Current status of ε_K and $|V_{cb}|$ in lattice QCD

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Contents II

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CKM matrix elements

Charged Current Lagrangian in Quark Sector of the SM

$$\mathcal{L}_W = \frac{g_w}{\sqrt{2}} \sum_{i=1,2,3} \sum_{k=1,2,3} \left[V_{jk} \bar{u}_{jL} \gamma^\mu d_{kL} W_\mu^+ + V_{jk}^* \bar{d}_{kL} \gamma^\mu u_{jL} W_\mu^- \right]$$

where

$$u_j = \begin{pmatrix} u \\ c \\ t \end{pmatrix}, \qquad d_k = \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

and

$$V_{jk} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

CKM matrix elements

Standard Parametrization:

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -s_{23}c_{12} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

Wolfenstein Parametrization:

$$s_{12} = \lambda, \qquad s_{23} = A\lambda^2, \qquad s_{13} = A\lambda^3\sqrt{\rho^2 + \eta^2},$$

$$V = \begin{pmatrix} 1 - \dfrac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \dfrac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4)$$

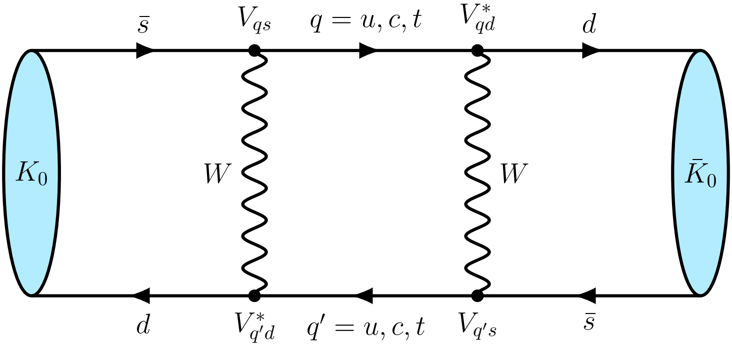
where $\lambda = |V_{us}| \cong 0.22, \quad A \cong 0.83, \quad \rho \cong 0.16, \quad \eta \cong 0.35$

[Meson]–[Anti-Meson] Mixing

$M - \overline{M}$ Mixing

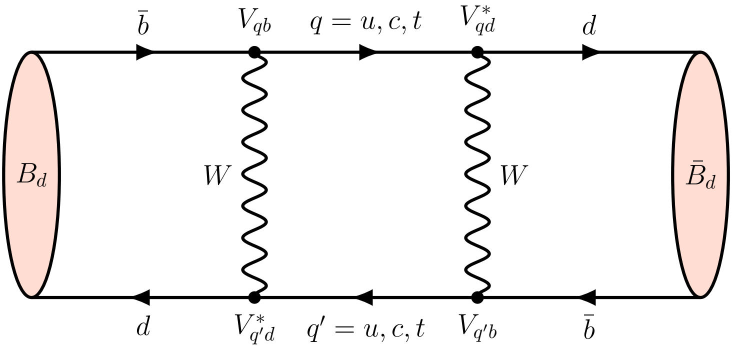
- It is possible only for the 4 neutral mesons.
- $K_0 \cong \bar{s}d \longleftrightarrow \overline{K}_0 \cong s\bar{d}$ $\qquad 497.611(13)\,\mathrm{MeV} \cong 0.5\,\mathrm{GeV}$
- $D_0 \cong c\bar{u} \longleftrightarrow \overline{D}_0 \cong \bar{c}u$ $\qquad 1864.83(5)\,\mathrm{MeV} \cong 1.9\,\mathrm{GeV}$
- $B_d \cong \bar{b}d \longleftrightarrow \overline{B}_d \cong b\bar{d}$ $\qquad 5279.64(13)\,\mathrm{MeV} \cong 5.3\,\mathrm{GeV}$
- $B_s \cong \bar{b}s \longleftrightarrow \overline{B}_s \cong b\bar{s}$ $\qquad 5366.88(17)\,\mathrm{MeV} \cong 5.4\,\mathrm{GeV}$

$K_0 - \overline{K}_0$ Mixing



- This is the main topic of the talk.
- Hence, we will discuss it later.

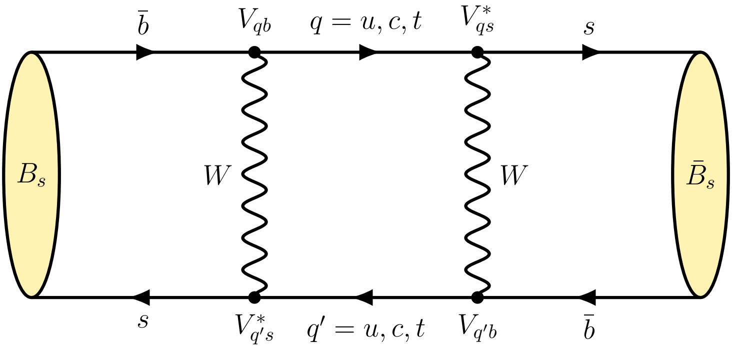
$B_d - \overline{B}_d$ Mixing



- $t - t$ box $\rightarrow x_t(V_{tb}V_{td}^*)^2 \cong x_t A^2\lambda^6(1-\rho+i\eta)^2$ with $x_t = (m_t/m_W)^2$
- $c - c$ box $\rightarrow x_c(V_{cb}V_{cd}^*)^2 \cong x_c A^2\lambda^6 \cong \frac{1}{16000} \times [t - t \text{ box}]$
- $c - t$ box $\rightarrow \sqrt{x_c x_t}(V_{cb}V_{cd}^* \cdot V_{tb}V_{td}^*) \cong -\sqrt{x_c x_t} A^2\lambda^6(1-\rho+i\eta)$

$$\Delta m_d = \frac{G_F^2}{6\pi^2} M_{B_d} f_{B_d}^2 \hat{B}_{B_d} M_W^2 S(x_t)(V_{tb}V_{td}^*)^2$$

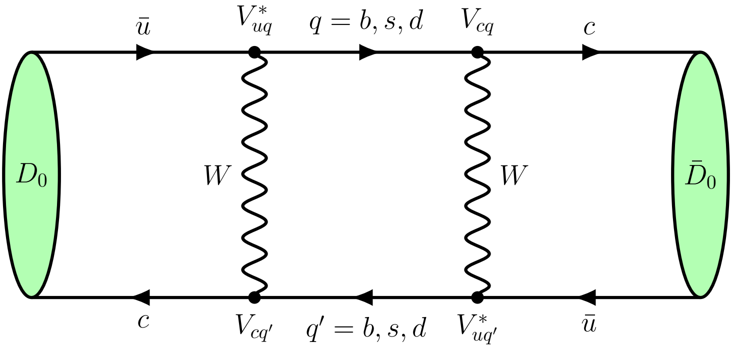
$B_s - \overline{B}_s$ Mixing



- $t - t$ box $\rightarrow x_t(V_{tb}V_{ts}^*)^2 \cong x_t A^2\lambda^4$ with $x_t = (m_t/m_W)^2$
- $c - c$ box $\rightarrow x_c(V_{cb}V_{cs}^*)^2 \cong x_c A^2\lambda^4 \cong \frac{1}{16000} \times [t - t$ box$]$
- $c - t$ box $\rightarrow \sqrt{x_c x_t}(V_{cb}V_{cs}^* \cdot V_{tb}V_{ts}^*) \cong -\sqrt{x_c x_t}A^2\lambda^4$

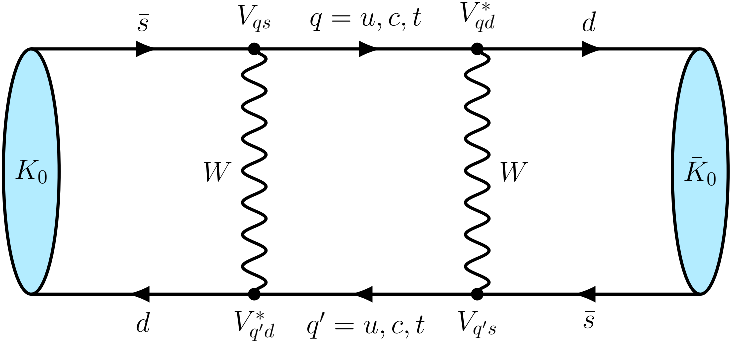
$$\Delta m_s = \frac{G_F^2}{6\pi^2} M_{B_s} f_{B_s}^2 \hat{B}_{B_s} M_W^2 S(x_t)(V_{tb}V_{ts}^*)^2$$

$D_0 - \overline{D}_0$ Mixing



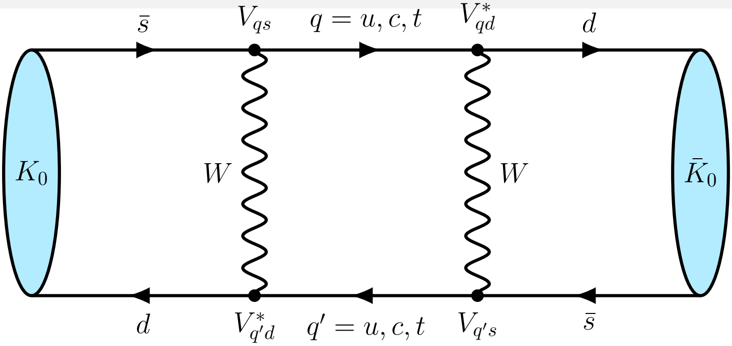
- $b - b$ box $\rightarrow x_b(V_{cb}V_{ub}^*)^2 \cong x_b A^4 \lambda^{10}(\rho + i\eta)^2$ with $x_b = (m_b/m_W)^2$
- $s - s$ box $\rightarrow x_s(V_{cs}V_{us}^*)^2 \cong x_s \lambda^2 \cong 200 \times [b - b$ box$]$
- $d - d$ box $\rightarrow x_d(V_{cd}V_{ud}^* \cdot V_{cd}V_{ud}^*) \cong x_d \lambda^2 \cong [b - b$ box$]$
- Hence, the long distance effect from the $s - s$ box becomes dominant and important. $\rightarrow$ Very tough in lattice QCD.

ΔM_K : Real Part of $K_0 - \overline{K}_0$ Mixing



- $t-t$ box $\rightarrow x_t(V_{ts}V^*_{td})^2 \cong x_t A^4\lambda^{10}(1-\rho+i\eta)^2$ with $x_t = (m_t/m_W)^2$
- $c-c$ box $\rightarrow x_c(V_{cs}V^*_{cd})^2 \cong x_c\lambda^2 \cong 25\times \mathrm{Re}[t-t \text{ box}]$
- $u-u$ box $\rightarrow x_u(V_{us}V^*_{ud})^2 \cong x_u\lambda^2 \cong \frac{1}{2800}\times \mathrm{Re}[t-t \text{ box}]$
- Hence, the $c-c$ box becomes dominant. Hence, the long distance effect ($\approx 30\%$) becomes important.

ε_K : Imaginary Part of $K_0 - \overline{K}_0$ Mixing

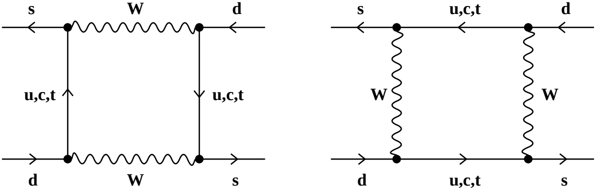


- $t - t \to x_t \mathrm{Im}(V_{ts}V^*_{td})^2 \cong 2x_t A^4 \lambda^{10}(1-\rho)\eta$ with $x_t = (m_t/m_W)^2$
- $c - c \to x_c \mathrm{Im}(V_{cs}V^*_{cd})^2 \cong -2x_c A^2 \lambda^6 \eta \cong -\frac{1}{25} \times \mathrm{Re}[t - t \text{ box}]$
- $c - t \to 2\sqrt{x_c x_t}\,\mathrm{Re}(V_{cs}V^*_{cd})\mathrm{Im}(V_{ts}V^*_{td}) \cong 2\sqrt{x_c x_t} A^2 \lambda^6 \eta \cong +\frac{1}{5} \times \mathrm{Re}[t - t \text{ box}]$
- Hence, the $t - t$ box is dominant (86%), the $c - t$ box is sub-dominant (17%), and the $c - c$ box is small and negative(-3.4%).

CP Violation in Neutral Kaons

Kaon Eigenstates and ε

- Flavor eigenstates, $K^0 = (\bar{s}d)$ and $\bar{K}^0 = (s\bar{d})$ mix via box diagrams.



(a) (b)

Kaon Eigenstates and ε

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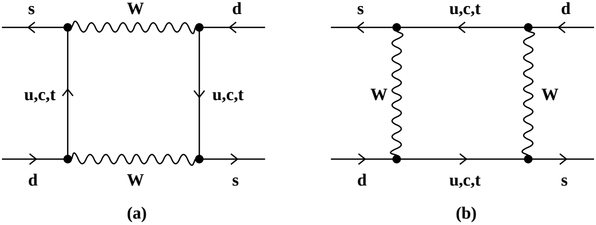
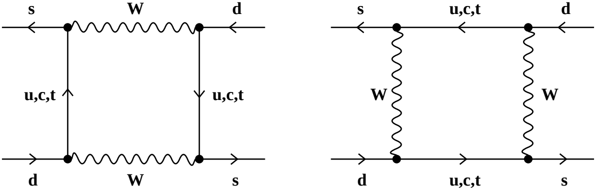


- CP eigenstates K_1(even) and K_2(odd).

$$K_1 = \frac{1}{\sqrt{2}}(K^0 - \bar{K}^0) \qquad K_2 = \frac{1}{\sqrt{2}}(K^0 + \bar{K}^0)$$

Kaon Eigenstates and ε

- Flavor eigenstates, $K^0 = (\bar{s}d)$ and $\bar{K}^0 = (s\bar{d})$ mix via box diagrams.



(a) (b)

- CP eigenstates K_1(even) and K_2(odd).

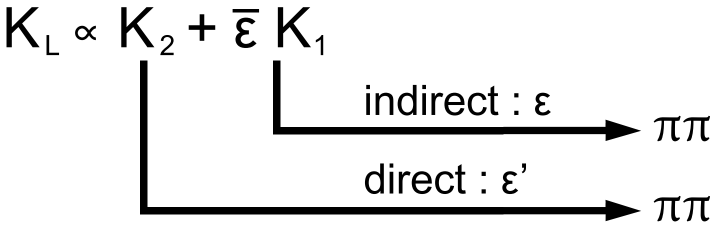
$$K_1 = \frac{1}{\sqrt{2}}(K^0 - \bar{K}^0) \qquad K_2 = \frac{1}{\sqrt{2}}(K^0 + \bar{K}^0)$$

- Neutral Kaon eigenstates K_S and K_L.

$$K_S = \frac{1}{\sqrt{1+|\bar{\varepsilon}|^2}}(K_1 + \bar{\varepsilon} K_2) \qquad K_L = \frac{1}{\sqrt{1+|\bar{\varepsilon}|^2}}(K_2 + \bar{\varepsilon} K_1)$$

Indirect CP violation and direct CP violation

- $\Gamma_{K_L} \cong 500 \times \Gamma_{K_S} \to$ only for neutral Kaons.
- It is possible to produce a high quality beam of K_L.



- $|\varepsilon_K| = |\varepsilon| \cong 2.2 \times 10^{-3}$.
- $|\varepsilon'/\varepsilon| \cong 1.7 \times 10^{-3}$.

ε_K and $\hat{B}_K$, V_{cb} I

- Definition of ε_K

$$\varepsilon_K \equiv \frac{A[K_L \to (\pi\pi)_{I=0}]}{A[K_S \to (\pi\pi)_{I=0}]}, \quad |\varepsilon_K| = 2.228(11) \times 10^{-3}$$

- Master formula for ε_K in the Standard Model.

$$\varepsilon_K = \exp(i\theta)\,\sqrt{2}\,\sin(\theta)\left(C_\varepsilon\, X_{\mathrm{SD}}\, \hat{B}_K + \frac{\xi_0}{\sqrt{2}} + \xi_{\mathrm{LD}}\right) + \mathcal{O}(\omega\varepsilon') + \mathcal{O}(\xi_0\Gamma_2/\Gamma_1)$$

$$X_{\mathrm{SD}} = \mathrm{Im}\lambda_t\Big[\mathrm{Re}\,\lambda_c\,\eta_{cc}\,S_0(x_c) - \mathrm{Re}\,\lambda_t\,\eta_{tt}\,S_0(x_t) - (\mathrm{Re}\,\lambda_c - \mathrm{Re}\,\lambda_t)\,\eta_{ct}\,S_0(x_c, x_t)\Big]$$

ε_K and $\hat{B}_K$, V_{cb} II

$$\lambda_i = V_{is}^* V_{id}, \qquad x_i = m_i^2 / M_W^2, \qquad C_\varepsilon = \frac{G_F^2 F_K^2 m_K M_W^2}{6\sqrt{2}\,\pi^2 \Delta M_K}$$

$$\frac{\xi_0}{\sqrt{2}} = \frac{1}{\sqrt{2}} \frac{\mathrm{Im} A_0}{\mathrm{Re} A_0} \approx -5\% \quad \rightarrow \quad \text{Absorptive Long Distance Effect}$$

$$\xi_{\mathrm{LD}} = \text{Dispersive Long Distance Effect} \approx 2\% \;\rightarrow\; \text{explain it later.}$$

- Inami-Lim functions:

$$S_0(x_i) = x_i \left[\frac{1}{4} + \frac{9}{4(1-x_i)} - \frac{3}{2(1-x_i)^2} - \frac{3x_i^2 \ln x_i}{2(1-x_i)^3} \right],$$

$$S_0(x_i, x_j) = \left\{ \frac{x_i x_j}{x_i - x_j} \left[\frac{1}{4} + \frac{3}{2(1-x_i)} - \frac{3}{4(1-x_i)^2} \right] \ln x_i - (i \leftrightarrow j) \right\} - \frac{3 x_i x_j}{4(1-x_i)(1-x_j)}$$

ε_K and $\hat{B}_K$, V_{cb} III

$$
\begin{array}{lcr}
t - t \longrightarrow & S_0(x_t) \longrightarrow & +72.4\% \\
c - t \longrightarrow & S_0(x_c, x_t) \longrightarrow & +45.4\% \\
c - c \longrightarrow & S_0(x_c) \longrightarrow & -17.8\%
\end{array}
$$

- Dominant contribution ($\approx$72%) comes with $|V_{cb}|^4$.

$$
\begin{aligned}
&\lambda_i \equiv V_{is}^* V_{id} \\
&\mathrm{Im}\lambda_t \cdot \mathrm{Re}\lambda_t = \bar{\eta}\lambda^2 |V_{cb}|^4 (1 - \bar{\rho}) \\
&\mathrm{Re}\lambda_c = -\lambda(1 - \frac{\lambda^2}{2}) + \mathcal{O}(\lambda^5) \\
&\mathrm{Re}\lambda_t = -(1 - \frac{\lambda^2}{2}) A^2 \lambda^5 (1 - \bar{\rho}) + \mathcal{O}(\lambda^7) \\
&\mathrm{Im}\lambda_t = \eta A^2 \lambda^5 + \mathcal{O}(\lambda^7) \\
&\mathrm{Im}\lambda_c = -\mathrm{Im}\lambda_t
\end{aligned}
$$

ε_K and $\hat{B}_K$, V_{cb} IV

- Definition of $\hat{B}_K$ in standard model.

$$B_K = \frac{\langle \bar{K}_0 | [\bar{s}\gamma_\mu(1-\gamma_5)d][\bar{s}\gamma_\mu(1-\gamma_5)d] | K_0 \rangle}{\frac{8}{3}\langle \bar{K}_0 | \bar{s}\gamma_\mu\gamma_5 d | 0 \rangle \langle 0 | \bar{s}\gamma_\mu\gamma_5 d | K_0 \rangle}$$

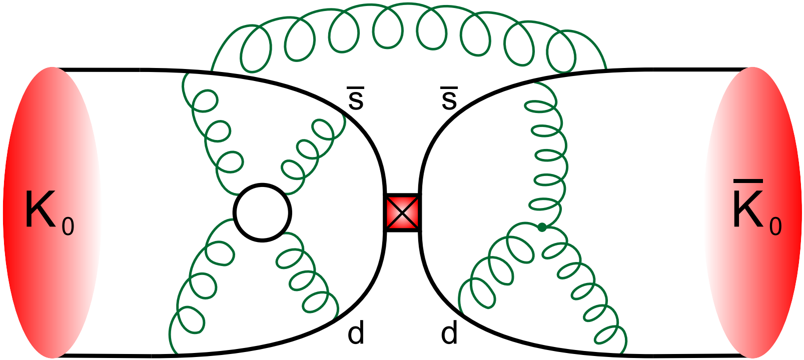
$$\hat{B}_K = C(\mu) B_K(\mu), \qquad C(\mu) = \alpha_s(\mu)^{-\frac{\gamma_0}{2b_0}} [1 + \alpha_s(\mu) J_3]$$

- Experiment:

$$\varepsilon_K = (2.228 \pm 0.011) \times 10^{-3} \times e^{i\phi_\varepsilon}$$

$$\phi_\varepsilon = 43.52(5)^\circ$$

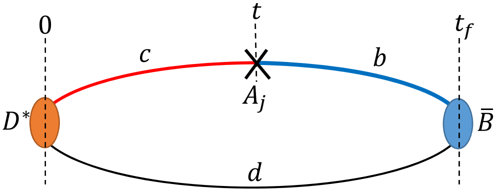
$\hat{B}_K$ on the lattice



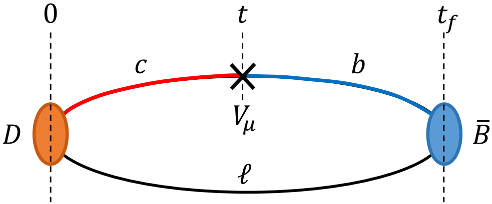
- This is one of the ∞ number of the Feynman diagrams that we need to calculate using lattice QCD tools (K. Jansen; C. Sachrajda).

$|V_{cb}|$ on the lattice

- $\bar{B} \to D^* \ell \bar{\nu}$ decay form factors:

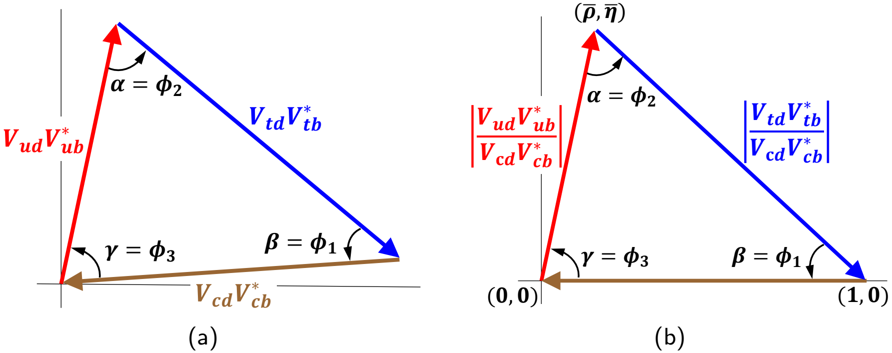


- $\bar{B} \to D \ell \bar{\nu}$ decay form factors:



ε_K with lattice QCD inputs

Unitarity Triangle → $(\bar{\rho}, \bar{\eta})$



Global UT Fit and Angle-Only-Fit (AOF)

Global UT Fit

- Input: $|V_{ub}|/|V_{cb}|$, Δm_d, $\Delta m_s/\Delta m_d$, ε_K, and $\sin(2\beta)$.
- Determine the UT apex $(\bar{\rho}, \bar{\eta})$.
- Take λ from

$$|V_{us}| = \lambda + \mathcal{O}(\lambda^7)\,,$$

  which comes from K_{l3} and $K_{\mu 2}$.
- Disadvantage: unwanted correlation between $(\bar{\rho}, \bar{\eta})$ and ε_K.

AOF

- Input: $\sin(2\beta)$, $\cos(2\beta)$, $\sin(\gamma)$, $\cos(\gamma)$, $\sin(2\beta+\gamma)$, $\cos(2\beta+\gamma)$, and $\sin(2\alpha)$.
- Determine the UT apex $(\bar{\rho}, \bar{\eta})$.
- Take λ from $|V_{us}| = \lambda + \mathcal{O}(\lambda^7)\,,$ which comes from K_{l3} and $K_{\mu 2}$.
- Use $|V_{cb}|$ to determine A.

$$|V_{cb}| = A\lambda^2 + \mathcal{O}(\lambda^7)$$

- Advantage: NO correlation between $(\bar{\rho}, \bar{\eta})$ and ε_K.

Inputs of Angle-Only-Fit (AOF)

- $A_{\rm CP}(J/\psi\ K_s) \to S_{\psi K_s} = \sin(2\beta)$ with assumption of $S_{\psi K_s} \ggg C_{\psi K_s}$.
- $(B \to DK) + (B \to [K\pi]_D K) +$ (Dalitz method) give $\sin(\gamma)$ and $\cos(\gamma)$.
- $S(D^-\pi^+)$ and $S(D^+\pi^-)$ give $\sin(2\beta+\gamma)$ and $\cos(2\beta+\gamma)$.
- $(B^0 \to \pi^+\pi^-) + (B^0 \to \rho^+\rho^-) + (B^0 \to (\rho\pi)^0)$ give $\sin(2\alpha)$.
- Combining all of these gives β, γ, and α, which leads to the UT apex $(\bar\rho,\ \bar\eta)$.

Wolfenstein Parameters

Input Parameters for Angle-Only-Fit (AOF)

- ε_K, $\hat{B}_K$, and $|V_{cb}|$ are used as inputs to determine the UT angles in the global fit of `UTfit` and `CKMfitter`.
- Instead, we can use angle-only-fit result for the UT apex $(\bar{\rho}, \bar{\eta})$.
- Then, we can take λ independently from

$$|V_{us}| = \lambda + \mathcal{O}(\lambda^7)\,,$$

which comes from K_{l3} and $K_{\mu 2}$.

- Use $|V_{cb}|$ instead of A.

$$|V_{cb}| = A\lambda^2 + \mathcal{O}(\lambda^7)$$

λ	0.22475(25)	[1] CKMfitter 2018
	0.22500(100)	[2] UTfit 2018
	0.2243(5)	[3] $|V_{us}|$ (AOF)
$\bar{\rho}$	0.1577(96)	[1] CKMfitter 2018
	0.148(13)	[2] UTfit 2018
	0.146(22)	[4] UTfit (AOF)
$\bar{\eta}$	0.3493(95)	[1] CKMfitter 2018
	0.348(10)	[2] UTfit 2018
	0.333(16)	[4] UTfit (AOF)

Input Parameter: $\hat{B}_K$ (FLAG 2019)

$\hat{B}_K$ in lattice QCD with $N_f = 2 + 1$.

Collaboration	Ref.	$\hat{B}_K$
SWME 15	[5]	0.735(5)(36)
RBC/UKQCD 14	[6]	0.7499(24)(150)
Laiho 11	[7]	0.7628(38)(205)
BMW 11	[8]	0.7727(81)(84)
FLAG 19	[9]	0.7625(97)

Input Parameter: Exclusive $|V_{cb}|$ in units of 1.0×10^{-3}

(a) HFLAV 2017 (CLN)

channel	value	Ref.
$B \to D^* \ell \bar{\nu}$	39.05(47)(58)	[10, 11]
$B \to D \ell \bar{\nu}$	39.18(94)(36)	[10, 12]
$\lvert V_{ub}\rvert/\lvert V_{cb}\rvert$	0.080(4)(4)	[10, 13]
ex-combined	39.13(59)	[10]

(b) BABAR and BELLE 2019

channel	value	Ref.
CLN	39.05(47)(58)	HFLAV 17 [10]
BGL	38.36(90)	BABAR 19 [14]
CLN	38.4(2)(6)(6)	BELLE 19 [15]
BGL	38.3(3)(7)(6)	BELLE 19 [15]

- There is no difference between the CLN and BGL analyses.
- Refer to BABAR 2019 [14] and BELLE 2019 [15].
- Hence, the CLN method turns out to be correct and OK within our limited knowledge.

$\bar{B} \to D^* \ell \bar{\nu}$ Form Factor Parametrization: CLN vs. BGL I

- Consider the $\bar{B} \to D^* \ell \bar{\nu}$ decays:

$$\frac{d\Gamma(\bar{B} \to D^* \ell \bar{\nu})}{dw} = \frac{G_F^2 m_{D^*}^3}{48\pi^3} (m_B - m_{D^*})^2 \, \chi(w) \, \eta_{\text{EW}}^2 \, \mathcal{F}^2(w) \, |V_{cb}|^2$$

- In order that the experiments determine $|\mathcal{F}(w)| \cdot |V_{cb}|$, they must know a specific functional form of $\mathcal{F}(w)$.
- The theory provides this parametrization for $\mathcal{F}(w)$.
- Popular parameterizations are CLN and BGL.
- CLN depends on the HQET, but BGL does NOT.
- HQET is the heavy quark effective theory, as if the chiral perturbation theory is the low energy effective theory of QCD.

$\bar{B} \to D^* \ell \bar{\nu}$ Form Factor Parametrization: CLN vs. BGL II

- CLN: Caprini, Lellouch, and Neubert [16]

$$\mathcal{F}(w) = h_{A_1}(w) \times \frac{1}{Y(w)} \times X(w)$$

$$h_{A_1}(w) = h_{A_1}(1)\Big[1 - 8\rho^2 z + (53\rho^2 - 15)z^2 - (231\rho^2 - 91)z^3\Big]$$

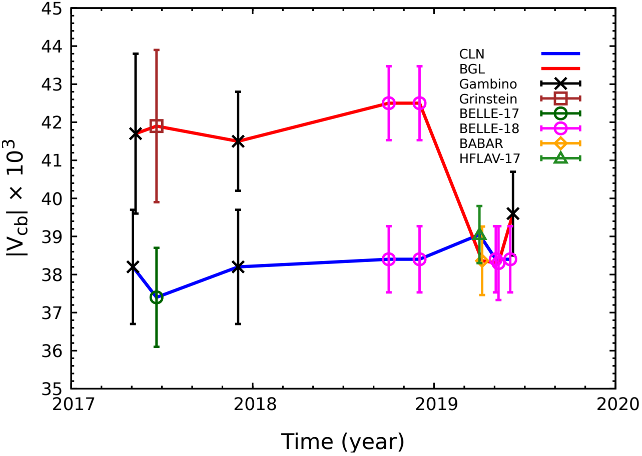
$$z = \frac{\sqrt{w+1} - \sqrt{2}}{\sqrt{w+1} + \sqrt{2}}, \qquad w \equiv v_B \cdot v_{D^*} = \frac{E_{D^*}}{m_{D^*}}$$

where z is a conformal mapping variable. $\rightarrow$ z expansion.

- BGL: Boyd, Grinstein, and Lebed [17]

$$\mathcal{F}(w) = \frac{1}{\phi(z)P(z)} \sum_{n=0}^{\infty} a_n z^n(z)$$

CLN vs. BGL in $B \to D^* \ell \bar{\nu}$ decays



- At present, we find that there is no difference in exclusive $|V_{cb}|$ between CLN and BGL. $\Longrightarrow$ Resolved ???

Input Parameter: Inclusive $|V_{cb}|$ in units of 1.0×10^{-3}

$|V_{cb}|$ in units of 1.0×10^{-3}.

(a) Exclusive $|V_{cb}|$

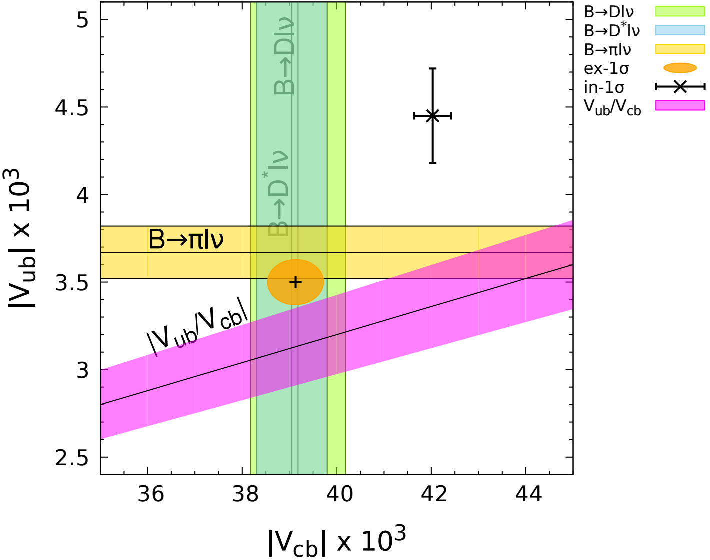
channel	value	Ref.
CLN	39.05(47)(58)	HFLAV 17 [10]
BGL	38.36(90)	BABAR 19 [14]
CLN	38.4(2)(6)(6)	BELLE 19 [15]
BGL	38.3(3)(7)(6)	BELLE 19 [15]

(b) Inclusive $|V_{cb}|$

channel	value	Ref.
kinetic scheme	42.19(78)	[10]
1S scheme	41.98(45)	[10]

- There is $3\sigma \sim 4\sigma$ difference in $|V_{cb}|$ between the exclusive and inclusive decay channels.
- This issue remains unresolved yet.

Current Status of $|V_{cb}|$ in 2018



Input Parameter: ξ_0

Indirect Method

$$\xi_0 = \frac{\mathrm{Im} A_0}{\mathrm{Re}\, A_0}, \quad \xi_2 = \frac{\mathrm{Im} A_2}{\mathrm{Re}\, A_2}.$$

ξ_0	$-1.63(19) \times 10^{-4}$	RBC-UK-2015 [18]

where $\mathcal{A}(K_0 \to \pi\pi(I)) \equiv A_I e^{i\delta_I} = |A_I| e^{i\xi_I} e^{i\delta_I}$

- RBC-UKQCD calculated $\mathrm{Im} A_2$: $\mathrm{Im} A_2 \to \xi_2 \to \varepsilon'_K/\varepsilon_K \to \xi_0$

$$\mathrm{Re}\left(\frac{\varepsilon'_K}{\varepsilon_K}\right) = \frac{1}{\sqrt{2}|\varepsilon_K|}\omega\left(\xi_2 - \xi_0\right).$$

  Other inputs ω, ε_K and $\varepsilon'_K/\varepsilon_K$ are taken from the experimental values.

- Here, we choose an approximation of $\cos(\phi_{\epsilon'} - \phi_\epsilon) \approx 1$.
- $\phi_\epsilon = 43.52(5)$, $\phi_{\epsilon'} = 42.3(1.5)$
- Isopspin breaking effect: (at most 15% of ξ_0) $\to$ (1% in ε_K) $\to$ neglected!

Input Parameter: ξ_0

Direct Method

- RBC-UKQCD calculated $\mathrm{Im} A_0$. $\mathrm{Im} A_0 \rightarrow \xi_0$.

$$\xi_0 = \frac{\mathrm{Im} A_0}{\mathrm{Re}\, A_0} = -0.57(49) \times 10^{-4}$$

  Other input $\mathrm{Re}\, A_0$ is taken from the experimental value.
- RBC-UKQCD also calculated δ_0

$$\delta_0 = 23.8(49)(12)^\circ [2015] \rightarrow 23.8(49)(112)^\circ [2018]$$

  This value is within 2σ from the experimental value: $\delta_0 = 39.1(6)^\circ$.
- This puzzle might be resolved by multi-state fitting with new operators: RBC-UKQCD, Tianle Wang [Lattice 2019].
- Here, we use the indirect method to determine ξ_0.

Input Parameter: ξ_0

Summary

Input Parameters: ξ_0

Method	Value	Ref.
Indirect	$-1.63(19) \times 10^{-4}$	RBC-UK-2015 [18]
Direct	$-0.57(49) \times 10^{-4}$	RBC-UK-2015 [19]

- Here, we use the results for ξ_0 obtained using the indirect method.

Input Parameter: $\xi_{\rm LD}$

$$\xi_{\rm LD} = \frac{m'_{\rm LD}}{\sqrt{2}\,\Delta M_K}$$

$$m'_{\rm LD} = -\mathrm{Im}\left[\mathcal{P}\sum_{C}\frac{\langle\overline{K}^0|H_{\rm w}|C\rangle\langle C|H_{\rm w}|K^0\rangle}{m_{K^0}-E_C}\right]$$

- RBC-UKQCD rough estimate [PRD 88, 014508] gives

$$\xi_{\rm LD} = (0 \pm 1.6)\% \quad \text{of } |\varepsilon_K|$$

- BGI estimate [PLB 68, 309, 2010] gives

$$\xi_{\rm LD} = -0.4(3) \times \frac{\xi_0}{\sqrt{2}}$$

- Precision measurement of lattice QCD is not available yet.

Input Parameter: Charm Quark Mass $m_c(m_c)$

$m_c(m_c)$ in lattice QCD.

Collaboration	N_f	$m_c(m_c)$	Ref.
FLAG 2019	$2+1$	1.275(5)	[9]
FLAG 2019	$2+1+1$	1.280(13)	[9]

- The results for $m_c(m_c)$ with $N_f = 2+1+1$ are inconsistent with each other.
- Hence, we use the results for $m_c(m_c)$ with $N_f = 2+1$.

Input Parameter: top quark mass $m_t(m_t)$

$m_t(m_t)$ in the $\overline{\text{MS}}$ scheme in units of GeV.

Collaboration	M_t	$m_t(m_t)$	Ref.
PDG 2016	173.5 ± 1.1	$163.65 \pm 1.05 \pm 0.17$	[20]
PDG 2018	173.0 ± 0.4	$163.17 \pm 0.38 \pm 0.17$	[3]

- M_t is the pole mass of top quarks.
- CMS and ATLAS have done a great job in reducing the error.
- Here, we use the results for $m_t(m_t)$ obtained from PDG 2018.

Other Input Parameters

Input	Value	Ref.
G_F	$1.1663787(6) \times 10^{-5}\,\mathrm{GeV}^{-2}$	PDG 18 [3]
M_W	$80.379(12)\,\mathrm{GeV}$	PDG 18 [3]
θ	$43.52(5)^\circ$	PDG 18 [3]
m_{K^0}	$497.611(13)\,\mathrm{MeV}$	PDG 18 [3]
ΔM_K	$3.484(6) \times 10^{-12}\,\mathrm{MeV}$	PDG 18 [3]
F_K	$155.7(3)\,\mathrm{MeV}$	FLAG 19 [9]

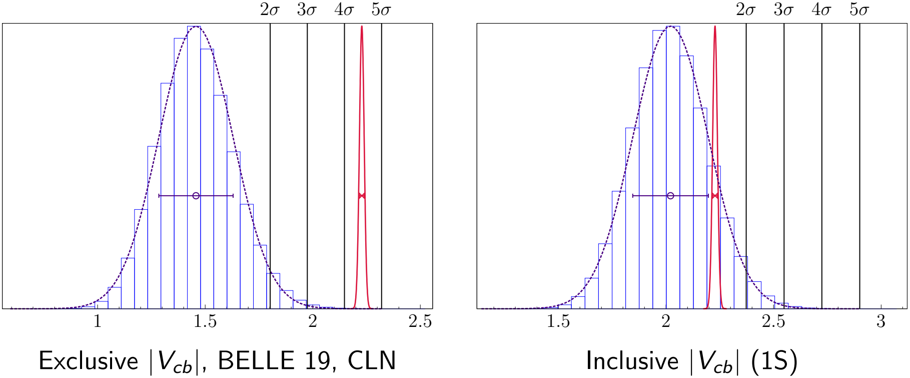
Higher order QCD corrections: η_{ij}.

Input	Value	Ref.
η_{cc}	1.72(27)	[21]
η_{tt}	0.5765(65)	[22]
η_{ct}	0.496(47)	[23]

Results for ε_K

ε_K from AOF, Exclusive $|V_{cb}|$ (BELLE 19, CLN)

RBC-UKQCD estimate for ξ_{LD}



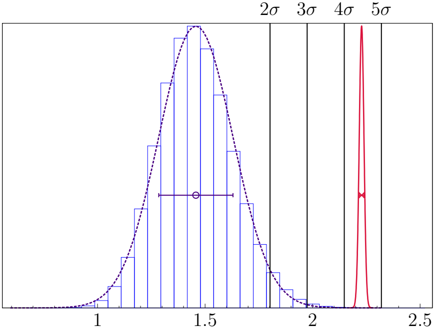
- With exclusive $|V_{cb}|$ (BELLE 19, CLN), it has 4.5σ tension.

$$|\varepsilon_K|^{\text{Exp}} = (2.228 \pm 0.011) \times 10^{-3}$$

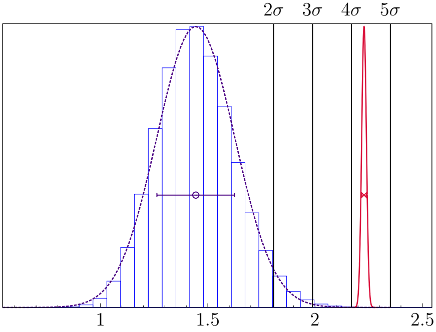
$$|\varepsilon_K|^{\text{SM}}_{\text{excl}} = (1.457 \pm 0.173) \times 10^{-3}$$

ε_K from AOF, Exclusive $|V_{cb}|$ (BELLE 19, CNL vs. BGL)

RBC-UKQCD estimate for ξ_{LD}



Exclusive $|V_{cb}|$ (BELLE 19, CNL)



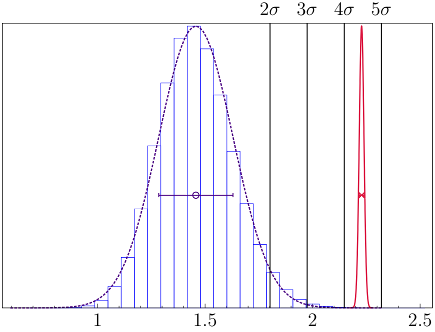
Exclusive $|V_{cb}|$ (BELLE 19, BGL)

- CLN has 4.5σ tension, and BGL has 4.3σ tension.

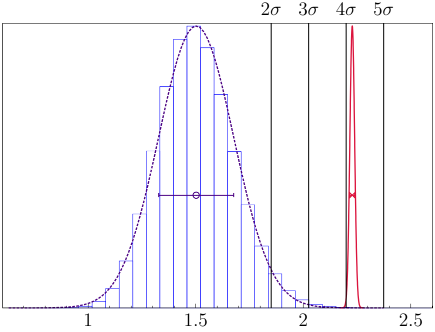
$$|\varepsilon_K|^{SM}_{excl} = (1.457 \pm 0.173) \times 10^{-3} \qquad \text{(CLN)}$$
$$|\varepsilon_K|^{SM}_{excl} = (1.444 \pm 0.181) \times 10^{-3} \qquad \text{(BGL)}$$

ε_K from AOF, Exclusive $|V_{cb}|$ (BELLE 19, CLN)

RBC-UKQCD vs. BGI estimate for ξ_{LD}



RBC-UKQCD estimate for ξ_{LD}



BGI estimate for ξ_{LD}

- RBC-UK estimate $\rightarrow$ 4.5σ tension, and BGI estimate $\rightarrow$ 4.2σ tension.

$$|\varepsilon_K|^{\text{SM}}_{\text{excl}} = (1.457 \pm 0.173) \times 10^{-3} \qquad \text{(RBC-UKQCD estimate for } \xi_{\text{LD}})$$

$$|\varepsilon_K|^{\text{SM}}_{\text{excl}} = (1.502 \pm 0.174) \times 10^{-3} \qquad \text{(BGI estimate for } \xi_{\text{LD}})$$

Current Status of ε_K

- FLAG 2019 + PDG 2018: (in units of 1.0×10^{-3}, AOF)

$$|\varepsilon_K|^{\rm SM}_{\rm excl} = 1.457 \pm 0.173 \quad \text{for Exclusive } V_{cb} \text{ (Lattice QCD + CLN)}$$
$$|\varepsilon_K|^{\rm SM}_{\rm incl} = 2.021 \pm 0.176 \quad \text{for Inclusive } V_{cb} \text{ (Heavy Quark Expansion)}$$

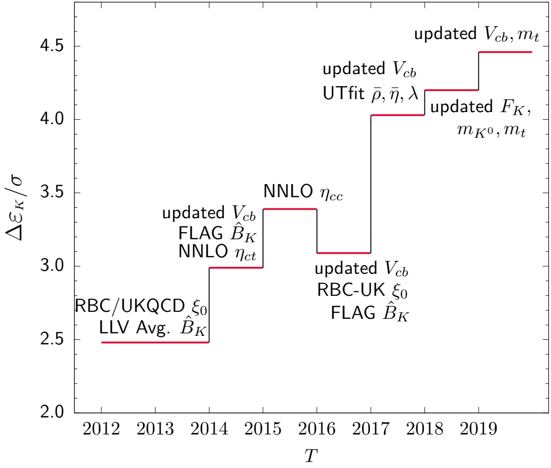
- Experiments:

$$|\varepsilon_K|^{\rm Exp} = 2.228 \pm 0.011$$

- Hence, we observe $4.5\sigma \sim 4.2\sigma$ difference between the SM theory (Lattice QCD) and experiments.

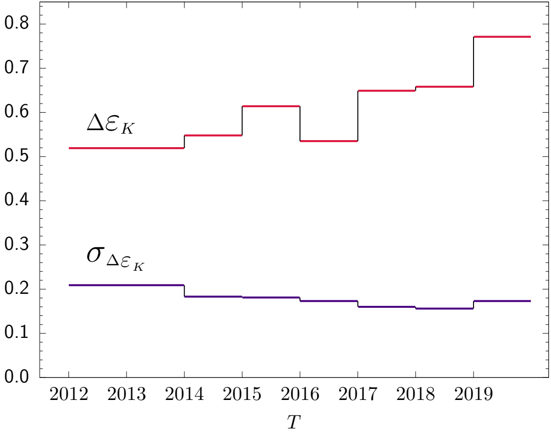
- What does this mean? $\longrightarrow$ Breakdown of SM ?

Time Evolution of $\Delta\varepsilon_K$ on the Lattice



- $\Delta\varepsilon_K \equiv |\varepsilon_K|^{\text{Exp}} - |\varepsilon_K|^{\text{SM}}_{\text{excl}} \Longleftarrow |V_{cb}|$ (CLN) & ξ_{LD} (RBC-UK)

Time Evolution of Average and Error for $\Delta\varepsilon_K$



- The average $\Delta\varepsilon_K$ has increased by 49% with some fluctuations.
- The error $\sigma_{\Delta\varepsilon_K}$ has decreased by 17% with some fluctuation: HFLAV 2017 $\rightarrow$ BELLE 2019.

Error Budget of $\Delta\varepsilon_K$: $|V_{cb}|$ (CLN), $\xi_{\rm LD}$ (RBC-UK)

source	error (%)	memo
$\|V_{cb}\|$	50.2	Exclusive (CLN)
$\bar\eta$	19.1	AOF
η_{ct}	16.3	$c - t$ Box
η_{cc}	6.9	$c - c$ Box
$\bar\rho$	2.8	AOF
$\xi_{\rm LD}$	1.7	Long-distance
$\hat{B}_K$	1.3	FLAG
ξ_0	0.58	Indirect
η_{tt}	0.54	$t - t$ Box
λ	0.16	$\|V_{us}\|$ (PDG)
⋮	⋮	⋮

- The error from $|V_{cb}|$ is dominant.

To Do List

- It would be highly desirable if the HFLAV group may perform a comprehensive reanalysis over the entire sets of the experimental data including both BABAR and BELLE for $\bar{B} \to D^* \ell \bar{\nu}$ using the BGL method and compare the results with those of CLN.
- It would be nice to reduce overall errors on $|V_{cb}|$: 1.4% $\to$ 0.8%. [OK action project: LANL-SWME: posters in Lattice 2019]
- It would be nice to reduce overall errors on $\bar{\eta}$. [BELLE2]
- It would be nice to reduce overall errors on ξ_0 and ξ_2 in lattice QCD. [RBC-UKQCD]
- It would be nice to reduce overall errors on $|V_{us}|$, $m_c(m_c)$, f_K in lattice QCD.

Summary and Conclusion

Summary

1. We find that

$$\Delta\varepsilon_K^{\text{excl}} = 4.5(3)\sigma \qquad \text{(Lattice QCD, CLN)} \qquad (1)$$

$$\Delta\varepsilon_K^{\text{incl}} = 1.2\sigma \qquad \text{(HQE, QCD Sum Rules)} \qquad (2)$$

2. It is too early to conclude that there might be something wrong with the SM yet.

3. Let us wait for the next round reanalysis of the HFLAV group on the entire data sets of the $\bar{B} \to D^* \ell \bar{\nu}$ decays, using both CLN and BGL.

4. Meanwhile, it would be very helpful to reduce the errors for $|V_{cb}|$, $|V_{us}|$, $\bar{\eta}$, ξ_0, ξ_2, $m_c(m_c)$, f_K, $\hat{B}_K$, and ξ_{LD} in lattice QCD.
$\bar{\eta} \quad \longleftarrow \quad \xi, \; f_{B_d}, \; f_{B_s}, \; B_{B_d}, \; B_{B_s}, \cdots$

5. Please stay tuned for the update.

$R(D)$ and $R(D^*)$
Other Anomalies in SM

$R(D)$ and $R(D^*)$

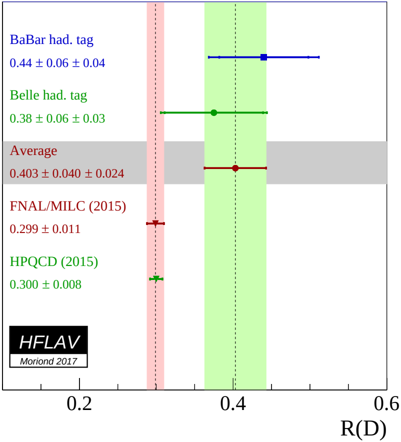
- Definition:

$$R(D) \equiv \frac{\mathcal{B}(B \to D\tau\nu_\tau)}{\mathcal{B}(B \to D\ell\nu_\ell)}$$
$$R(D^*) \equiv \frac{\mathcal{B}(B \to D^*\tau\nu_\tau)}{\mathcal{B}(B \to D^*\ell\nu_\ell)}$$

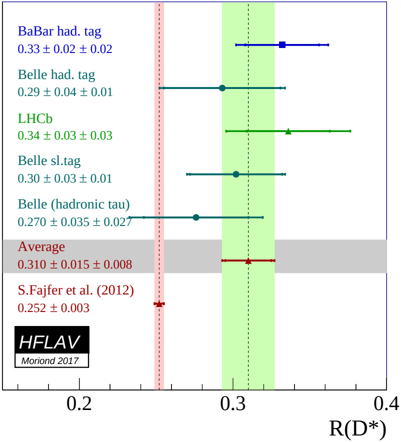
- Results:

channel	SM (Lattice QCD)	Experiment	Difference
$R(D)$	0.300(8)	0.403(40)(24)	2.2σ
$R(D^*)$	0.252(3)	0.310(15)(8)	3.4σ

$R(D)$ and $R(D^*)$

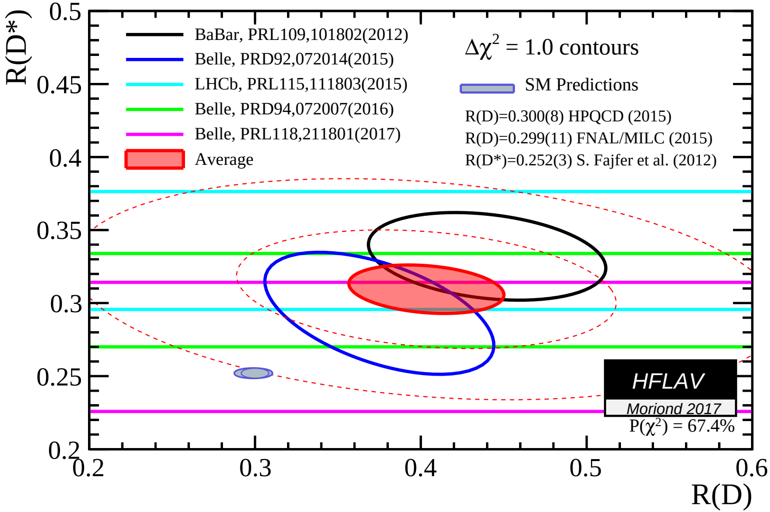


(a) $R(D) \to 2.2\sigma$



(b) $R(D^*) \to 3.4\sigma$

$R(D)$ and $R(D^*)$

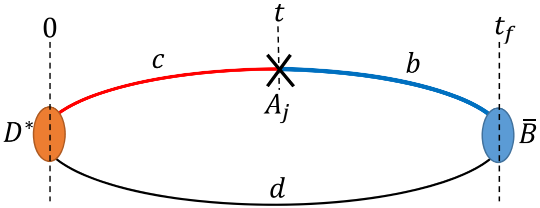


R(D*)
R(D)
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.2
0.3
0.4
0.5
0.6
BaBar, PRL109,101802(2012)
Belle, PRD92,072014(2015)
LHCb, PRL115,111803(2015)
Belle, PRD94,072007(2016)
Belle, PRL118,211801(2017)
Average
Δχ² = 1.0 contours
SM Predictions
R(D)=0.300(8) HPQCD (2015)
R(D)=0.299(11) FNAL/MILC (2015)
R(D*)=0.252(3) S. Fajfer et al. (2012)
HFLAV
Moriond 2017
P(χ²) = 67.4%

$|V_{cb}|$ on the lattice

$|V_{cb}|$ from the exclusive $\bar{B} \to D^* \ell \bar{\nu}$ at zero recoil

1. **Experiment**: $\frac{d\Gamma}{dw}(\bar{B} \to D^* \ell \bar{\nu}) \propto |V_{cb}|^2 \cdot |\mathcal{F}(w=1)|^2$
2. **Lattice QCD**: Calculate $\mathcal{F}(1) = h_{A_1}(1)$ from the matrix element



$$\frac{1}{\sqrt{m_B m_{D^*}}} \langle D^*(p', \epsilon) | A^\mu | \bar{B}(p) \rangle = -i h_{A_1}(w)(w+1)\epsilon^{*\mu} + i h_{A_2}(w)(\epsilon^* \cdot v) v^\mu + i h_{A_3}(w)(\epsilon^* \cdot v) v'^\mu$$

3. **Determine $|\mathbf{V_{cb}}|$ by combining experiment with lattice QCD results**

Calculation of V_{cb} on the lattice

1. Exclusive $B \to D^* \ell \bar{\nu}$ zero recoil [Fermilab-MILC (2014)]
   - Gold-plated: most precise experimental and lattice error
   - Form factor calculation using the 3-point function $\langle D^* | A^\mu | B \rangle$ on the lattice.

2. Exclusive $B \to D \ell \bar{\nu}$ non-zero recoil [Fermilab-MILC (2015), HPQCD (2015)]
   - Near the zero recoil, the experimental precision is poor due to to the phase space suppression.
   - Form factor calculation using the 3-point function $\langle D | V^\mu | B \rangle$ on the lattice.

3. Inclusive $B \to X_c \ell \bar{\nu}$ [S. Hashimoto (2017)]
   - Preliminary, Calculate the 4-point function on the lattice,

$$\langle B | T\{J^\dagger_\mu(q) J_\nu(0)\} | B \rangle, \qquad \text{where } J_\mu = \bar{c}\gamma_\mu(1-\gamma_5)b.$$

Decay mode	Method	$\lvert V_{cb}\rvert \times 10^3$ [HFLAV (2017)]
$\bar{B} \to D^* \ell \bar{\nu}$	Lattice	39.05(47)(58)
$\bar{B} \to D \ell \bar{\nu}$	Lattice	39.18(94)(36)
$B \to X_c \ell \bar{\nu}$	QCD sum rule	42.03(39)

Limitation of Fermilab action calculation

- On the lattice, we have inevitable systematic error: **discretization error**.
- For the $\bar{B} \to D^{(*)}\ell\bar{\nu}$ study, the heavy quark discretization error, especially for charm quark is dominant. $(\lambda \sim \Lambda/2m_Q)$
- Fermilab action calculation of h_{A_1} ($\bar{B} \to D^{*}\ell\bar{\nu}$ semileptonic form factor) has $\mathcal{O}(\alpha_s\lambda^2)$ and $\mathcal{O}(\lambda^3) \sim 1\%$ discretization error, basically.
- To achieve the precision below 1%, we have to use new action: **Oktay-Kronfeld action**, $\mathcal{O}(\lambda^3)$ improved action where its discretization error appears at $\mathcal{O}(\lambda^4) \sim 0.2\%$.

Limitation of Fermilab action calculation

- We expect the improvement in charm quark discretization error from the current Fermilab/MILC results [PRD89, 114504 (2014) and PRD92, 034506 (2015)] of the $\bar{B} \to D^{(*)}\ell\bar{\nu}$ semileptonic form factor.

	h_{A_1}	f_+
	$\bar{B} \to D^*\ell\bar{\nu}$	$\bar{B} \to D\ell\bar{\nu}$
statistics	0.4	0.7
matching	0.4	0.7
χPT	0.5	0.6
$g_{D^*D\pi}$	0.3	-
c discretization	$1.0 \to (0.2)_{\text{OK}}$	$0.4 \to (0.1)_{\text{OK}}$
others	0.1	0.2
total	$1.4 \to (0.8)_{\text{OK}}$	$1.2 \to (1.1)_{\text{OK}}$

- Belle2 has been running since Dec. 2018, and the target statistics is 50 times larger than Belle.

OK Action (mass form)

$$S_{\rm OK} = S_{\rm Fermilab} + S_{\rm new}\,, \qquad S_{\rm Fermilab} = S_0 + S_B + S_E$$

$$S_0 = m_0 \sum_x \bar\psi(x)\psi(x) + \sum_x \bar\psi(x)\gamma_4 D_4 \psi(x) - \tfrac{1}{2} a \sum_x \bar\psi(x)\triangle_4\psi(x)$$

$$+ \zeta \sum_x \bar\psi(x)\vec\gamma\cdot\vec D\psi(x) - \tfrac{1}{2} r_s \zeta a \sum_x \bar\psi(x)\triangle^{(3)}\psi(x)$$

$$= \mathcal{O}(1) + \mathcal{O}(\lambda) \qquad [\lambda \sim a\Lambda,\, \Lambda/m_Q]$$

$$S_B = -\tfrac{1}{2} c_B \zeta a \sum_x \bar\psi(x) i\vec\Sigma\cdot\vec B\psi(x) \to \mathcal{O}(\lambda)$$

$$S_E = -\tfrac{1}{2} c_E \zeta a \sum_x \bar\psi(x)\vec\alpha\cdot\vec E\psi(x) \to \mathcal{O}(\lambda^2) \qquad (c_E \neq c_B :\ \text{OK action})$$

$$m_0 = \frac{1}{2\kappa_t} - (1 + 3 r_s \zeta + 18 c_4)$$

[M. B. Oktay and A. S. Kronfeld, PRD **78**, 014504 (2008)]

[A. El-Khadra, A. S. Kronfeld and P. B. Mackenzie, PRD **55**, 3933 (1997)] .

OK Action (mass form)

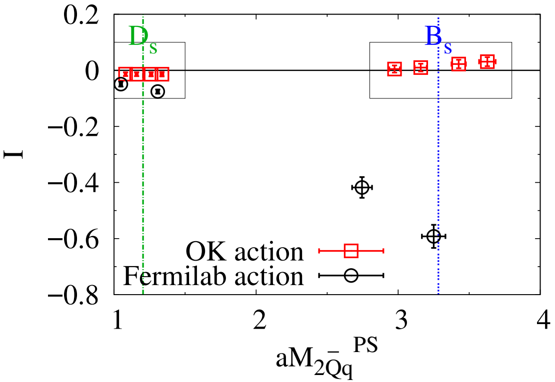
$$
\begin{aligned}
S_{\text{new}} = \mathcal{O}(\lambda^3) = c_1 a^2 \sum_x \bar{\psi}(x) \sum_i \gamma_i D_i \triangle_i \psi(x) \\
+ c_2 a^2 \sum_x \bar{\psi}(x) \{\vec{\gamma} \cdot \vec{D}, \triangle^{(3)}\} \psi(x) \\
+ c_3 a^2 \sum_x \bar{\psi}(x) \{\vec{\gamma} \cdot \vec{D}, i\vec{\Sigma} \cdot \vec{B}\} \psi(x) \\
+ c_{EE} a^2 \sum_x \bar{\psi}(x) \{\gamma_4 D_4, \vec{\alpha} \cdot \vec{E}\} \psi(x) \\
+ c_4 a^3 \sum_x \bar{\psi}(x) \sum_i {\triangle_i}^2 \psi(x) \\
+ c_5 a^3 \sum_x \bar{\psi}(x) \sum_i \sum_{j \neq i} \{\mathrm{i}\Sigma_i B_i, \triangle_j\} \psi(x)
\end{aligned}
$$

Improvements in OK action: Inconsistency

We calculate the inconsistency parameter I [Collins, Edwards, Heller, and Sloan, NPB Proc. Suppl. 47, 455 (1996)] to see $\mathcal{O}(\mathbf{p}^4)$ improvement in the OK action.

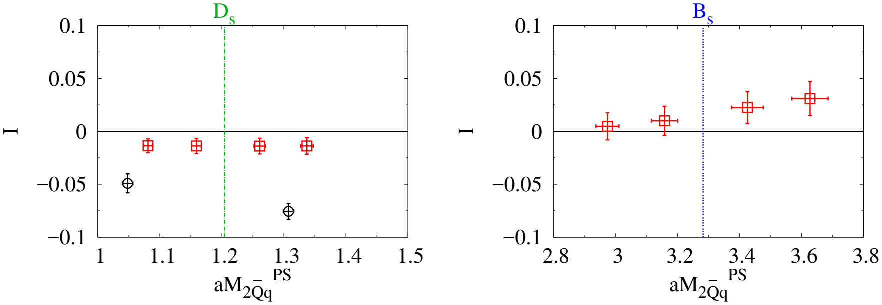
$$I \equiv \frac{2\delta M_{\bar{Q}q} - (\delta M_{\bar{Q}Q} + \delta M_{\bar{q}q})}{2M_{2\bar{Q}q}} = \frac{2\delta B_{\bar{Q}q} - (\delta B_{\bar{Q}Q} + \delta B_{\bar{q}q})}{2M_{2\bar{Q}q}}$$

where binding energy $M_{1\bar{Q}q} = m_{1\bar{Q}} + m_{1q} + B_{1\bar{Q}q}$ and so on.



Inconsistency

a12m310, $\kappa_{\text{crit}} = 0.051211$ (nonperturbative)



[Yong-Chull Jang et al., EPJC 77:768]

$\bar{B} \to D^* \ell \bar{\nu}$ Form Factor: $h_{A_1}(w = 1)$

$\bar{B} \to D^* \ell \bar{\nu}$ at zero recoil: $h_{A_1}(1)$ on the lattice

[C. Bernard et al. (Fermilab Lattice and MILC collab.), PRD79, 014506 (2009)]

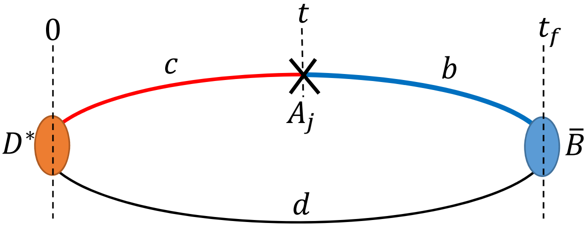
$$|h_{A_1}(1)|^2 = \frac{\langle D^* | A^j_{cb} | \bar{B} \rangle \langle \bar{B} | A^j_{bc} | D^* \rangle}{\langle D^* | V^4_{cc} | D^* \rangle \langle \bar{B} | V^4_{bb} | \bar{B} \rangle}$$

On the lattice, we calculate the double ratio R:

$$R(t, t_f) \equiv \frac{C^{B \to D^*}_{A_1}(t, t_f) C^{D^* \to B}_{A_1}(t, t_f)}{C^{B \to B}_{V_4}(t, t_f) C^{D^* \to D^*}_{V_4}(t, t_f)} \quad \xrightarrow[t \to \infty]{t_f \to \infty} \quad \left| \frac{h_{A_1}(1)}{\rho_{A_j}} \right|^2 \quad \xrightarrow[a \to 0]{V \to \infty} \quad |h_{A_1}(1)|^2$$

- $h_{A_1}(1)$: Semileptonic form factor for $\bar{B} \to D^* \ell \bar{\nu}$ at zero recoil
- $C^{X \to Y}_J(t, t_f)$: Lattice 3-point correlation functions
- ρ_{A_j}: Matching factor

3-point correlation function



$$C_{A_1}^{B \to D^*}(t, t_f) = \sum_{\vec{x}, \vec{y}} \langle O_{D^*}^\dagger(0) A_1^{cb}(\vec{y}, t) O_B(\vec{x}, t_f) \rangle \qquad (0 < t < t_f)$$

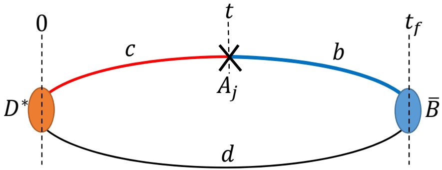
Interpolating operators for mesons

$$O_B = \bar{\psi}_b \gamma_5 \psi_l, \qquad O_{D^*} = \bar{\psi}_c \gamma_j \psi_l$$

Improved axial current operator

$$A_j^{cb} = \bar{\Psi}_c \gamma_j \gamma_5 \Psi_b,$$

3-point correlation function: current improvement



$$A_j^{cb} = \bar{\Psi}_c \gamma_j \gamma_5 \Psi_b,$$

The $\mathcal{O}(\lambda^3)$ improved field with 11 parameters (d_i): [Jaehoon Leem, Lattice 2017]

$$\begin{aligned}
\Psi(x) = e^{M_1/2}\Big[1 &+ d_1 \boldsymbol{\gamma}\cdot\boldsymbol{D} && \to \mathcal{O}(\lambda^1)\\
&+ d_2\Delta^{(3)} + d_B i\boldsymbol{\Sigma}\cdot\boldsymbol{B} + d_E \boldsymbol{\alpha}\cdot\boldsymbol{E} && \to \mathcal{O}(\lambda^2)\\
&+ d_{rE}\{\boldsymbol{\gamma}\cdot\boldsymbol{D}, \boldsymbol{\alpha}\cdot\boldsymbol{E}\} + d_3\sum_i \gamma_i D_i \Delta_i + d_4\{\boldsymbol{\gamma}\cdot\boldsymbol{D}, \Delta^{(3)}\}\\
&+ d_5\{\boldsymbol{\gamma}\cdot\boldsymbol{D}, i\boldsymbol{\Sigma}\cdot\boldsymbol{B}\} + d_{EE}\{\gamma_4 D_4, \boldsymbol{\alpha}\cdot\boldsymbol{E}\} && \to \mathcal{O}(\lambda^3)\\
&+ d_6[\gamma_4 D_4, \Delta^{(3)}] + d_7[\gamma_4 D_4, i\boldsymbol{\Sigma}\cdot\boldsymbol{B}]\Big]\psi(x).
\end{aligned}$$

Calculate $R = |h_{A_1}(1)/\rho_{A_j}|^2$ using two different analysis

1. **Direct analysis on $C_J^{X \to Y}$**

$$C_{A_1}^{B\to D^*}(t, t_f) = B^{B\to D^*} e^{-M_D^* t} e^{-M_B(t_f - t)} (1 + \hat{c}^{B\to D^*}(t, t_f))$$

where $B^{B\to D^*} = \langle D^* | A_1^{cb} | B \rangle$, and $\hat{c}^{B\to D^*}$ represent the contamination from the excited states of B and D^* mesons.

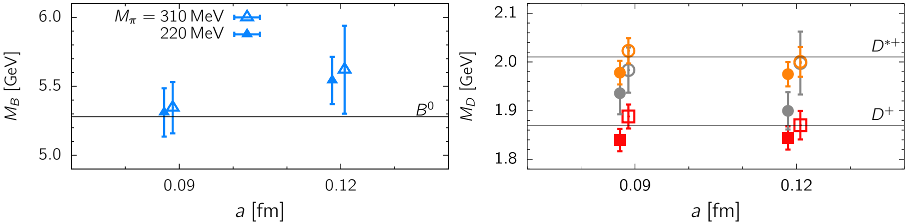
$$R = \frac{B^{B\to D^*} \cdot B^{D^*\to B}}{B^{B\to B} \cdot B^{D^*\to D^*}}$$

2. **Analysis on R**

$$\begin{aligned} R(t, t_f) &\equiv \frac{C_{A_1}^{B\to D^*}(t, t_f) C_{A_1}^{D^*\to B}(t, t_f)}{C_{V_4}^{B\to B}(t, t_f) C_{V_4}^{D^*\to D^*}(t, t_f)} \\ &= \frac{B^{B\to D^*} \cdot B^{D^*\to B}}{B^{B\to B} \cdot B^{D^*\to D^*}} [1 + \hat{c}^{B\to D^*}(t, t_f) + \hat{c}^{D^*\to B}(t, t_f) \\ &\qquad - \hat{c}^{B\to B}(t, t_f) - \hat{c}^{D^*\to D^*}(t, t_f) \cdots]. \end{aligned}$$

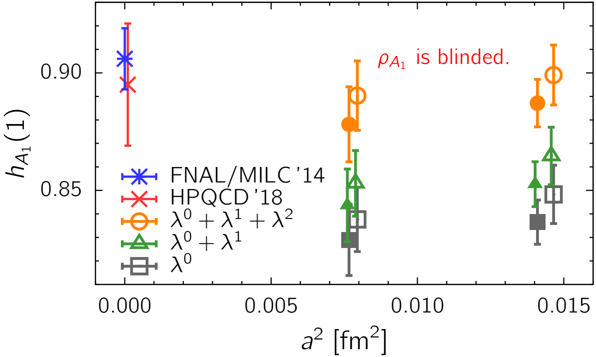
Meson Spectrum of B and $D^{(*)}$

$$E = M_1 + \frac{\boldsymbol{p}^2}{2M_2} - \frac{(\boldsymbol{p}^2)^2}{8M_4^3} - \frac{a^3 w_4}{6} \sum_i p_i^4 + \cdots .$$



- Meson masses (M_B, M_D) can be obtained from the kinetic mass M_2.
- M_{D^*} (gray) : kinetic mass (M_2).
- M_{D^*} (orange) : $M(D^*) = M_2(D) + M_1(D^*) - M_1(D) \to$ smaller errors.

$\bar{B} \to D^* \ell \bar{\nu}$ Form Factor at Zero Recoil : $h_{A_1}(w=1)$



- ρ_{A_j} is blinded: $\rho_{A_j} = \dfrac{Z^{bc}_{A_j} Z^{cb}_{A_j}}{Z^{bb}_{V_4} Z^{cc}_{V_4}} \to 1.$
- Non-perturbative calculation of ρ_{A_j} is underway.
- Preliminary results!

Summary

- This is the first numerical study using the OK action with currents improved up to $\mathcal{O}(\lambda^3)$.
- We have obtained preliminary results for $\dfrac{h_{A_1}(w=1)}{\rho_{A_j}}$ of $\bar{B} \to D^* \ell \bar{\nu}$ decays.

To be done

- Non-perturbative calculation of matching factor ρ_{A_j}.
- Extending measurement to superfine and ultrafine ensembles.
- Chiral-continuum extrapolation
- Accumulate more statistics

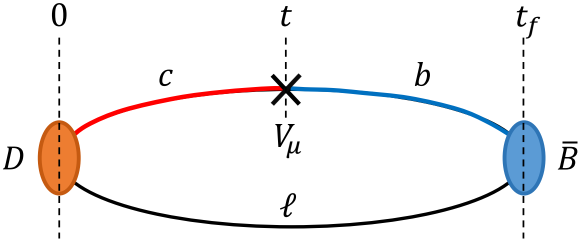
$\bar{B} \to D\ell\bar{\nu}$ Form Factors: $h_{\pm}(w)$

$\bar{B} \to D\ell\bar{\nu}$ Form Factors: $h_{\pm}(w)$ on the lattice

$$\frac{\langle D(M_D, \boldsymbol{p}')|V_\mu|B(M_B, \mathbf{0})\rangle}{\sqrt{2M_D}\sqrt{2M_B}} = \frac{1}{2}\left\{h_+(w)(v+v')_\mu + h_-(w)(v-v')_\mu\right\} ,$$

- B meson is at rest: $v = \dfrac{p}{M_B} = (1, \mathbf{0})$.
- D meson is moving with velocity: $v' = \dfrac{p'}{M_D} = (\dfrac{E_D}{M_D}, \dfrac{\boldsymbol{p}'}{M_D})$.
- Recoil parameter: $w = v \cdot v'$.

3-point correlation function



$$C_{V_\mu}^{B\to D}(t,t_f) = \sum_{\vec{x},\vec{y}} \langle O_D^\dagger(0) V_\mu^{cb}(\vec{y},t) O_B(\vec{x},t_f)\rangle \qquad (0 < t < t_f)$$

Interpolating operators for mesons

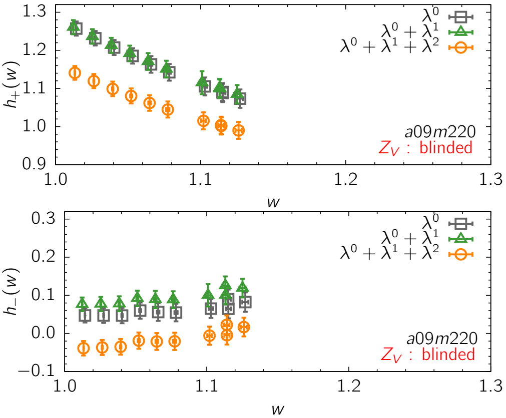
$$O_B = \bar{\psi}_b \gamma_5 \psi_\ell, \qquad O_D = \bar{\psi}_c \gamma_5 \psi_\ell$$

Improved vector current operator

$$V_\mu^{cb} = \bar{\Psi}_c \gamma_\mu \Psi_b,$$

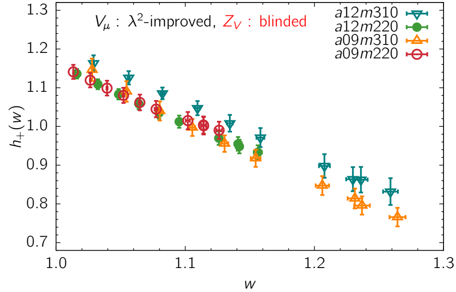
$\bar{B} \to D\ell\bar{\nu}$ Form Factors $h_\pm(w)$

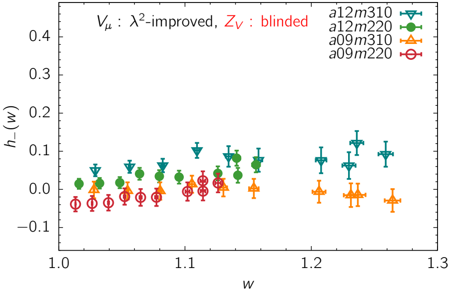
$a \cong 0.09\,\mathrm{fm}$ & $m_\pi \cong 220\,\mathrm{MeV}$



- MILC HISQ lattice at $a \cong 0.09\,\mathrm{fm}$ and $m_\pi \cong 220\,\mathrm{MeV}$.
- Z_V is blinded.

$\bar{B} \to D\ell\bar{\nu}$ Form Factors $h_\pm(w)$





- MILC HISQ lattices at $a \cong 0.12\,\mathrm{fm}$ and $a \cong 0.09\,\mathrm{fm}$
- Z_V is blinded. NPR is underway.
- The vector current is improved up to the λ^2 order.
- Preliminary results!

Summary

- This is the first numerical study using the OK action with currents improved up to $\mathcal{O}(\lambda^2)$.
- We produced 3-point correlation functions, and obtained preliminary results for $\frac{h_\pm(w)}{Z_V}$ of $\bar{B} \to D\ell\bar{\nu}$ decay.

To do list

- Non-perturbative calculation of matching factors: Z_V.
- Extending measurement to superfine and ultrafine ensembles.
- Chiral-continuum extrapolation
- Accumulate more statistics

Thank God for your help !!!

Back-up Slides

CLN

CLN: Caprini, Lellouch, Neubert I

- Consider $\bar{B} \to D^*\ell\bar{\nu}$ decays.

$$\frac{d\Gamma(\bar{B} \to D^*\ell\bar{\nu})}{dw} = \frac{G_F^2 m_{D^*}^3}{48\pi^3}(m_B - m_{D^*})^2 \, \chi(w) \, \eta_{\text{EW}}^2 \, \mathcal{F}^2(w) \, |V_{cb}|^2$$

- Here, G_F is Fermi constant, η_{EW} is a small electroweak correction, and $\mathcal{F}(w)$ is the form factor.
- The kinematic factor $\chi(w)$ is

$$\chi(w) = \sqrt{w^2 - 1}(w+1)^2 \times Y(w)$$

$$Y(w) = \left[1 + \frac{4w}{w+1}\frac{1 - 2wr + r^2}{(1-r)^2}\right]$$

CLN: Caprini, Lellouch, Neubert II

- The form factor can be rewritten as follows,

$$\mathcal{F}^2(w) = h_{A_1}^2(w) \times \frac{1}{Y(w)} \times \left\{ 2\frac{1-2wr+r^2}{(1-r)^2}\left[1+\frac{w-1}{w+1}R_1^2(w)\right] + \left[1+\frac{w-1}{1-r}\Big(1-R_2(w)\Big)\right]^2 \right\}$$

- So far the formalism is quite general.
- CLN method [16]: ($\approx$ model-dependent approximation)

$$h_{A_1}(w) = h_{A_1}(1)\left[1-8\rho^2 z+(53\rho^2-15)z^2-(231\rho^2-91)z^3\right] \quad (3)$$

$$R_1(w) = R_1(1) - 0.12(w-1) + 0.05(w-1)^2 \quad (4)$$

$$R_2(w) = R_2(1) + 0.11(w-1) - 0.06(w-1)^2 \quad (5)$$

CLN: Caprini, Lellouch, Neubert III

where z is a conformal mapping variable:

$$z = \frac{\sqrt{w+1} - \sqrt{2}}{\sqrt{w+1} + \sqrt{2}} \quad (6)$$

- The trouble is that the slopes and curvatures of $R_1(w)$ and $R_2(w)$ are fixed by the HQET perturbation theory (zero-recoil expansion). The HQET results for the slopes and curvatures has about 10% uncertainty of order $\mathcal{O}(\Lambda^2/m_c^2)$ and $\mathcal{O}(\alpha_s \Lambda/m_c)$.
- Hence, CLN can NOT have precision better than 2% by construction.
- The trouble is that the experimental results have errors less than 2% and that the lattice QCD results for the form factors have such a high precision that the errors are below the 2% level.

CLN: Caprini, Lellouch, Neubert IV

- At any rate, the experimental group (HFLAV 2017) use CLN to fit the experimental data to determine four parameters: $\eta_{\text{EW}}\mathcal{F}(1)|V_{cb}|$, ρ^2, $R_1(1)$, $R_2(1)$.
- Lattice QCD determines $\mathcal{F}(1)$ very well.
- η_{EW} is very well known.
- Hence, we can determine exclusive $|V_{cb}|$ out of this.

BGL

BGL: Boyd, Grinstein, Lebed I

- BGL is model-independent.
- BGL is constructed on three building blocks:
  1. Dispersion relation
  2. Crossing symmetry
  3. Analytic continuation: analyticity
- Consider the 2-point function:

$$\begin{aligned}\Pi_J^{\mu\nu}(q) &= (q^\mu q^\nu - q^2 g^{\mu\nu})\Pi_J^T(q^2) + g^{\mu\nu}\Pi_J^L(q^2) \\ &\equiv i\int d^4x e^{iq\cdot x}\langle 0|TJ^\mu(x)[J^\nu(0)]^\dagger|0\rangle \end{aligned} \tag{7}$$

- In general, $\Pi_J^{T,L}(q^2)$ is not finite.

BGL: Boyd, Grinstein, Lebed II

- Hence, we need to make one or two subtractions to obtain finite dispersion relations:

$$\chi_J^L(q^2) = \frac{\partial \Pi_J^L}{\partial q^2} = \frac{1}{\pi}\int_0^\infty dt \frac{\mathrm{Im}\Pi_J^L(t)}{(t-q^2)^2} \quad (8)$$

$$\chi_J^T(q^2) = \frac{\partial \Pi_J^T}{\partial q^2} = \frac{1}{\pi}\int_0^\infty dt \frac{\mathrm{Im}\Pi_J^T(t)}{(t-q^2)^2} \quad (9)$$

- Källen-Lehmann spectral decomposition:

$$(q^\mu q^\nu - q^2 g^{\mu\nu})\mathrm{Im}\Pi_J^T(q^2) + g^{\mu\nu}\mathrm{Im}\Pi_J^L(q^2)$$
$$= \frac{1}{2}\sum_X (2\pi)^4 \delta^4(q-p_X)\langle 0|J^\mu(0)|X\rangle\langle X|[J^\nu(0)]^\dagger|0\rangle \quad (10)$$

BGL: Boyd, Grinstein, Lebed III

- Multiply $\xi_\mu \xi^*_\nu$ on both sides:

$$\left[(q^\mu q^\nu - q^2 g^{\mu\nu})\mathrm{Im}\Pi^T_J(q^2) + g^{\mu\nu}\mathrm{Im}\Pi^L_J(q^2)\right]\xi_\mu \xi^*_\nu \geq 0 \tag{11}$$

for any complex 4-vector ξ_μ.

- From this we can prove the positivity:

$$\mathrm{Im}\Pi^T_J(q^2) \geq 0 \tag{12}$$

$$\mathrm{Im}\Pi^L_J(q^2) \geq 0 \tag{13}$$

BGL: Boyd, Grinstein, Lebed IV

- Consider the two body state of $X = H_b(p_1)H_c(p_2)$.

$$\begin{aligned}\mathrm{Im}\Pi_J^{ii}(q^2) =& \frac{1}{2}\int \frac{d^3p_1 d^3p_2}{(2\pi)^3 4E_1E_2}\delta^4(q-p_1-p_2) \\ &\times \sum_{\text{pol}} \langle 0|J^i|H_b(p_1)H_c(p_2)\rangle\langle H_b(p_1)H_c(p_2)|[J^i]^\dagger|0\rangle \\ &+\cdots\end{aligned} \tag{14}$$

- Here, the ellipsis $(\cdots)$ represents strictly positive contributions from the higher resonances and multi-particle states.
- We may assume that $H_b = B, B^*$ meson states, and $H_c = D, D^*$ meson states.

BGL: Boyd, Grinstein, Lebed V

- Let us consider a simple example of $H_b = B$ and $H_c = D^*$.

$$\mathrm{Im}\Pi_J^{ii}(t) \geq k(t)|\mathcal{F}(t)|^2 \tag{15}$$

where $t = q^2$, $k(t)$ is a calculable kinematic function arising from two-body phase space.

- Let us use the crossing symmetry and analytic continuation:

$$\langle 0|J^i|H_b(p_1)H_c(p_2)\rangle = \mathcal{F}(t) \qquad (t_+ \leq t < \infty) \tag{16}$$

$$\langle \bar{H}_b(-p_1)|J^i|H_c(p_2)\rangle = \mathcal{F}(t) \qquad (m_\ell^2 \leq t < t_-) \tag{17}$$

BGL: Boyd, Grinstein, Lebed VI

- Hadronic moments $\chi_J^{(n)}$:

$$\chi_J^{(n)} \equiv \frac{1}{\Gamma(n+3)} \left. \frac{\partial^{n+2}\Pi_J^{ii}}{\partial^{n+2}q^2} \right|_{q^2=0}$$
$$= \frac{1}{\pi} \int_0^{\infty} dt \left. \frac{\mathrm{Im}\Pi_J^{ii}(t)}{(t-q^2)^{n+3}} \right|_{q^2=0} \quad (18)$$

- Hence, the inequality is

$$\chi_J^{(n)} \geq \frac{1}{\pi} \int_{t_+}^{\infty} dt \frac{k(t)|\mathcal{F}(t)|^2}{t^{n+3}} \quad (19)$$
$$\longrightarrow \quad \frac{1}{\pi} \int_{t_+}^{\infty} dt |h^{(n)}(t) F(t)|^2 \leq 1 \quad (20)$$

BGL: Boyd, Grinstein, Lebed VII

where

$$[h^{(n)}(t)]^2 = \frac{k(t)}{t^{n+3}\chi_J^{(n)}} \geq 0\,. \tag{21}$$

- Let us introduce the conformal mapping function:

$$z(t, t_s) = \frac{\sqrt{t_+ - t} - \sqrt{t_+ - t_s}}{\sqrt{t_+ - t} + \sqrt{t_+ - t_s}} \tag{22}$$

- The inequality can be rewritten as follows,

$$\frac{1}{\pi}\int_{t_+}^{\infty} dt \left|\frac{dz(t, t_0)}{dt}\right| |\phi(t, t_0)P(t)F(t)|^2 \leq 1\,, \tag{23}$$

BGL: Boyd, Grinstein, Lebed VIII

- Here, the outer function ϕ is

$$\phi(t, t_0) = \tilde{P}(t) \frac{h^{(n)}(t)}{\sqrt{\left| \frac{dz(t, t_0)}{dt} \right|}} \tag{24}$$

- Here, the factor $\tilde{P}(t)$ removes the sub-threshold poles and branch cuts in $h^{(n)}(t)$.

$$\tilde{P}(t) = \prod_{i=1}^{\tilde{N}} z(t, t_{s_i}) \prod_{j=1}^{\tilde{M}} \sqrt{z(t, t_{s_j})} \tag{25}$$

BGL: Boyd, Grinstein, Lebed IX

- The Blaschke factor $P(t)$ removes all the sub-threshold poles in $\mathcal{F}(t)$.

$$P(t) \equiv \prod_{i=1}^{N} \frac{z - z_{P_i}}{1 - z z_{P_i}^*} = \prod_{i=1}^{N} \frac{z - z_{P_i}}{1 - z z_{P_i}} \qquad (26)$$

$$z_{P_i} \equiv z(t_{P_i}, t_-) = \frac{\sqrt{t_+ - t_{P_i}} - \sqrt{t_+ - t_-}}{\sqrt{t_+ - t_{P_i}} + \sqrt{t_+ - t_-}} \qquad (27)$$

  where $t_{P_i} = M_{P_i}^2$ represents the pole positions of $F(t)$ below the threshold $(t_{P_i} < t_+)$.
- $|\tilde{P}(t)| = 1$ and $|P(t)| = 1$ for $t_+ \leq t < \infty$.
- Hence, $\phi(t, t_0)P(t)\mathcal{F}(t)$ is analytic even in the sub-threshold region.

BGL: Boyd, Grinstein, Lebed X

- BGL method for the form factor parametrization:

$$F(t) = \frac{1}{\phi(t, t_0)P(t)} \sum_{n=0}^{\infty} a_n z^n(t, t_0) \tag{28}$$

- After the Fourier analysis, the inequality is

$$\sum_{n=0}^{\infty} |a_n|^2 \leq 1\,. \tag{29}$$

- This is called the unitarity conditions (the weak version).

B_s meson mass

Motivation

- In heavy flavor physics, $|V_{cb}|$ has 4.1σ tension between in and ex.
- The dominant error in ε_K comes from $|V_{cb}|$.

$$\begin{cases} 30.1\% & \leftarrow |V_{cb}| \\ 1.8\% & \leftarrow \hat{B}_K \end{cases}$$

- 4.0σ tension is observed using most up to date input parameters.

$$|\varepsilon_K|^{\text{Exp}} = 2.228(11) \times 10^{-3} \qquad \text{(PDG)}$$
$$|\varepsilon_K|^{\text{SM}} = 1.58(16) \times 10^{-3} \qquad (\text{FLAG } \hat{B}_K,\ \text{Exclusive } |V_{cb}|)$$

- More precision in $|V_{cb}|$ might lead to larger tension.
- The dominant error in $|V_{cb}| \longleftrightarrow$ heavy quark discretization effect.
- We use the OK action to calculate form factors for the semi-leptonic decays: $\bar{B} \to D^* \ell \bar{\nu}_\ell\,, \qquad \bar{B} \to D \ell \bar{\nu}_\ell$.
- Here, we will verify the improvement in B meson spectrum.

OK Action (mass form)

$$S_{\rm OK} = S_{\rm Fermilab} + S_{\rm new}\,, \qquad S_{\rm Fermilab} = S_0 + S_B + S_E$$

$$\begin{aligned} S_0 &= m_0 \sum_x \bar\psi(x)\psi(x) + \sum_x \bar\psi(x)\gamma_4 D_4\psi(x) - \tfrac{1}{2}a\sum_x \bar\psi(x)\triangle_4\psi(x) \\ &+ \zeta\sum_x \bar\psi(x)\vec\gamma\cdot\vec D\psi(x) - \tfrac{1}{2}r_s\zeta a\sum_x \bar\psi(x)\triangle^{(3)}\psi(x) \\ &= \mathcal{O}(1) + \mathcal{O}(\lambda) \qquad [\lambda \sim a\Lambda,\ \Lambda/m_Q] \end{aligned}$$

$$S_B = -\tfrac{1}{2}c_B\zeta a\sum_x \bar\psi(x)i\vec\Sigma\cdot\vec B\psi(x) \to \mathcal{O}(\lambda)$$

$$S_E = -\tfrac{1}{2}c_E\zeta a\sum_x \bar\psi(x)\vec\alpha\cdot\vec E\psi(x) \to \mathcal{O}(\lambda^2) \qquad (c_E \neq c_B:\ \text{OK action})$$

$$m_0 = \frac{1}{2\kappa_t} - (1 + 3r_s\zeta + 18c_4)$$

[M. B. Oktay and A. S. Kronfeld, PRD **78**, 014504 (2008)]

[A. El-Khadra, A. S. Kronfeld and P. B. Mackenzie, PRD **55**, 3933 (1997)] .

OK Action (mass form)

$$
\begin{aligned}
S_{\text{new}} = \mathcal{O}(\lambda^3) = c_1 a^2 \sum_x \bar{\psi}(x) \sum_i \gamma_i D_i \triangle_i \psi(x) \\
+ c_2 a^2 \sum_x \bar{\psi}(x) \{\vec{\gamma} \cdot \vec{D}, \triangle^{(3)}\} \psi(x) \\
+ c_3 a^2 \sum_x \bar{\psi}(x) \{\vec{\gamma} \cdot \vec{D}, i \vec{\Sigma} \cdot \vec{B}\} \psi(x) \\
+ c_{EE} a^2 \sum_x \bar{\psi}(x) \{\gamma_4 D_4, \vec{\alpha} \cdot \vec{E}\} \psi(x) \\
+ c_4 a^3 \sum_x \bar{\psi}(x) \sum_i {\triangle_i}^2 \psi(x) \\
+ c_5 a^3 \sum_x \bar{\psi}(x) \sum_i \sum_{j \neq i} \{\mathrm{i}\Sigma_i B_i, \triangle_j\} \psi(x)
\end{aligned}
$$

OK Action: Tadpole Improvement (hopping form)

$$c_5 a^3 \bar{\psi}(x) \sum_i \sum_{j\neq i} \{\mathrm{i}\Sigma_i B_{i\,\mathrm{lat}}, \triangle_{j\,\mathrm{lat}}\}\psi(x)$$

$$= \mathrm{i}\frac{2\tilde{c}_5\tilde{\kappa}_t}{4u_0^2}\bar{\psi}_x \sum_i \Sigma_i T_i^{(3)}\psi_x - \mathrm{i}\frac{32\tilde{c}_5\tilde{\kappa}_t}{2u_0^3}\bar{\psi}_x \overrightarrow{\Sigma}\cdot\overrightarrow{B}\psi_x$$

$$+ \mathrm{i}\frac{2\tilde{c}_5\tilde{\kappa}_t}{u_0^4}\bar{\psi}_x \sum_i \Big(-\frac{1}{4}\Sigma_i T_i^{(3)} + \sum_{j\neq i}\{\Sigma_i B_i, (T_j + T_{-j})\}\Big)\psi_x$$

$$T_i^{(3)} \equiv \sum_{j,k=1}^{3} \epsilon_{ijk}\Big(T_{-k}(T_j - T_{-j})T_k - T_k(T_j - T_{-j})T_{-k}\Big)$$

Measurement

Gauge Ensemble, Heavy Quark κ, Meson Momentum

- MILC asqtad $N_f = 2 + 1$

a(fm)	$N_L^3 \times N_T$	β	am_l'	am_s'	u_0	a^{-1}(GeV)	N_{conf}	$N_{t_{\text{src}}}$
0.12	$20^3 \times 64$	6.79	0.02	0.05	0.8688	1.683^{+43}_{-16}	500	6

- 11 momenta $|\boldsymbol{p}a| = 0, 0.099, \cdots, 1.26$

Measurement: Interpolating Operator

- Meson correlator

$$C(t,\boldsymbol{p}) = \sum_{\boldsymbol{x}} e^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} \langle \mathcal{O}^\dagger(t,\boldsymbol{x})\mathcal{O}(0,\boldsymbol{0})\rangle$$

- Heavy-light meson interpolating operator

$$\mathcal{O}_{\mathrm{t}}(x) = \bar{\psi}_\alpha(x)\Gamma_{\alpha\beta}\Omega_{\beta\mathrm{t}}(x)\chi(x)$$

$$\Gamma = \begin{cases} \gamma_5 & \text{(Pseudo-scalar)} \\ \gamma_\mu & \text{(Vector)} \end{cases}, \quad \Omega(x) \equiv \gamma_1^{x_1}\gamma_2^{x_2}\gamma_3^{x_3}\gamma_4^{x_4}$$

- Quarkonium interpolating operator

$$\mathcal{O}(x) = \bar{\psi}_\alpha(x)\Gamma_{\alpha\beta}\psi_\beta(x)$$

[Wingate *et al.*, PRD **67**, 054505 (2003) , C. Bernard *et al.*, PRD **83**, 034503 (2011)]

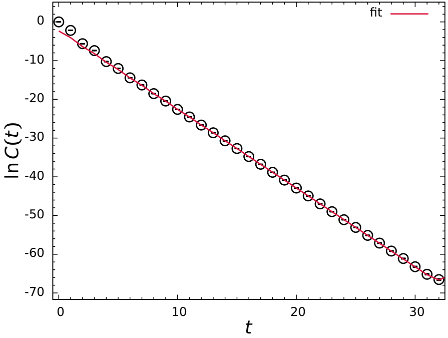
Correlator Fit

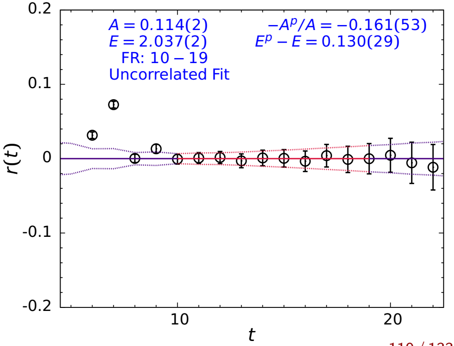
- fit function

$$f(t) = A\{e^{-Et} + e^{-E(T-t)}\} + (-1)^t A^p\{e^{-E^p t} + e^{-E^p(T-t)}\}$$

- fit residual

$$r(t) = \frac{C(t) - f(t)}{|C(t)|}, \text{where } C(t) \text{ is data.}$$





Correlator Fit: Effective Mass

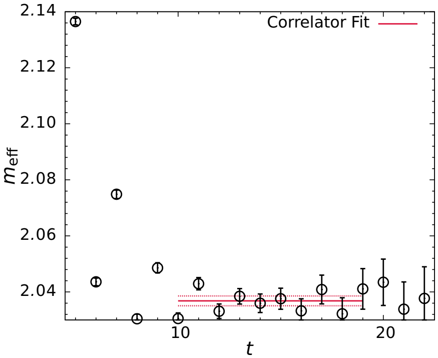
$$m_{\text{eff}}(t) = \frac{1}{2} \ln \left(\frac{C(t)}{C(t+2)} \right)$$

For small t,

$$\begin{aligned} C(t) \cong& A(e^{-Et} + \beta e^{-(E+\Delta E)t}) \\ =& A e^{-Et}(1 + \beta e^{-(\Delta E)t}) \ , \end{aligned}$$

$$\begin{cases} \beta > 0 & \text{(excited state)} \\ \beta \sim -(-1)^t & \text{(time parity state)} \end{cases}$$

$$m_{\text{eff}} \approx E + \beta(\Delta E) e^{-(\Delta E)t}$$

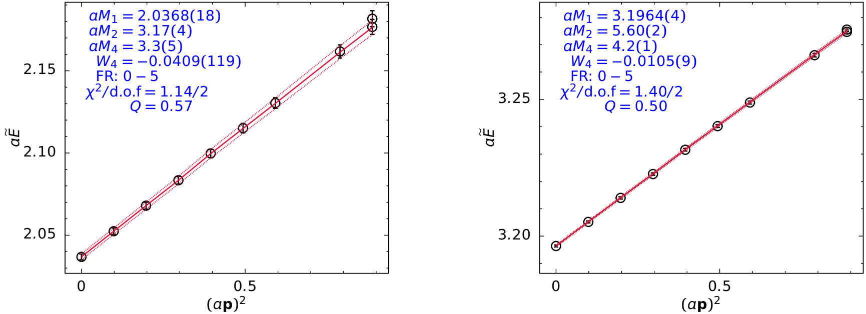


$[\overline{Q}q$, PS, $\kappa = 0.041$, $\boldsymbol{p} = \boldsymbol{0}]$

Dispersion Relation

$$E = M_1 + \frac{\boldsymbol{p}^2}{2M_2} - \frac{(\boldsymbol{p}^2)^2}{8M_4^3} - \frac{a^3 W_4}{6}\sum_i p_i^4$$

$$\widetilde{E} = E + \frac{a^3 W_4}{6}\sum_i p_i^4\,, \quad \boldsymbol{n} = (2,2,1)\,,(3,0,0)$$



$[\overline{Q}q,\ \text{PS},\ \kappa = 0.041]$ $[\overline{Q}Q,\ \text{PS},\ \kappa = 0.041]$

Improvement Test: Inconsistency Parameter

$$I \equiv \frac{2\delta M_{\overline{Q}q} - (\delta M_{\overline{Q}Q} + \delta M_{\overline{q}q})}{2M_{2\overline{Q}q}} = \frac{2\delta B_{\overline{Q}q} - (\delta B_{\overline{Q}Q} + \delta B_{\overline{q}q})}{2M_{2\overline{Q}q}}$$

$$M_{1\overline{Q}q} = m_{1\overline{Q}} + m_{1q} + B_{1\overline{Q}q} \qquad \delta M_{\overline{Q}q} = M_{2\overline{Q}q} - M_{1\overline{Q}q}$$

$$M_{2\overline{Q}q} = m_{2\overline{Q}} + m_{2q} + B_{2\overline{Q}q} \qquad \delta B_{\overline{Q}q} = B_{2\overline{Q}q} - B_{1\overline{Q}q}$$

[S. Collins *et al.*, NPB **47**, 455 (1996) , A. S. Kronfeld, NPB **53**, 401 (1997)]

- Inconsistency parameter I can be used to examine the improvements by $\mathcal{O}(\boldsymbol{p}^4)$ terms in the action. OK action is designed to improve these terms and matched at tree-level.
- Binding energies B_1 and B_2 are of order $\mathcal{O}(\boldsymbol{p}^2)$. Because the kinetic meson mass M_2 appears with a factor $\boldsymbol{p}^2$, the leading contribution of binding energy B_2 generated by $\mathcal{O}(\boldsymbol{p}^4)$ terms in the action.

$$E = M_1 + \frac{\boldsymbol{p}^2}{2M_2} + \cdots = M_1 + \frac{\boldsymbol{p}^2}{2(m_{2\overline{Q}} + m_{2q})}\left[1 - \frac{B_{2\overline{Q}q}}{(m_{2\overline{Q}} + m_{2q})} + \cdots\right] + \cdots$$

Improvement Test: Inconsistency Parameter

$$I \cong \frac{2\delta M_{\overline{Q}q} - \delta M_{\overline{Q}Q}}{2M_{2\overline{Q}q}} \cong \frac{2\delta B_{\overline{Q}q} - \delta B_{\overline{Q}Q}}{2M_{2\overline{Q}q}}$$

- Considering non-relativistic limit of quark and anti-quark system, for S-wave case $(\mu_2^{-1} = m_{2\overline{Q}}^{-1} + m_{2q}^{-1})$,

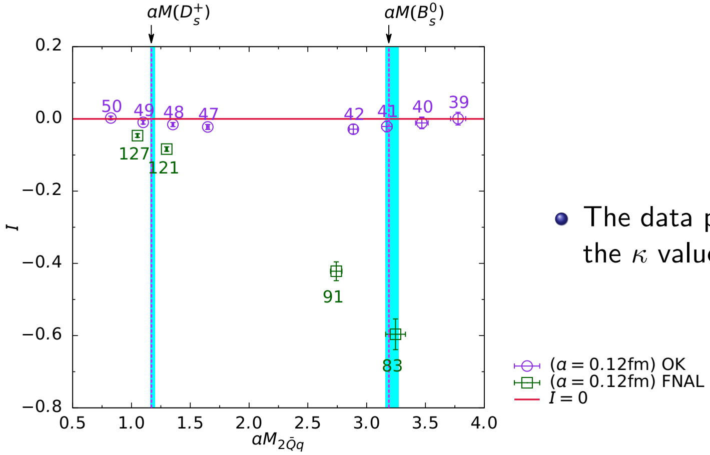
$$\begin{aligned} \delta B_{\overline{Q}q} &= \frac{5}{3}\frac{\langle \boldsymbol{p}^2\rangle}{2\mu_2}\left[\mu_2\left(\frac{m_{2\overline{Q}}^2}{m_{4\overline{Q}}^3} + \frac{m_{2q}^2}{m_{4q}^3}\right) - 1\right] \quad (m_4 : c_1, c_3) \\ &+\frac{4}{3}a^3\frac{\langle \boldsymbol{p}^2\rangle}{2\mu_2}\mu_2(w_{4\overline{Q}}m_{2\overline{Q}}^2 + w_{4q}m_{2q}^2) \quad (w_4 : c_2, c_4) \\ &+\mathcal{O}(p^4) \end{aligned}$$

[A. S. Kronfeld, NPB **53**, 401 (1997) , C. Bernard *et al.*, PRD **83**, 034503 (2011)]

- Leading contribution of $\mathcal{O}(\boldsymbol{p}^2)$ in δB vanishes when $w_4 = 0$, $m_2 = m_4$, not only for S-wave states but also for higher harmonics.
- This condition is satisfied exactly at tree-level, and we expect I is close to 0.

Improvement Test: Inconsistency Parameter

- The coarse ($a = 0.12$fm) ensemble data covers the B_s^0 mass and shows significant improvement compared to the Fermilab action.



- The data point labels denote the κ values.

Improvement Test: Hyperfine Splitting Δ

$$\Delta_1 = M_1^* - M_1\,,\ \Delta_2 = M_2^* - M_2$$

Recall,

$$\begin{aligned} M^{(*)}_{1\overline{Q}q} &= m_{1\overline{Q}} + m_{1q} + B^{(*)}_{1\overline{Q}q} \\ M^{(*)}_{2\overline{Q}q} &= m_{2\overline{Q}} + m_{2q} + B^{(*)}_{2\overline{Q}q} \\ \delta B^{(*)} &= B_2^{(*)} - B_1^{(*)} \end{aligned}$$

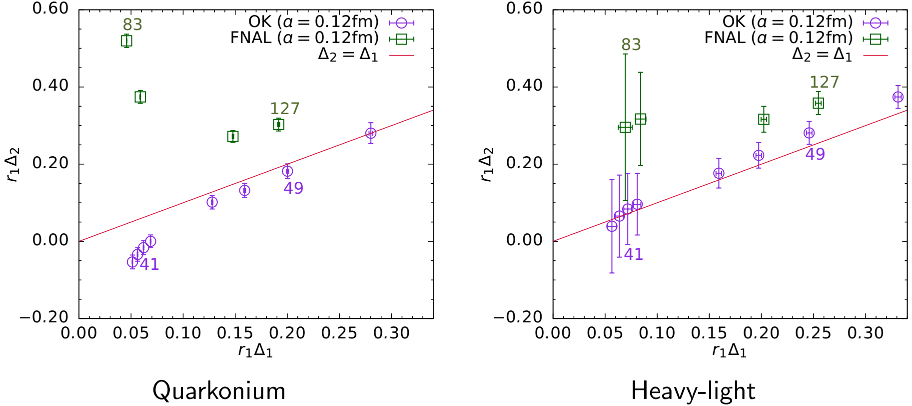
Then,

$$\Delta_2 = \Delta_1 + \delta B^* - \delta B$$

- The difference in hyperfine splittings $\Delta_2 - \Delta_1$ also can be used to examine the improvement from $\mathcal{O}(\boldsymbol{p}^4)$ terms in the action.

Improvement Test: Hyperfine Splitting Δ

$$\Delta_2 = \Delta_1 + \delta B^* - \delta B$$



Conclusion and Outlook

- Inconsistency parameter shows that the OK action clearly improves $\mathcal{O}(\boldsymbol{p}^4)$ terms.
- Hyperfine splitting shows that the OK action clearly improves the higher dimension magnetic effects for the quarkonium.
- We plan to calculate V_{cb} with the highest precision possible.
- Improved current relevant to the decay $\bar{B} \to D^* l\nu$ at zero recoil is needed. (Jon A. Bailey and J. Leem)
- We plan to calculate the 1-loop coefficients for c_B and c_E in the OK action. (Y.C. Jang)
- Highly optimized inverter using QUDA will be available soon. (Y.C. Jang)

κ Tuning

$N_f = 2+1+1$ MILC HISQ lattice

a (fm)	Volume	$\hat{m}'/m'_s$	$M_\pi L$	M_π (MeV)	N_{conf}
0.12	$24^3 \times 64$	1/5	4.54	305.3(4)	1040
	$24^3 \times 64$	1/10	3.22	218.1(4)	1020
	$32^3 \times 64$	1/10	4.29	216.9(2)	1000
	$40^3 \times 64$	1/10	5.36	217.0(2)	1028
	$48^3 \times 64$	1/27	3.88	131.7(1)	1000
0.09	$32^3 \times 96$	1/5	4.50	312.7(6)	1011
	$48^3 \times 96$	1/10	4.71	220.3(2)	1000
	$64^3 \times 96$	1/27	3.66	128.2(1)	1047
0.06	$48^3 \times 144$	1/5	4.51	319.3(5)	1016
	$64^3 \times 144$	1/10	4.30	229.2(4)	1246
	$96^3 \times 192$	1/27	3.69	135.5(2)	858
0.042	$64^3 \times 192$	1/5	4.35	309.3(9)	1133
	$144^3 \times 288$	1/27	4.17	134.2(2)	381
0.03	$96^3 \times 288$	1/5	4.84	308.7(1.2)	609

$|V_{cb}|$ from the exclusive decay $\bar{B} \to D^* \ell \bar{\nu}$

$$\frac{d\Gamma}{dw}(\bar{B} \to D^* l \bar{\nu}) = \frac{G_F^2 |V_{cb}|^2 M_B^5}{4\pi^3} r^{*3} (1 - r^*)^2 (w^2 - 1)^{\frac{1}{2}} \eta_C |\eta_{EW}|^2 \chi(w) |\mathcal{F}(w)|^2$$

- $w = v_B \cdot v_{D^*}$, $r^* = \frac{M_{D^*}}{M_B}$
- η_C: Coulomb attraction, $\eta_{EW} = 1.0066$: the one-loop electroweak correction
- $\chi(w)$: Phase-space factor
- $\mathcal{F}(w)$: Form factor ($\leftarrow$LATTICE)

Heavy quarks on the lattice: Fermilab method

The most updated version of V_{cb} calculation is done using the Fermilab action to control the c, b heavy quark discretization errors. It is generalized version of the Wilson clover action [El-Khadra, Kronfeld, and Mackenzie, PRD55, 3933 (1997)]

$$S_{\text{Fermilab}} = S_0 + S_E + S_B$$

$$S_0 = a^4 \sum_x \bar{\psi}(x)\Big[m_0 + \gamma_4 D_4 - \frac{a}{2}\Delta_4 + \zeta\Big(\boldsymbol{\gamma}\cdot\boldsymbol{D} - \frac{r_s a}{2}\Delta^{(3)}\Big)\Big]\psi(x)$$

$$S_E = -\frac{1}{2}c_E\zeta a^5 \sum_x \bar{\psi}(x)\boldsymbol{\alpha}\cdot\boldsymbol{E}\psi(x), \quad S_B = -\frac{1}{2}c_B\zeta a^5 \sum_x \bar{\psi}(x)i\boldsymbol{\Sigma}\cdot\boldsymbol{B}\psi(x).$$

- The Wilson term breaks the chiral symmetry explicitly, and the mass gets additive renormalization. $\rightarrow$ We tune κ and κ_{crit} to the physical quark.

$$am_0 = \frac{1}{2u_0}\left(\frac{1}{\kappa} - \frac{1}{\kappa_{\text{crit}}}\right)$$

Oktay-Kronfeld action

The OK action is an improved version of the Fermilab action such that the bilinear operators are tree-level matched to QCD through $\mathcal{O}(\lambda^3)$ in HQET power counting where $\lambda \sim a\Lambda \sim \Lambda/(2m_Q)$ [Oktay and Kronfeld, PRD78, 014504 (2008)]

$$S_{\mathrm{OK}} = S_{\mathrm{Fermilab}} + S_{\mathrm{new}}$$

$$S_{\mathrm{new}} = a^6 \sum_x \bar{\psi}(x)\Big[c_1 \sum_i \gamma_i D_i \Delta_i + c_2\{\boldsymbol{\gamma}\cdot\boldsymbol{D}, \Delta^{(3)}\} + c_3\{\boldsymbol{\gamma}\cdot\boldsymbol{D}, i\boldsymbol{\Sigma}\cdot\boldsymbol{B}\}$$
$$+ c_{EE}\{\gamma_4 D_4, \boldsymbol{\alpha}\cdot\boldsymbol{E}\} + c_4 \sum_i \Delta_i^2 + c_5 \sum_{i\neq j}\{i\Sigma_i B_i, \Delta_j\}\Big]\psi(x)$$

- The matching determines c_B, c_E, $c_{1,\cdots,5}$ and c_{EE} as a function of m_0. We have a tree-level value for the κ_{crit}

$$\kappa_{\mathrm{crit}}^{\mathrm{tree}} = [2u_0(1+3\zeta r_s + 18c_4)]^{-1} = 0.053850 \qquad (\zeta = r_s = 1)$$

where $u_0 = 0.86372$ for MILC HISQ lattice (a12m310, $24^3 \times 64$)

Fermilab method

We write non-relativistic dispersion relation,

$$E(\mathbf{p}) = M_1 + \frac{\mathbf{p}^2}{2M_2} - \frac{(\mathbf{p}^2)^2}{8M_4^3} - \frac{a^3 W_4}{6} \sum_i p_i^4 + \cdots .$$

- M_1: rest mass
- M_2: kinetic mass → Tuning to the physical mass
- M_4: quartic mass
- W_4: Lorentz symmetry breaking term

(Example) Tree-level relation between the bare quark mass m_0 and the kinetic quark mass m_2

$$\frac{1}{am_2} = \frac{2\zeta^2}{am_0(2 + am_0)} + \frac{r_s \zeta}{1 + am_0}$$

Nonperturbative determination of κ_{crit}

- $M_2(\kappa, \kappa_{\text{crit}})$: Light kinetic meson mass (600∼950 MeV)
- $m_2(\kappa, \kappa_{\text{crit}})$: kinetic quark mass

Let us suppose the meson mass relation

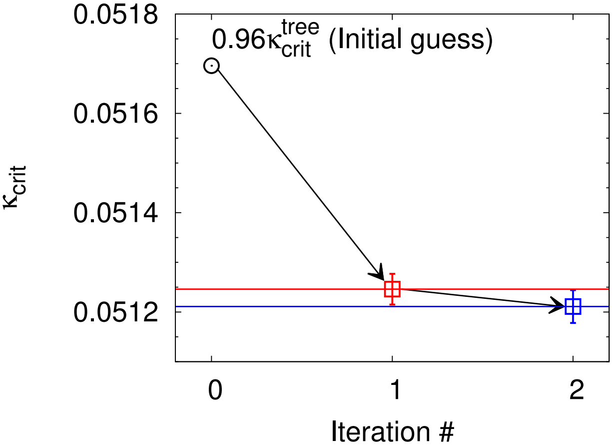
$$M_2^2 = A + Bm_2 + Cm_2^2.$$

The fitting using the true value of κ_{crit} will give $A = 0$. Note that the action depends on both κ and κ_{crit}. We determine κ_{crit} iteratively, as follows.

1. Start with an initial guess $\kappa'_{\text{crit}} = 0.96\kappa_{\text{crit}}^{\text{tree}}$
2. Determine the OK action coefficients using κ'_{crit}
3. Produce 2-pt pion correlators, and determine kinetic meson mass $M_2(\kappa, \kappa'_{\text{crit}})$ using various κ in the range (600∼950 MeV)
4. Find κ_{crit} such that fitting in terms of $m_2(\kappa, \kappa_{\text{crit}})$ gives $A = 0$.
5. Update $\kappa'_{\text{crit}} = \kappa_{\text{crit}}$ and go to the step 2.

Nonperturbative determination of κ_{crit}: result

$N_f = 2 + 1 + 1$ MILC HISQ ensemble (a12m310),
$N_{\text{conf}} = 130$, point source



$\kappa_{\text{crit}} = 0.051211(33)(4)$

κ tuning using D_s and B_s masses

- $M_2(\kappa, \kappa_{\text{crit}})$: Heavy-light meson mass
- $m_2(\kappa, \kappa_{\text{crit}})$: kinetic quark mass

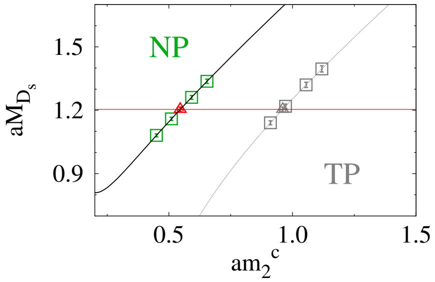
We use the HQET expansion of heavy-light meson masses M_2 as a fitting function:

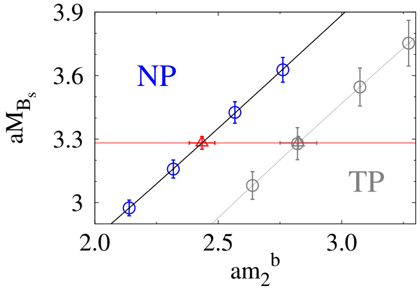
$$aM_2 = am_2 + d_0 + \frac{d_1}{am_2} + \frac{d_2}{(am_2)^2}.$$

1. Determine the OK action coefficients using charm and bottom type κ values with nonperturbative κ_{crit}.
2. Produce 2-pt B_s, D_s correlators, and determine $M_2(\kappa, \kappa_{\text{crit}})$
3. Determine the coefficients d_0, d_1 and d_2 using least-χ^2 fitting
4. Find m_2^{tuned} that gives the physical meson mass $M^{\text{Phys}} = M_2(m_2^{\text{tuned}})$.
5. obtain κ^{tuned} such that $m_2^{\text{tuned}} = m_2(m_0^{\text{tuned}})$ and $m_0^{\text{tuned}} = m_0(\kappa^{\text{tuned}}, \kappa_{\text{crit}})$.

κ tuning using D_s and B_s masses: results

- $N_f = 2 + 1 + 1$ MILC HISQ ensemble (a12m310)
- HISQ propagators ($am_s = 0.0509$) with point source
- OK propagators ($\kappa_{\text{crit}} = 0.051211$ and $\kappa_{\text{crit}}^{\text{tree}}$) with covariant Gaussian smearing.





$$\kappa_c = 0.048524(33)(43), \qquad \kappa_b = 0.04102(14)(9)$$

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