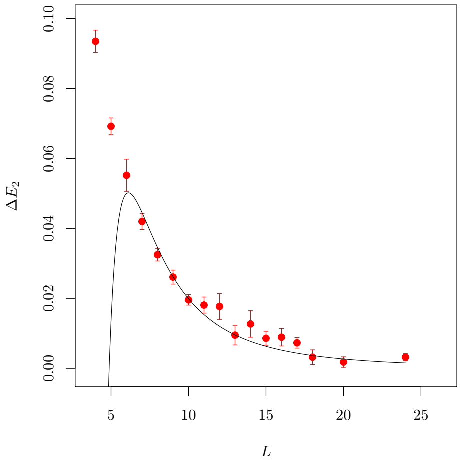
$$C_1(t) = \sum_{t'} \sum_{x,y} \langle \hat{\mathcal{O}}_{2\varphi}(\mathbf{x}, t') \hat{\mathcal{O}}_{2\varphi}^\dagger(\mathbf{y}, t + t') \rangle \quad \overset{t\to\infty}{\propto} \quad e^{-E_2 t} + \text{thermal pollutions}$$

- thermal pollutions due to finite time extend T and periodic BCs

$\Rightarrow$ have to be taken care of

Example: Complex ϕ^4 Theory

- compute ΔE as function of L
- for chosen bare parameters: repulsive interaction
- depending on fit range sensitive to a_0 or r
- for too small L description breaks down



$\Rightarrow$ ΔE_2 gives access to a_0 and r

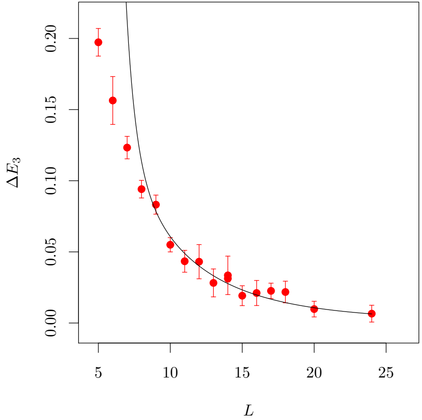
Example: Complex ϕ^4 Theory

- three particle formula (zero total momentum)

$$\Delta E_3 = E_3 - 3M_1 = -\frac{12\pi a_0}{M_1 L^3}(1 + \ldots) - \frac{D}{48 M_1^3 L^6}$$

[see e.g. Sharpe 2017]

- D encodes three body interaction
- data well described
- a_0, r input from ΔE_2
- clear evidence for non-zero three particle interaction!



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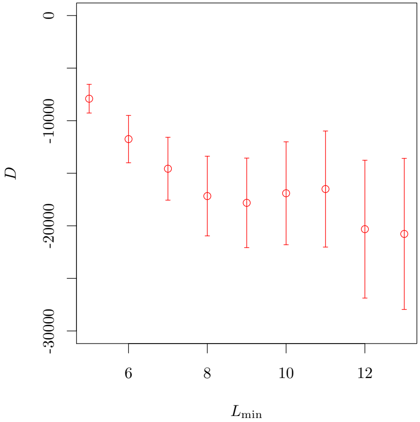
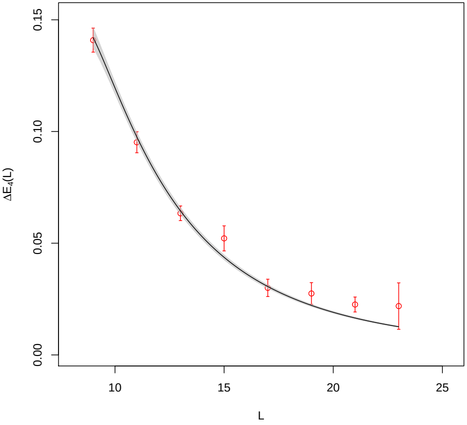
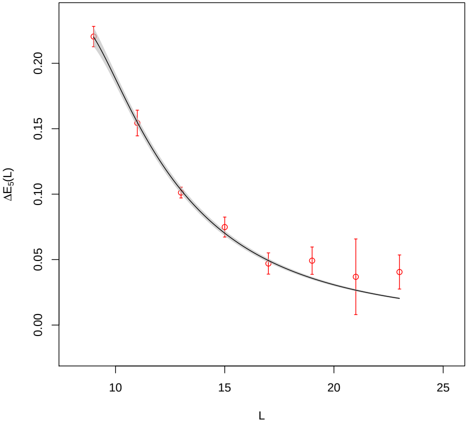


Four- and Five-Particles





Lüscher Formula More General

- in general Lüscher formula is a matrix equation

$$\det\left(\mathcal{M}_{lm,l'm'}(k) - \delta_{ll'}\delta_{mm'}\cot(\delta_l)\right) = 0$$

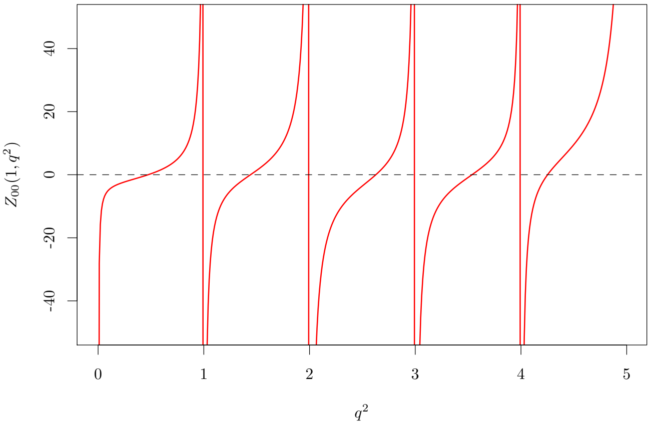
- matrix $\mathcal{M}_{lm,l'm'}(k)$ known analytically
- scattering momentum k from

$$E_2 = 2\sqrt{k^2 + M_1^2}$$

- $\mathcal{M}$ contains the so-called Lüscher Z-function $Z(q^2)$
- with

$$q = \frac{L}{2\pi}k$$

Lüscher Z-Function



- Singularities at free energy levels

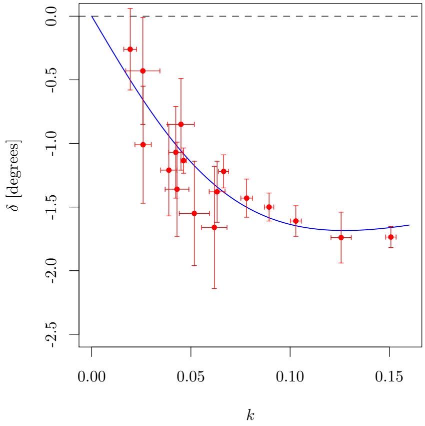
Example: Complex ϕ^4 Theory

- for the example: consider only s-wave
- determinant equation reduces to

$$\Rightarrow \quad \cot(\delta_0) = \frac{Z_{00}(1,q^2)}{\pi^{3/2} q}$$

$\Rightarrow$ every volume translates into one pair

$$(\delta_0(k), k)$$



- blue line reconstructed from

$$\cot\delta = \frac{1}{a_0} + \frac{r}{2}k^2$$

$\pi - \pi$ Scattering with $I = 2$

Action Details

- most of the following results based on
  - Wilson twisted mass ensembles by ETMC
  - $N_f = 2 + 1 + 1$ dynamical quark flavours
  - three values of the lattice spacing

  [ETMC, 2010, 2011]

- has its weaknesses, which we can discuss later

- weakly repulsive channel
- very interesting check of chiral perturbation theory
- at small momenta $k \to 0$ use effective range expansion

$$k^{2\ell+1} \cot \delta_\ell \;=\; \frac{1}{a_\ell} + \mathcal{O}(k^2)$$

  scattering length a_ℓ

- only S-waves ($\ell = 0$) contribute (to a good approximation)
- precise results from experiment plus Roy equations available

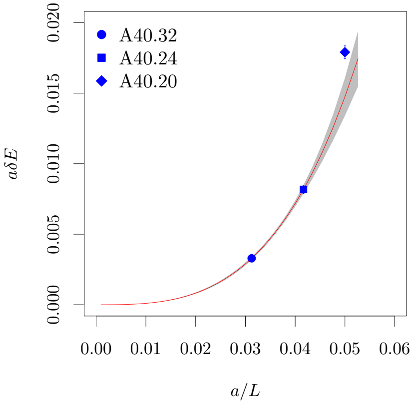
$\Rightarrow$ benchmark quantity for lattice QCD

$\pi - \pi$ Scattering with $I = 2$: Finite Volume Dependence

Lüscher formula (known constants c_i)

$$\Delta E = E_2 - 2M_\pi = -\frac{4\pi a_0}{M_\pi L^3}\left(1 + c_1\frac{a_0}{L} + c_2\frac{a_0^2}{L^2}\right) + \mathcal{O}(L^{-6})\,,$$

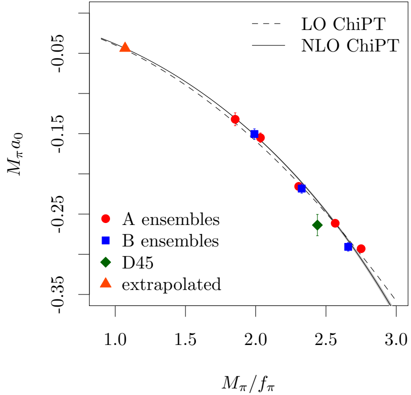
- valid, if other FS corrections small
- three ensembles with identical parameters but L
- smallest L deviates a few sigma
- smallest L too small
- all other ensembles have comparably larger L-values



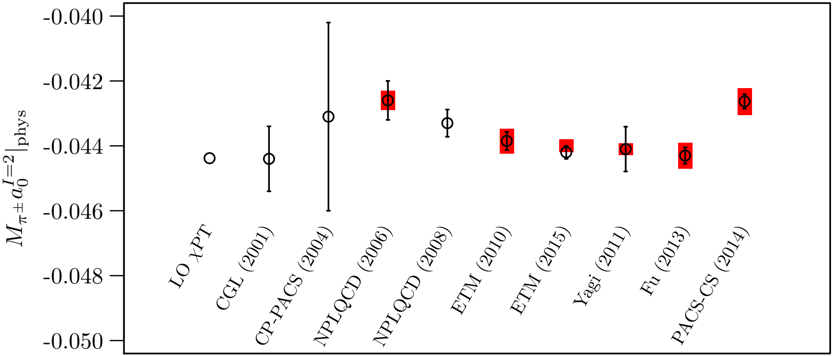
- ChPT formula at NLO [Beane et al, (2005,2007)]

$$M_\pi a_0 = -\frac{M_\pi^2}{8\pi f_\pi^2}\left\{1 + \frac{M_\pi^2}{16\pi^2 f_\pi^2}\left[3\ln\frac{M_\pi^2}{f_\pi^2} - 1 - \ell_{\pi\pi}^{I=2}(\mu_R = f_{\pi,\mathrm{phys}})\right]\right\}$$

- functional form highly constraining
- surprisingly small deviations from LO ChPT
- lattice artefacts small (in fact $\mathcal{O}(a^2 m_q)$)
- see JHEP 1509 (2015) 109



$\pi - \pi$ Scattering with $I = 2$: Summary



- result:

$$M_\pi a_0^{I=2} \;=\; -0.0442(2)_{\text{stat}}\left({}^{+4}_{-0}\right)_{\text{sys}}\,, \qquad \ell_{\pi\pi}^{I=2} \;=\; 3.79(0.61)_{\text{stat}}\left({}^{+1.34}_{-0.11}\right)_{\text{sys}}$$

[ETMC, Helmes, CU, et al, (2015)]

Two Goldstone Bosons at Maximal Isospin

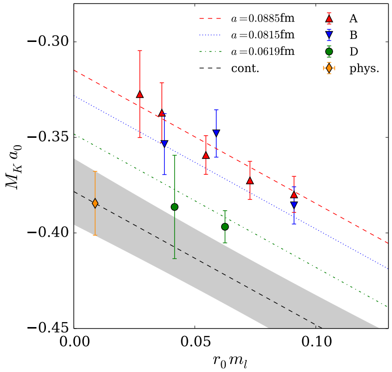
$\pi - \pi$, $\pi - K$, $K - K$

Two Goldstone Bosons at Maximal Isospin

- we have studied also $\pi - K$ and $K - K$ at maximal isospin

  [ETMC, 2017; ETMC, 2018]

  - for all three systems:
    - carefully studied the continuum limit
    - performed chiral extrapolation using ChPT
  - how far do we get with chiral perturbation theory?
  - relevant energy scale: $M_1 + M_2 +$ scattering energy



ChPT for Goldstone Boson Interaction with Maximal Isospin

- all two GB systems share

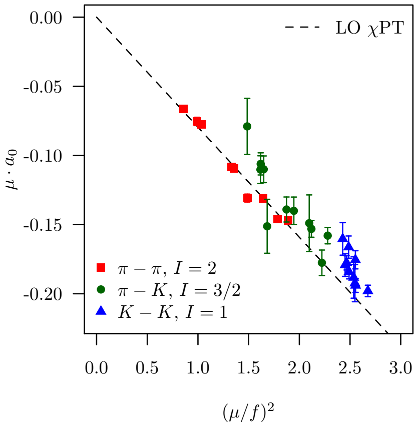
$$\mu a_0^{I_{\rm max}} = \frac{\mu^2}{4\pi f^2}$$

  at leading order ChPT

- reduced mass μ
  decay constant f (f_π or f_K)

- all weakly interacting

-



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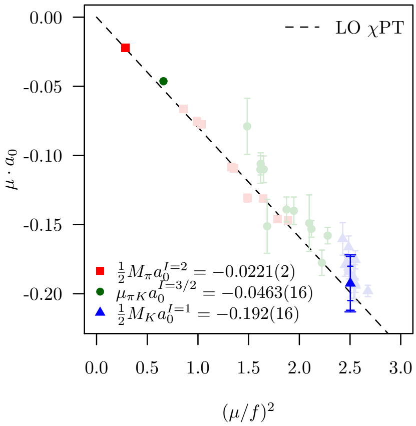


$\pi - \pi$ Scattering with $I = 1$:

the ρ-Resonance

How to map out the phase shift in QCD...

- unlike ϕ^4-theory:
  in LQCD often only few volumes available
  (actually mostly a single one)

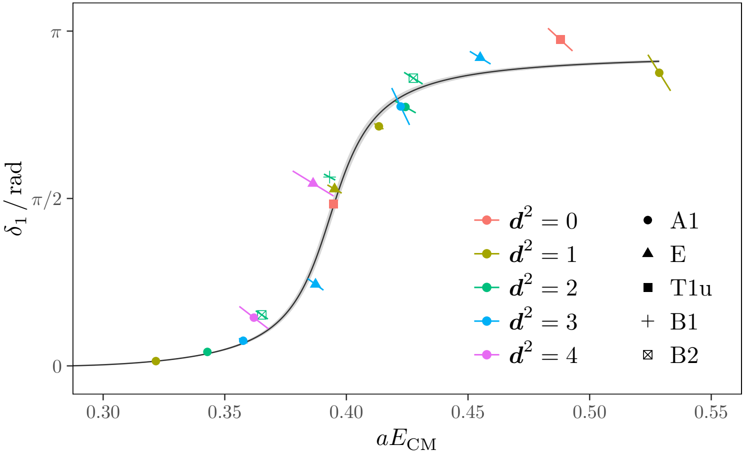
- use moving frames instead

  [Rummukainen, Gottlieb, 1995]

- requires reformulation of Lüscher formalism for moving frames
  $\rightarrow$ Steve Sharpe

- need group theory for lattice symmetry group O_h
  and irreducible representations

P-Wave Phase Shift (Example)



Extracting Mass and Width

- Extraction of mass and width requires phase shift parametrisation
- the functional form is

$$\tan\delta_1 = \frac{g_{\rho\pi\pi}^2}{6\pi}\frac{p^3}{E_{\mathrm{CM}}(M_\rho^2 - E_{\mathrm{CM}}^2)}\,, \qquad p = \sqrt{E_{\mathrm{CM}}^2/4 - M_\pi^2}$$

- E_{CM} is the center of mass energy
- the width is given by

$$\Gamma_\rho = \frac{g_{\rho\pi\pi}^2 \sqrt{M_\rho^2/4 - M_\pi^2}^{\,3}}{6\pi M_\rho^2}$$

$\Rightarrow$ model dependence!

Chiral Extrapolation

- chiral extrapolation using EFT with complex mass renormalisation
- introduce complexification

$$Z = (M_\rho + i\Gamma_\rho/2)^2$$

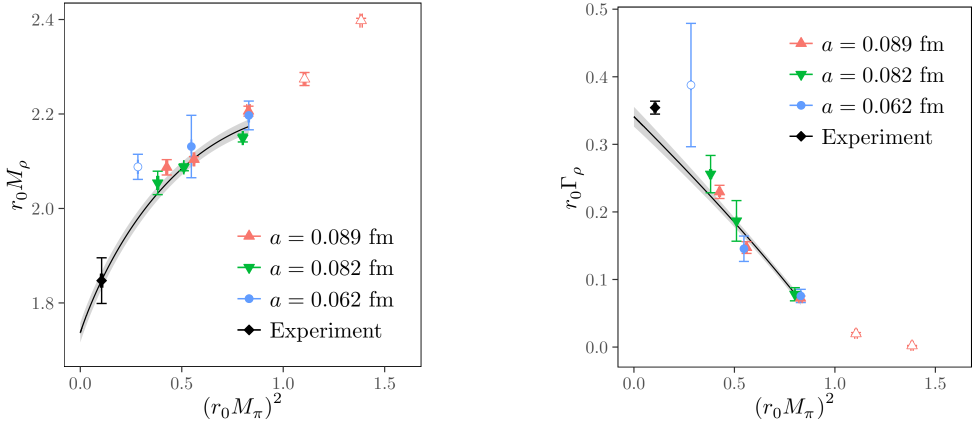
- extrapolate

$$Z(M_\pi) = Z_\chi + C_\chi M_\pi^2 - \frac{g^2}{16\pi^2} Z_\chi^{1/2} M_\pi^3$$

[Djukanovic et *al* (2009,2010)]

- lattice artefacts come at $\mathcal{O}(a^2)$ for our data

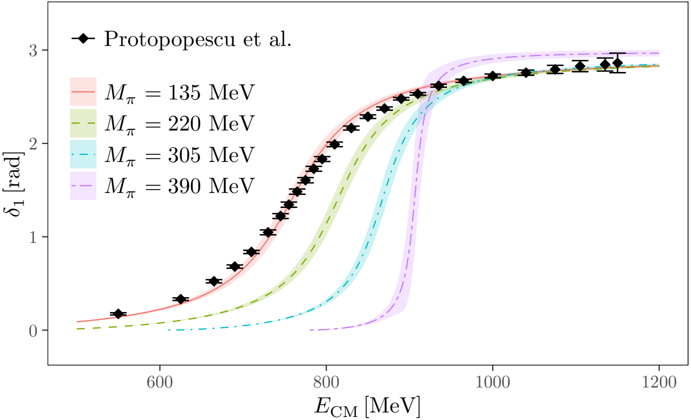
Chiral Extrapolation



- result (Breit-Wigner mass):

$$M_\rho \; = \; 769(19) \text{ MeV}\,, \qquad \Gamma_\rho \; = \; 129(7) \text{ MeV}$$

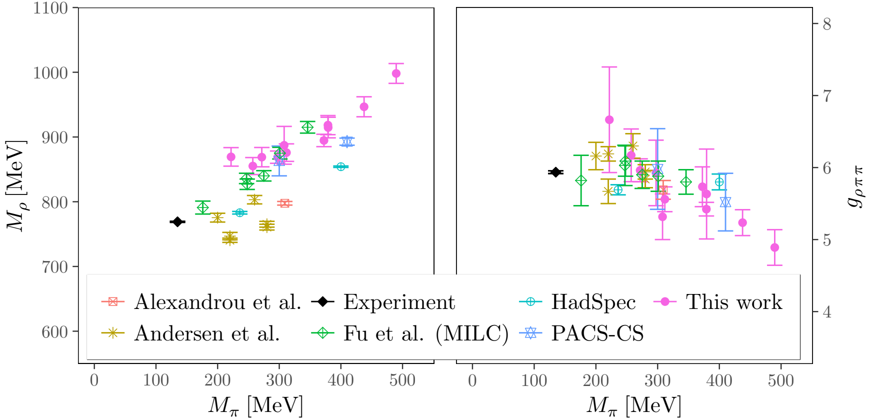
Comparison to Experiment



$\Rightarrow$ Reasonable agreement with experimental data

- ρ becomes stable around $M_\pi = 400$ MeV

Comparison to Lattice Data with $N_f = 2+1(+1)$



$\Rightarrow$ in general good agreement!

- Some actions show lattice artefacts in M_ρ

yet another “Ruler” Plot...?

- only data by Fu and ours
- M_ρ data well fitted by

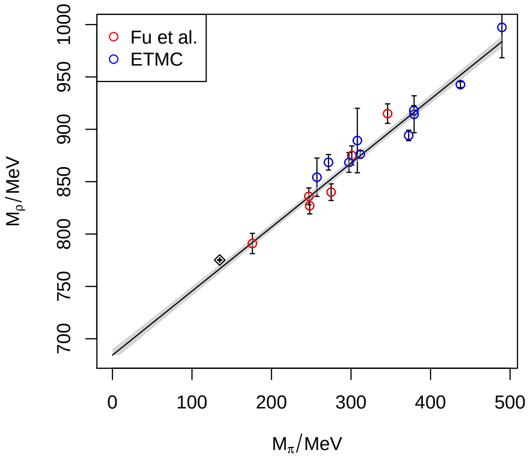
$$M_\rho \;=\; 680\ \mathrm{MeV} + 0.6 M_\pi$$

- on rather general grounds from EFT

$$M_\rho = M_\rho^0 + c_1 M_\pi^2 + c_2 M_\pi^3 + \ldots$$

[Bruns, Meißner, 2004]

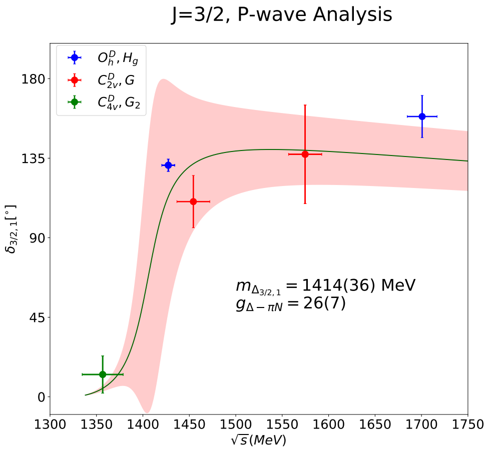
- linear term must not be there
- a cancellation!?



- see also the “Ruler” plot for the nucleon mass → A. Walker-Loud

$\pi - N$ scattering and the Δ^{++} -Resonance

- Lüscher formula can be extended to particles with spin
- here: $\pi - N$ scattering with $I = 3/2$
- Δ^{++} resonance
- example for $M_\pi \approx 250$ MeV



[Srijit Paul et *al.*, Lattice 2018]

Summary

- one can thoroughly test the Lüscher formalism in complex ϕ^4 theory

$\Rightarrow$ consistent 2-5 particle energy shifts

- examples in Lattice QCD
  - two Goldstone bosons at maximal isospin
  - the ρ-resonance
  - $\pi - N$ scattering

- studies of three-particle systems very active field

Outlook

- compute the binding energy of, say, Carbon directly from Quantum Chromodynamics?

$\Rightarrow$ clearly, a long way to go

- there are many challenges to deal with

- among others:
  - binding energy tiny compared to mass of the nucleus
  - very large volumes needed
  - enormous number of contractions

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