1 Internal Symmetries, Generators and Charges

In this section we discuss a generic, global, internal symmetry and derive several important quantities: the conserved currents associated to the symmetry and the corresponding charges that act as generators of the internal symmetry.

First of all, we recall the derivation of the Noether's theorem (which is a theorem valid for *classical* symmetries but can fail in relativistic quantum field theories because of the anomalies). Let us consider a Lagrangian which depend on a set of fields which belong to a representation (reducible or irreducible) of a given symmetry group

$$\mathcal{L} = \mathcal{L}\left(\phi^a, \partial_\mu \phi^a\right) \,. \tag{1}$$

Our Lagrangian is a function of some field ϕ^a, which has an internal index a, and its derivative. Note that fields with a different indices a are independent from each other. Thus we have

$$\begin{aligned} \left[\phi^a\left(\vec{x},t\right), \phi^b\left(\vec{y},t\right)\right] &= \left[\pi^a\left(\vec{x},t\right), \pi^b\left(\vec{y},t\right)\right] = 0 \\ \left[\phi^a\left(\vec{x},t\right), \pi^b\left(\vec{y},t\right)\right] &= i\delta^{ab}\delta^{(3)}\left(\vec{x}-\vec{y}\right) \,, \end{aligned} \tag{2}$$

with,

$$\pi^a\left(\vec{x},t\right) = \frac{\delta\mathcal{L}}{\delta(\partial_0\phi^a(\vec{x},t))} \,. \tag{3}$$

The classical equations of motion are

$$\partial_\mu\left[\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi^a)}\right] = \frac{\partial\mathcal{L}}{\partial\phi^a} \,. \tag{4}$$

Let us study the transformation properties of the Lagrangian under an infinitesimal rotation in the internal group space (for the moment we consider the rotation to be global)

$$\phi^{a\,\prime} = \phi^a + i\,\left(T^A\right)^a_b \alpha^A \phi^b + O\left(\alpha^2\right) \,. \tag{5}$$

We have

$$\begin{aligned}
\delta\mathcal{L} &= \frac{\delta\mathcal{L}}{\delta\left(\partial_\mu\phi^a\right)}\delta\left(\partial_\mu\phi^a\right)+\frac{\delta\mathcal{L}}{\delta\phi^a}\delta\phi^a \\
&= \frac{\delta\mathcal{L}}{\delta\left(\partial_\mu\phi^a\right)}\partial_\mu\left(\delta\phi^a\right)+\partial_\mu\left[\frac{\delta\mathcal{L}}{\delta\left(\partial_\mu\phi^a\right)}\right]\delta\phi^a \\
&= \partial_\mu\left[\frac{\partial\mathcal{L}}{\delta\left(\partial_\mu\phi^a\right)}\delta\phi^a\right]=\partial_\mu\left(i\left[\frac{\partial\mathcal{L}}{\delta\left(\partial_\mu\phi^a\right)}\right]\left(T^A\right)^a_b\,\alpha^A\phi^b\right). \qquad (6)
\end{aligned}$$

We now introduce a set of currents $J^{A\,\mu}$, one for any of the generators T^A of the transformation, defined as

$$J^{A\,\mu}=i\left[\frac{\partial\mathcal{L}}{\delta\left(\partial_\mu\phi^a\right)}\right]\left(T^A\right)^a_b\,\phi^b \qquad (7)$$

from which we can write

$$\left(\partial_\mu J^{A\,\mu}\right)\alpha^A=\delta\mathcal{L}\,, \qquad (8)$$

or

$$\left(\partial_\mu J^{A\,\mu}\right)=\frac{\delta\mathcal{L}}{\delta\alpha^A}\,. \qquad (9)$$

If the Lagrangian is invariant under the full set of the generators, $\delta\mathcal{L}=0$ whatever is the choice of α^A. Thus we can choose only a given α^A different from zero at a time and obtain, for the corresponding current, the continuity equation (one for any of the generators)

$$\partial_\mu J^{A\,\mu}=\frac{\partial}{\partial t}J^A_0+\vec{\nabla}\cdot\vec{J}^A=0\,. \qquad (10)$$

The above equation implies that the charge associated to J^A_μ is a constant of motion or, which is equivalent, that the charge commutes with the Hamiltonian $\mathcal{H}$

$$\frac{dQ^A\left(t\right)}{dt}=0 \quad \leftrightarrow \quad \left[\mathcal{H},Q^A\right]=0\,, \qquad (11)$$

where

$$\begin{aligned}
Q^A\left(t\right)=\int d^3x\,J^A_0\left(\overline{x},t\right) &= i\int d^3x\left[\frac{\delta\mathcal{L}}{\delta\left(\partial_0\phi^a\right)}\right]\left(T^A\right)^a_b\,\phi^b \\
&= i\int d^3x\,\pi_a\left(\vec{x},t\right)\left(T^A\right)^a_b\,\phi^b\,. \qquad (12)
\end{aligned}$$

The conservation of the charge has several implications. First it means that, given a certain rank of the group, you may classify the physical states as eigenstates of the energy-momentum, of the invariant mass and of as many internal quantum numbers as the rank of the group is. For example, if the symmetry is $SU(3)$ of flavour, corresponding to the up, down and strange quarks, which has rank two, we can classify the physical states on the basis of their isotopic spin and hypercharge (or electric charge). This includes obviously the single particles states. Moreover, in any physical process, since Q^A is conserved the charge of the initial states must be the same of the final ones.

We now compute the commutator of the charge with a field:

$$\begin{aligned}\left[Q^A(t), \phi^d(\vec{x},t)\right] &= i\int d^3y\, \left[\pi_a(\vec{y},t)\left(T^A\right)^a_b \phi^b(\vec{y},t), \phi^d(\vec{x},t)\right] \\ &= i\int d^3y\, \left(T^A\right)^a_b \pi_a(\vec{y},t)\left[\phi^b(\vec{y},t), \phi^d(\vec{x},t)\right] \\ &+ i\int d^3y\, \left(T^A\right)^a_b \left[\pi_a(\vec{y},t), \phi^d(\vec{x},t)\right]\phi^b(\vec{y},t)\ . \end{aligned} \tag{13}$$

Using eqs.(2) we then have

$$\left[Q^A(t), \phi^d(\vec{x},t)\right] = \left(T^A\right)^d_b \phi^b(\vec{x},t)\ , \tag{14}$$

which shows that we can write the transformation of the field in two completely equivalent ways

$$\phi^{a\,\prime} = \left(e^{iT^A\alpha^A}\right)^a_b \phi^b = \left(e^{iQ^A\alpha^A}\,\phi\, e^{-iQ^A\alpha^A}\right)^a\ , \tag{15}$$

where $e^{iT^A\alpha^A}$ is a matrix and ϕ a complex vector, whereas in the last term on the right hand side ϕ can be seen as a quantum field and $e^{iQ^A\alpha^A}$ is function of the quantum fields and their derivatives. To demonstrate this statement it is sufficient to consider an infinitesimal transformation

$$\begin{aligned}\phi^{a\,\prime}(\vec{x},t) &= \left(1+iQ^A\alpha^A\right)\phi^a(\vec{x},t)\left(1-iQ^A\alpha^A\right) \\ &\sim \phi^a(\vec{x},t) + i\,\alpha^A\,\left[Q^A(t), \phi^a(\vec{x},t)\right] + O(\alpha^2) \\ &= \phi^a(\vec{x},t) + i\,\alpha^A\,\left(T^A\right)^a_b \phi^b(\vec{x},t) + O(\alpha^2)\,. \end{aligned} \tag{16}$$

2 Complex Scalar Theory

The Lagrangian of a complex scalar field is:

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial_\mu \phi - m^2 \phi^\dagger \phi - \lambda \left(\phi^\dagger \phi\right)^2 \tag{17}$$

Obviously the canonical variables of this Lagrangian are ϕ and $\phi^\dagger$ and the kinetic term depends on their derivatives. The Lagrangian is invariant under the infinitesimal transformation

$$\phi' = \phi + i\alpha\,\phi\,, \quad \phi^{\dagger\prime} = \phi^\dagger - i\alpha\,\phi^\dagger\,. \tag{18}$$

Using the general Nöther theorem discussed in the previous section we obtain the current

$$\begin{aligned} J^\mu &= i\left(\left[\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\right]\phi - \left[\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^\dagger}\right]\phi^\dagger\right) \\ &= i\left(\partial_\mu\phi^\dagger\phi - \phi^\dagger\partial_\mu\phi\right)\,. \end{aligned} \tag{19}$$

This recalls something that you have already seen in non relativistic quantum mechanics where the three dimensional current and density probability are defined as

$$\vec{J} = -i\left(\phi^*\vec{\nabla}\phi - \phi\vec{\nabla}\phi^*\right)\,, \quad \rho = \phi^*\phi \tag{20}$$

and the continuity equation is given by

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0\,. \tag{21}$$

2.1 Dirac Theory and Scalar Electro-dynamics

The Lagrangian is given by

$$\begin{aligned} \mathcal{L} &= \bar{\psi}(i\,\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &= \bar{\psi}(i\,\not{\partial} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + q\,J^\mu A_\mu \end{aligned} \tag{22}$$

where

$$J^\mu = \bar{\psi}\gamma^\mu\psi\,, \tag{23}$$

and q denotes the charge of the Dirac particle. As seen in the previous lectures, the Feynman rules for the vertex is $-i\,q\,\gamma^{\mu}_{\alpha\beta}$ where α and β corresponds to the spinor index of the outgoing or incoming particle respectively.

In the case of a scalar charged field, the electromagnetic Lagrangian is written as

$$\mathcal{L} = (D_\mu\phi)^\dagger(D^\mu\phi) - m^2\phi^\dagger\phi - \lambda\left(\phi^\dagger\phi\right)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad (24)$$

from which we can derive the conserved electromagnetic current that, by gauge invariance, must have the form

$$J^\mu_{\text{scalar}} = i\left[(D_\mu\phi)^\dagger\,\phi - \phi^\dagger\,(D_\mu\phi)\right]\,, \qquad (25)$$

By gauge invariance the coupling between the scalar field and the vector potential also contains a quadratic term in A_μ corresponding to the Feynman rules

$$V_\mu = i\,q\,(p_\mu + p'_\mu)\,, \quad V_{\mu\nu} = 2\,q^2\,g_{\mu\nu}\,. \qquad (26)$$

Exercise 1

Given the QCD Lagrangian:

$$\begin{aligned}
\mathcal{L}_{QCD} &= i\left(\bar{u}\,\not{D}u + \bar{d}\,\not{D}d + \bar{s}\,\not{D}s\right) - m_u\,\bar{u}u - m_d\,\bar{d}d - m_s\,\bar{s}s - \frac{1}{4}\sum_A G^A_{\mu\nu}G^{A\,\mu\nu} \quad A = 1,\ldots,8\,; \\
&= i\sum_f \bar{q}^f\,\not{D}q_f - \sum_{f,f'}\bar{q}^f M_f^{f'}q_{f'} - \frac{1}{4}\sum_A G^A_{\mu\nu}G^{A\,\mu\nu} \\
&= i\bar{q}\,\not{D}q - \bar{q}\hat{M}q - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} \quad f, f' = 1,\ldots,\text{number of light flavours}
\end{aligned}$$

and the transformation rules

$$q'_f = U_f{}^i\,q_i \qquad \bar{q}'^{\,f} = \bar{q}^i\,\left(U^\dagger\right)_i{}^f \qquad U_f{}^i = \left(e^{i\alpha^A\lambda^A}\right)_f{}^i \quad \left(U^\dagger\right)_i{}^f = \left(e^{-i\alpha^A\lambda^A}\right)_i{}^f\,,$$

$$q'_f = \left(e^{i\beta^A\lambda^A\gamma_5}\right)_f{}^i\,q_i \qquad \bar{q}'^{\,f} = \bar{q}^i\left(e^{i\beta^A\lambda^A\gamma_5}\right)_i{}^f$$

with

$$\begin{aligned}
\bar{q}\hat{M}q &= \frac{m_u + m_d + m_s}{3}\left(\bar{u}u + \bar{d}d + \bar{s}s\right) \\
&+ \frac{m_u - m_d}{2}\left(\bar{u}u - \bar{d}d\right) + \frac{m_u + m_d - 2\,m_s}{6}\left(\bar{u}u + \bar{d}d - 2\,\bar{s}s\right) \\
&= \bar{m}\,\bar{q}\hat{I}q + \Delta m_{ud}\bar{q}\lambda^3 q + \sqrt{\frac{1}{3}}\Delta m_{uds}\bar{q}\lambda^8 q \\
\Delta m_{ud} &= m_u - m_d \quad \Delta m_{uds} = m_u + m_d - 2\,m_s\,,
\end{aligned}$$

catalogue the symmetries of the theory in the massless limit and compute the Ward ids for the currents,

$$\left(\partial_\mu J^{A\,\mu}\right) = \frac{\delta\mathcal{L}}{\delta\alpha^A}\,. \tag{27}$$

with $m_u \neq m_d \neq m_s$.

Exercise 2

Given the baryon and antibaryon SU(3) matrices in the adjoint representation ($\Sigma m = \Sigma^-$, $\Xi 0 = \Xi^0$, $pd = \bar{p}$, $\Lambda 0d = \bar{\Lambda}^0$, etc.) , fig. 1 and fig. 2, show that one can write the baryon matrix B as $\sum_A B^A\lambda^A$, where the λ^A are the Gell-Mann matrices, and compute the coefficients B^A. Compute the commutator, $[B, \lambda^8]$ and the anti-commutator $\{B, \lambda^8\}$.

xForm=

$$\begin{pmatrix} \frac{\Lambda 0}{\sqrt{6}} + \frac{\Sigma 0}{\sqrt{2}} & \Sigma p & p \\ \Sigma m & \frac{\Lambda 0}{\sqrt{6}} - \frac{\Sigma 0}{\sqrt{2}} & n \\ \Xi m & \Xi 0 & -\sqrt{\frac{2}{3}}\,\Lambda 0 \end{pmatrix}$$

Figura 1: baryon octet

xForm=

$$\begin{pmatrix} \frac{\Lambda 0d}{\sqrt{6}} + \frac{\Sigma 0d}{\sqrt{2}} & \Sigma md & \Xi md \\ \Sigma pd & \frac{\Lambda 0d}{\sqrt{6}} - \frac{\Sigma 0d}{\sqrt{2}} & \Xi 0d \\ pd & nd & -\sqrt{\frac{2}{3}}\,\Lambda 0d \end{pmatrix}$$

Figura 2: anti-baryon octet

Exercise 3

Given the Fermi Hamiltonian

$$\mathcal{H}_F = -\frac{G_F}{\sqrt{2}} \left(\bar{\nu}_\mu \gamma^\rho \left(1-\gamma_5\right) \mu\right) \left(\bar{e} \gamma_\rho \left(1-\gamma_5\right) \nu_e\right) , \tag{28}$$

prove the identity with the Fiery rearranged Hamiltonian

$$\mathcal{H}_F = -\frac{G_F}{\sqrt{2}} \left(\bar{e} \gamma^\rho \left(1-\gamma_5\right) \mu\right) \left(\bar{\nu}_\mu \gamma_\rho \left(1-\gamma_5\right) \nu_e\right) . \tag{29}$$

Exercise 4

Compute in dimensional regularisation the one loop electromagnetic corrections to the Hamiltonian in eq. (29) and show that there is no logarithmic divergence. You may also compute the corrections in the W-regulrisation and show that there is no $\log[M_W^2]$.

Exercise 5

Draw all the diagrams, virtual and real photon emission) contributing to the $O(\alpha_{em})$ corrections to the semileptonic rate. Draw the diagrams for the decays $K \to \mu \nu_\mu \ell^+ \ell^-$.