Electromagnetic Corrections to Decay Amplitudes:
Real Emissions and Semileptonic Decays -3

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Frontiers in Lattice QCD
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PLAN OF THE LECTURES

Leptonic decays e.g. $\pi^+ \to \mu^+ \nu_\mu (\gamma)$ *& heavier*

Real Emissions (RM123 method)

Infinite volume reconstruction method (hints
(see Xu Feng @ lattice 2019)

Semileptonic decays

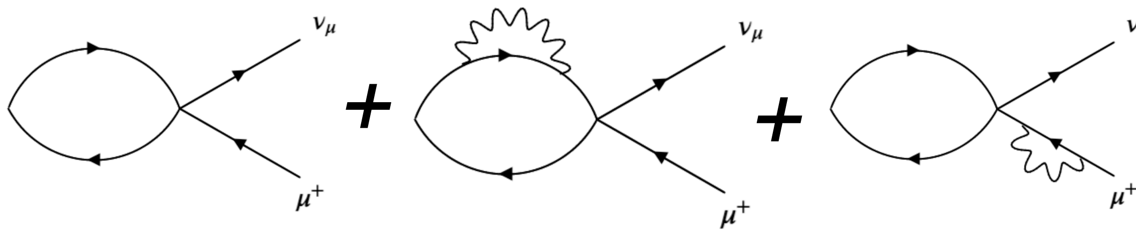
Conclusion & Outlook

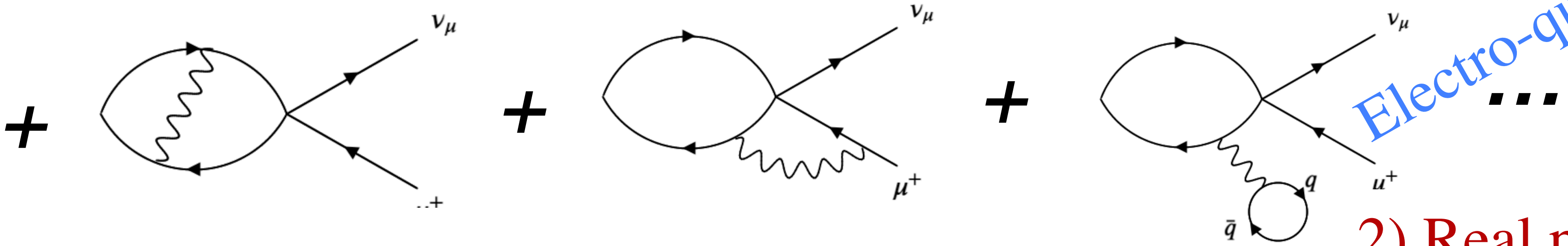
Rate at O(α)

$$\Gamma(\Delta E) = \int_0^{\Delta E} dE_\gamma \frac{d\Gamma}{dE_\gamma} = \Gamma_0 + \Gamma_1(\Delta E)$$

1) Virtual photons

$$\Gamma(\Delta E) = \int d\Omega_2 \, \Big|$$



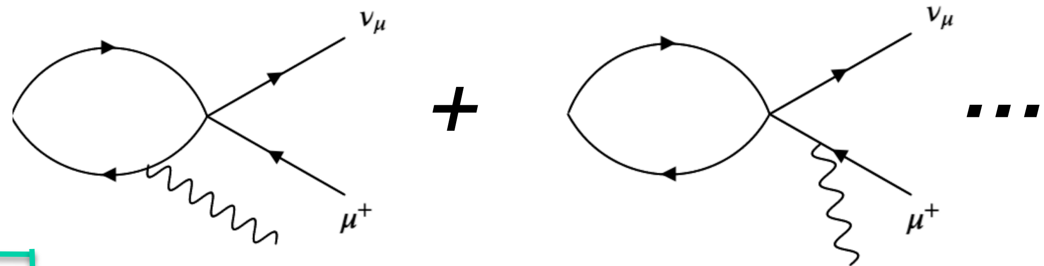


Electro-quenched

$$\Big|^2$$

2) Real photons

$$+ \int_{E_\gamma \leq \Delta E} d\Omega_3 \, \Big|$$



$$\Big|^2$$

$d\Omega_{2,3} = 2-3$ body phase $-$ space

The Infrared Problem

$$\Gamma(\Delta E) = \Gamma_0 + \Gamma_1(\Delta E)$$

QED Corrections to Hadronic Processes in Lattice QCD,
N.Carrasco, V.Lubicz, G.Martinelli, C.T.Sachrajda, N.Tantalo, C.Tarantino and M.Testa,
Phys. Rev. D **91** (2015) no.7, 07450 [arXiv:1502.00257 [hep-lat]].

Finite-Volume QED Corrections to Decay Amplitudes in Lattice QCD,
V.Lubicz, G.Martinelli, C.T.Sachrajda, F.Sanfilippo, S.Simula and N.Tantalo,
Phys. Rev. D **95** (2017) no.3, 034504 [arXiv:1611.08497 [hep-lat]].

First Lattice Calculation of the QED Corrections to Leptonic Decay Rates,
D.Giusti, V.Lubicz, G.Martinelli, C.T.Sachrajda, F.Sanfilippo, S.Simula, N.Tantalo and C.Tarantino,
Phys. Rev. Lett. **120** (2018) 072001 [arXiv:1711:06537]

Light-meson leptonic decay rates in lattice QCD+QED
M.Di Carlo, D.Giusti, V.Lubicz, G.Martinelli, C.T.Sachrajda, F.Sanfilippo, S.Simula and N.Tantalo,
arXiv:1904.08731.

MASTER FORMULA for the rate at O(α)

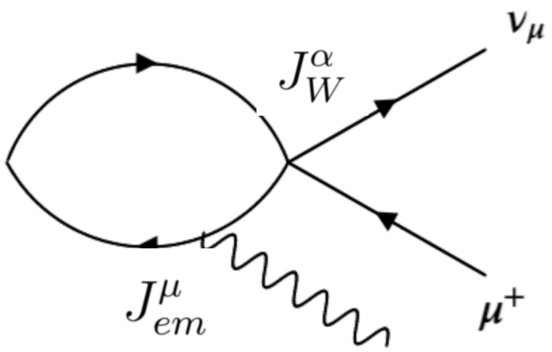
$$\Gamma(\Delta E) = \lim_{V\to\infty}(\Gamma_0 - \Gamma_0^{\text{pt}}) + \lim_{V\to\infty}(\Gamma_0^{\text{pt}} + \Gamma_1(\Delta E))$$

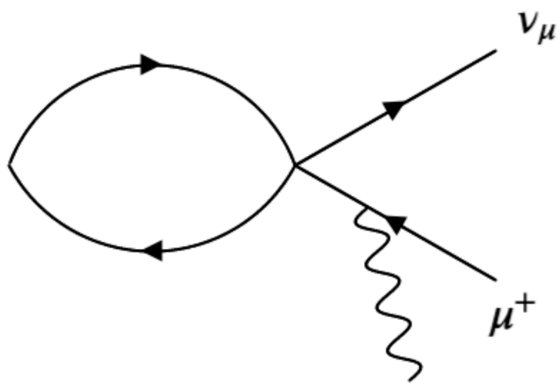
pt = point-like & perturbative

- the infrared divergences in Γ_0 and Γ_0^{pt} are universal and cancel in the difference
- $\Gamma(\Delta E) = \Gamma_0^{pt} + \Gamma_1(\Delta E)$ is infrared finite since is a physical, well defined quantity *F. Bloch, A. Nordsieck Phys.Rev. 52 (1937) T.D. Lee, M. Nauenberg Phys.Rev. 133 (1964)*
- the infrared divergences in $\Delta\Gamma_0(L) = \Gamma_0 - \Gamma_0^{pt}$ and
- $\Gamma(\Delta E) = \Gamma_0^{pt} + \Gamma_1(\Delta E)$ cancel separately hence they can be regulated with different infrared cutoff
- Γ_0 and Γ_0^{pt} are also ultraviolet finite

We now discuss the non-perturbative determination of $\Gamma_1(\Delta E)$

Real photons





- the leptonic part factorizes;
- the photon is not really there, we just give a pinch that carries away some momentum at the vertex where the (conserved) electromagnetic current is inserted;
- therefore finite volume effects are exponentially suppressed.

- can be computed in perturbation theory;
- it has no relation with the non perturbative structure of the hadron.

The relevant hadronic amplitude (T-product)

$$H_W^{\alpha\, r}(k,p) = \varepsilon_\mu^r(k)\, H_W^{\alpha\,\mu}(k,p) = \varepsilon_\mu^r(k) \int d^4y\, e^{ik\cdot y}\, \langle 0| T\{ j_W^\alpha(0) j_{em}^\mu(y)\} |P(p)\rangle$$

- $j_{em}^\mu(y)$ electromagnetic current
- $j_W^\alpha(0)$ weak current
- $P(p)$ pseudoscalar meson with momentum p
- $\epsilon_\mu^r(k)$ polarisation vector of the photon with momentum *k*

Decomposition of the amplitude in form-factors

$$
\begin{aligned}
H_W^{\alpha\mu}(k,p) &= H_1\left[k^2 g^{\alpha\mu} - k^\alpha k^\mu\right] + H_2\left[\left(p\cdot k - k^2\right)k^\mu - k^2\left(p-k\right)^\mu\right]\left(p-k\right)^\alpha \\
&- i\frac{F_V}{m_P}\epsilon^{\mu\alpha\gamma\beta}k_\gamma p_\beta + \frac{F_A}{m_P}\left[\left(p\cdot k - k^2\right)g^{\alpha\mu} - k^\alpha\left(p-k\right)^\mu\right] \\
&+ f_P\left[g^{\alpha\mu} + \frac{(2p-k)^\mu\,(p-k)^\alpha}{2p\cdot k - k^2}\right]
\end{aligned}
$$

The last term is dictated from the Ward id

$$k_\mu H_W^{\alpha\mu}(k,p) = i\langle 0|j_W^\alpha|P(p)\rangle = f_P\,p^\alpha$$

it is the same of a point-like scalar particle

it is related to the i.r. divergence

- 4 independent form-factors;
- the form factors only depend on k^2 e p.k;
- we will only discuss the real photon case, $k^2 = 0$ $\varepsilon^r.k = 0$;
- in the future it may be interesting to consider $\Gamma^{exp}(K \to \ell\nu_\ell\, l^+ l^-)$

Amplitude: decomposition in scalar form-factors

$$H_W^{\alpha\mu}(k,p) = H_1\left[k^2 g^{\alpha\mu} - k^\alpha k^\mu\right] + H_2\left[\left(p\cdot k - k^2\right)k^\mu - k^2\left(p-k\right)^\mu\right]\left(p-k\right)^\alpha$$
$$- i\frac{F_V}{m_P}\epsilon^{\mu\alpha\gamma\beta}k_\gamma p_\beta + \frac{F_A}{m_P}\left[\left(p\cdot k - k^2\right)g^{\alpha\mu} - k^\alpha\left(p-k\right)^\mu\right]$$
$$+ f_P\left[g^{\alpha\mu} + \frac{\left(2p-k\right)^\mu\left(p-k\right)^\alpha}{2p\cdot k - k^2}\right]$$

Real Photon Emission

$$H_W^{\alpha\mu}(k,p) = -i\frac{F_V}{m_P}\epsilon^{\mu\alpha\gamma\beta}k_\gamma p_\beta + \left(\frac{F_A}{m_P} + \frac{f_P}{p\cdot k}\right)\left(p\cdot k\, g^{\alpha\mu} - k^\alpha p^\mu\right) + \frac{f_P}{p\cdot k}p^\alpha p^\mu$$

1. different tensors can be separated by using suitable projectors, we then have to disentangle F_A from f_P ;
2. the point-like term is related to the infrared divergence in 1/p.k that has to cancel the corresponding divergence of the virtual correction in the rate;

It is F_A (together with F_V) the relevant structure dependent quantity to be determined.

Lattice Calculation (Euclidean)

Relevant Correlator

$E_\gamma = |\mathbf{k}|$

$$C_W^{\alpha r}(t,\mathbf{p},\mathbf{k}) = \epsilon_\mu^r(\mathbf{k}) \int d^4y\, d^3x\, e^{t_y E_\gamma - i\mathbf{k}\cdot\mathbf{y} + i\mathbf{p}\cdot\mathbf{x}}\, \mathrm{T}\langle 0|j_W^\alpha(t) j_{em}^\mu(y) P(0,\mathbf{x})|0\rangle$$

suitable projector on the different form factors

Euclidean extraction of the photon momentum k

source of the pseudoscalar meson with momentum p

the convergence of the integral over t_y is ensured by the safe analytic continuation from Minkowsky to Euclidean, namely by the absence of intermediate states lighter than the pseudoscalar meson. The physical form factors can be extracted directly from the Euclidean correlation functions

$$\sum_n \int_{-\infty}^0 dt\, e^{tE_\gamma} \langle 0|j_W^\alpha(0)|n\rangle\langle n|e^{tH} j_{em}^\mu(0,\mathbf{k}) e^{-tH}|P\rangle \;\to e^{t(E_\gamma+E_n-E_P)}$$

$$E_\gamma + E_n > E_P$$

Define

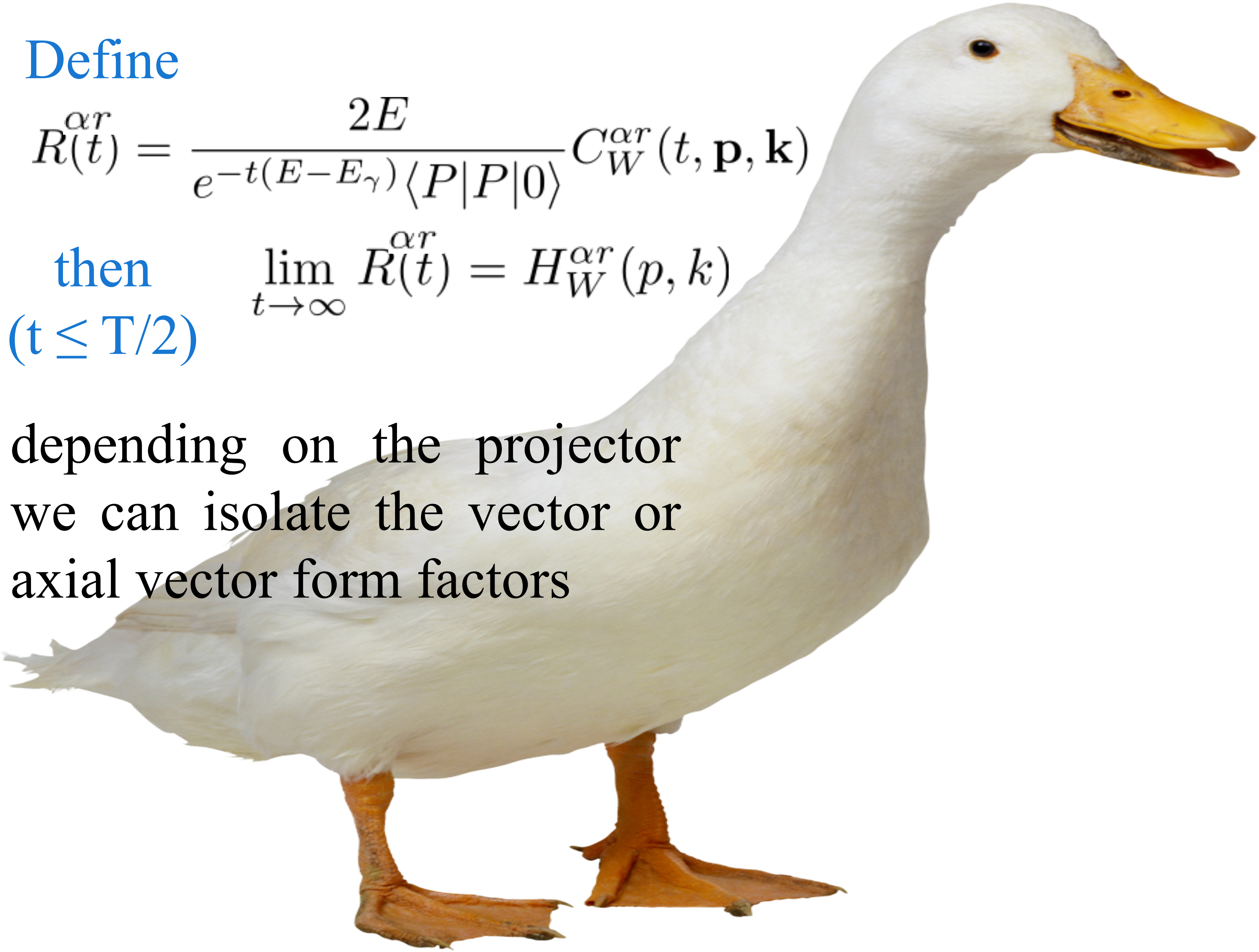
$$R^{\alpha r}(t) = \frac{2E}{e^{-t(E-E_\gamma)}\langle P|P|0\rangle} C_W^{\alpha r}(t, \mathbf{p}, \mathbf{k})$$

then

(t ≤ T/2)

$$\lim_{t\to\infty} R^{\alpha r}(t) = H_W^{\alpha r}(p, k)$$

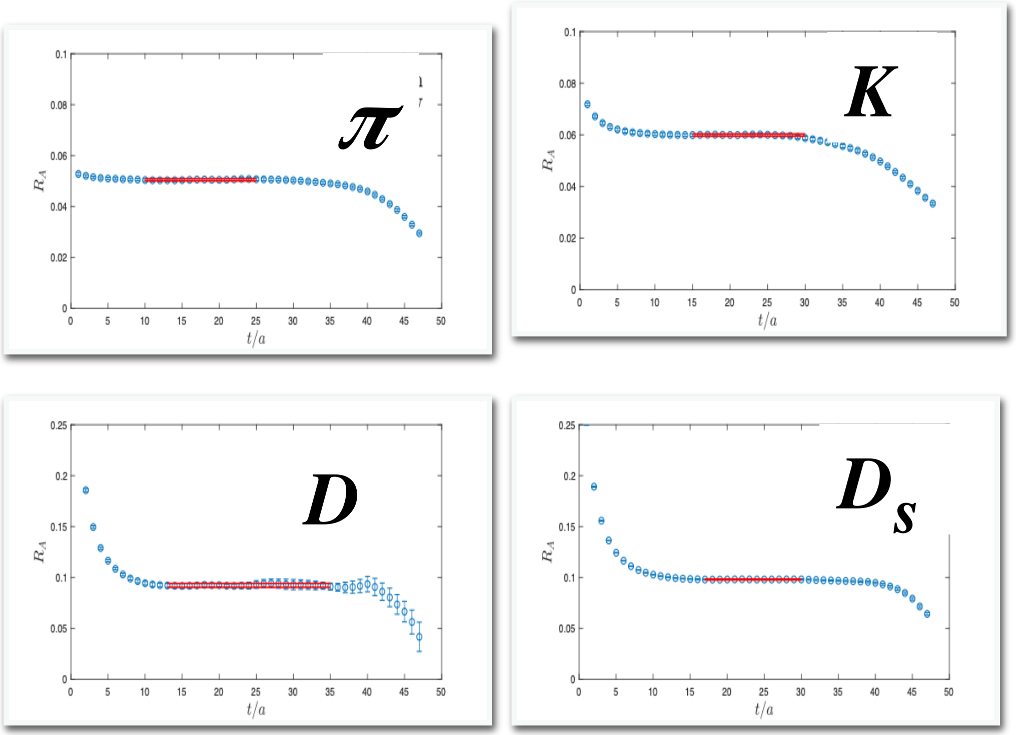
depending on the projector we can isolate the vector or axial vector form factors



Numerical Results

(all the results in the following are preliminary)

- *2+1+1 twisted mass fermions*
- *Three different lattice spacings*
- *3-4 different pion masses for each lattice spacing*
- *Pion masses down to ~230 MeV*
- *100 different momentum configurations*

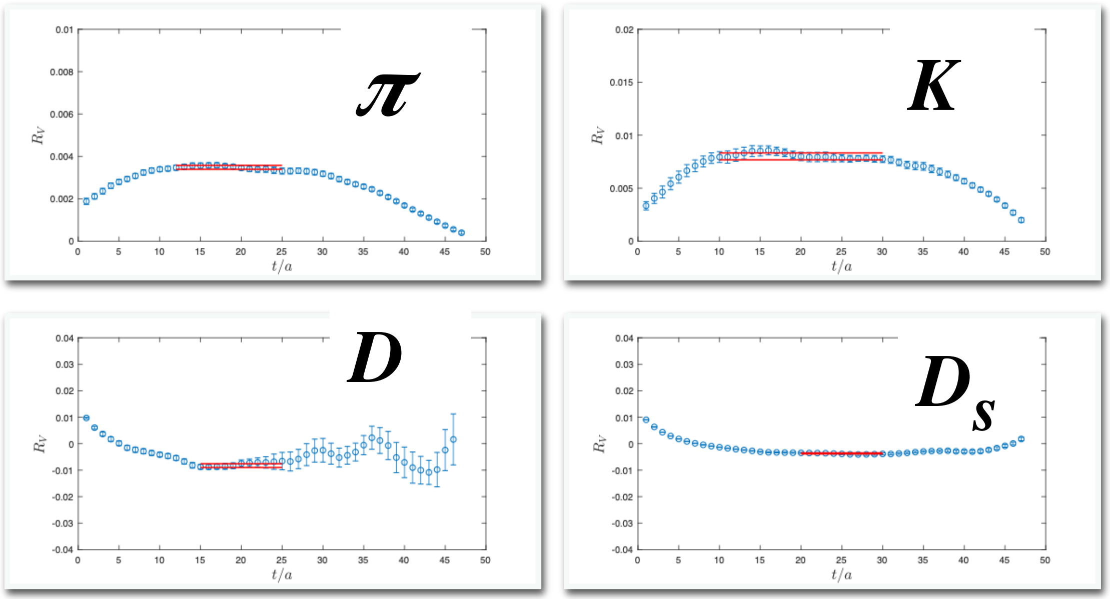


for the plateaus we find rather good signals

$$\lim_{t\to\infty} R^{\alpha r}(t) = H_W^{\alpha r}(p,k)$$

Axial vectors

Vectors



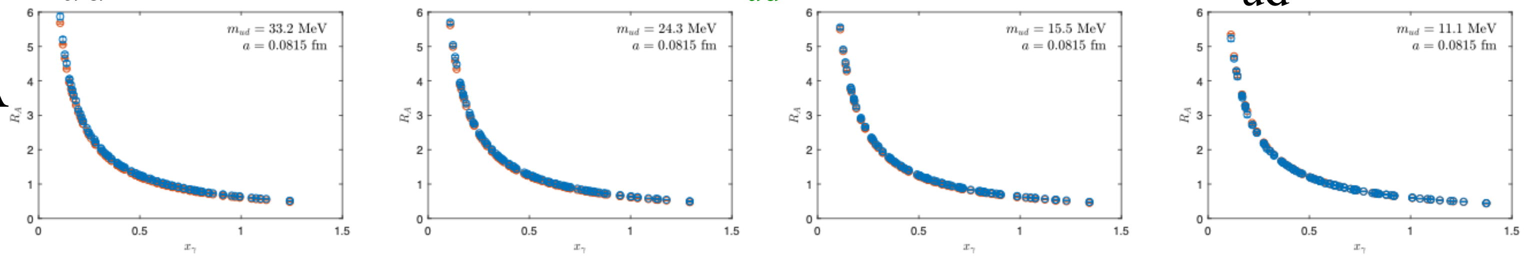
the plateaux correspond to m_{ud} ~ 24 MeV and a ~ 0.08 fm found in the other panels. The masses of K, D, Ds are practically the physical once since the masses of the charm and the strange quarks have been tuned using them as. The pion is, of course, heavier and $m\pi$ ~ 360 MeV

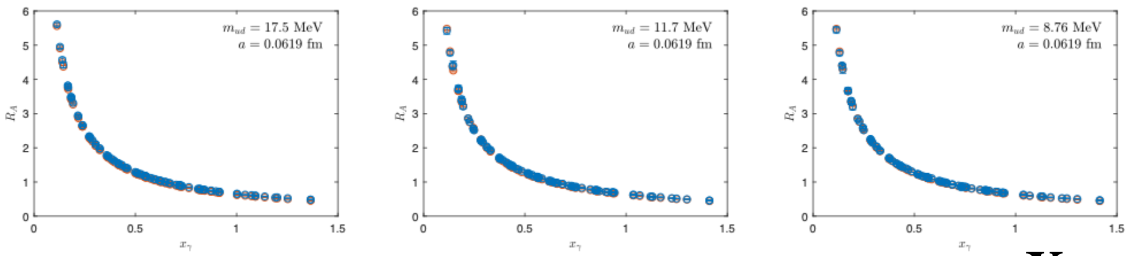
Axial-vector channel for the Kaon

$$R_A = \frac{F_A}{m_P} + \frac{f_P}{p \cdot k}$$

m_{ud} = 33.2 MeV -> decreasing m_{ud} @ a=0.0815 fm m_{ud} = 11.1 MeV

R_A





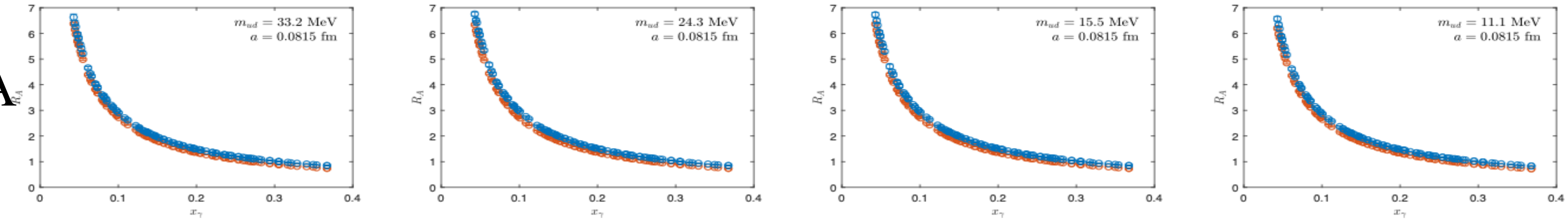
a=0.0619 fm

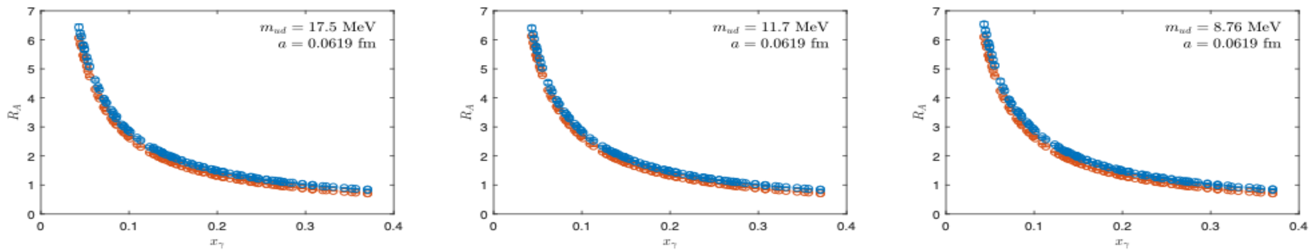
x_γ

- dominated by the point-like vertex
- no important discretization errors

Axial-vector channel for the D_s

R_A



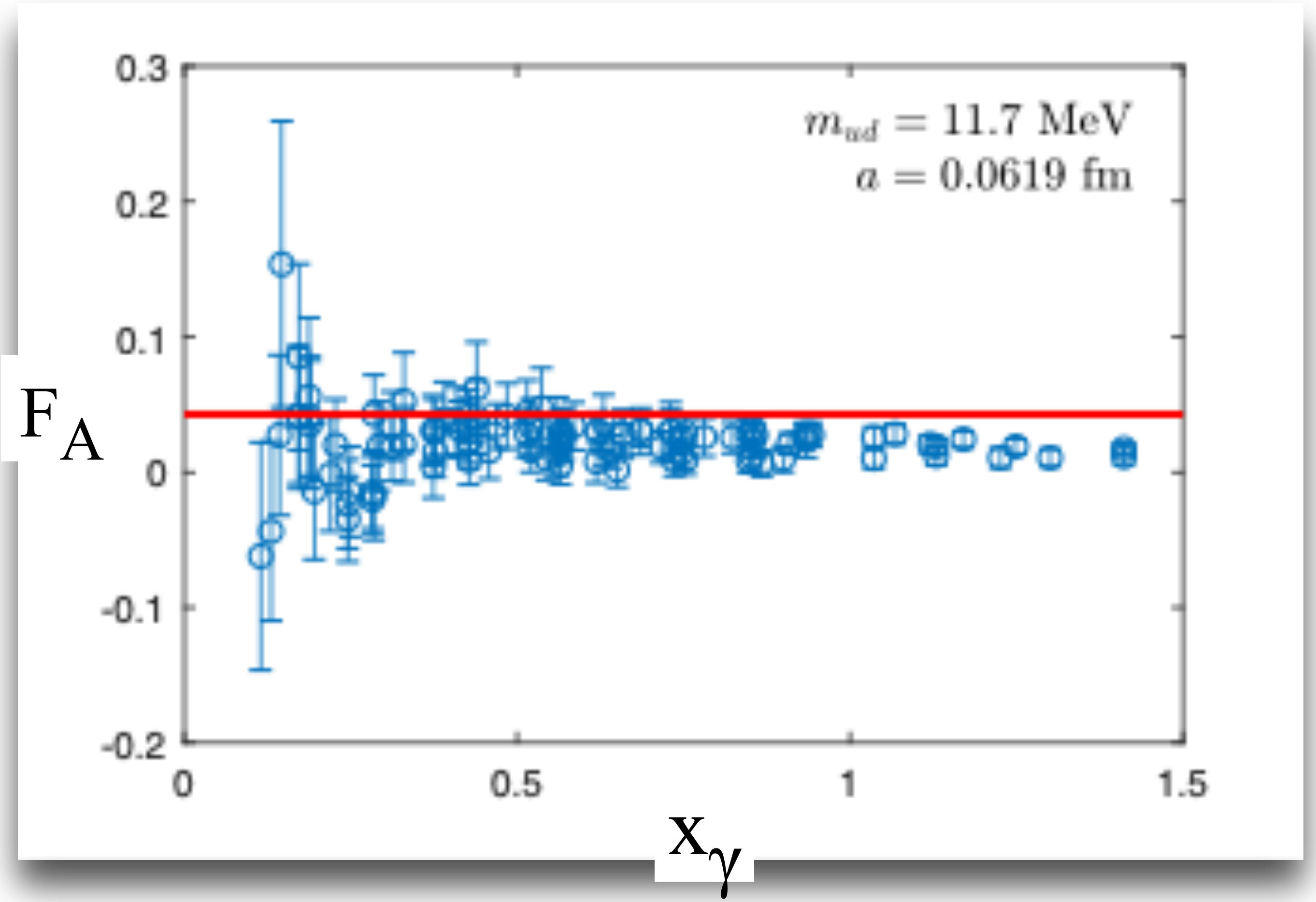


x_γ

$$x_\gamma = \frac{2p \cdot k}{m_P^2}$$

we may compare with χPT that at $O(p^4)$ gives without any momentum dependence

$$F_A = \frac{8 m_K}{f_\pi} \left(L_9^r + L_{10}^r \right)$$

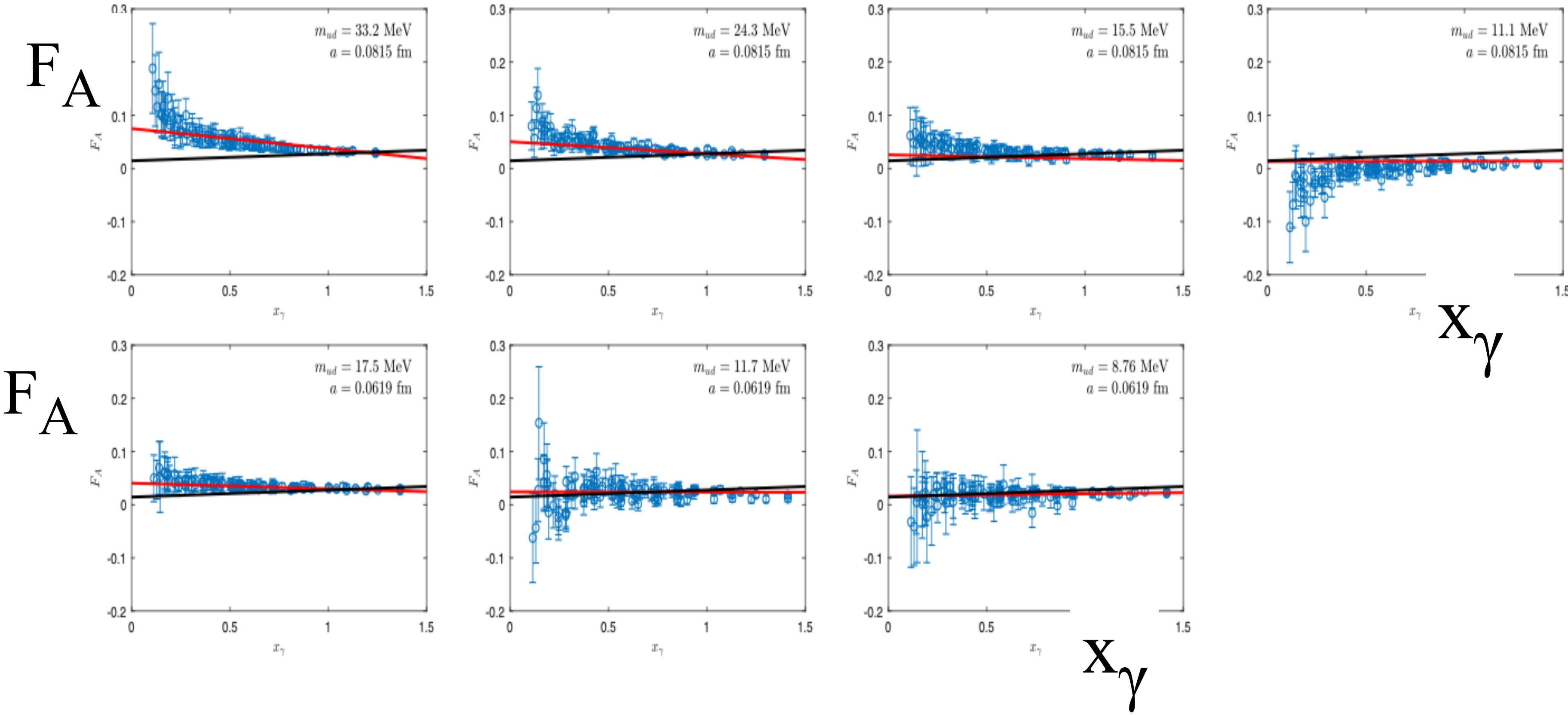


Kaon F_A form factor

with an improved analysis we will be able to extract the momentum dependence too

Extrapolation to the physical point F_A for the Kaon

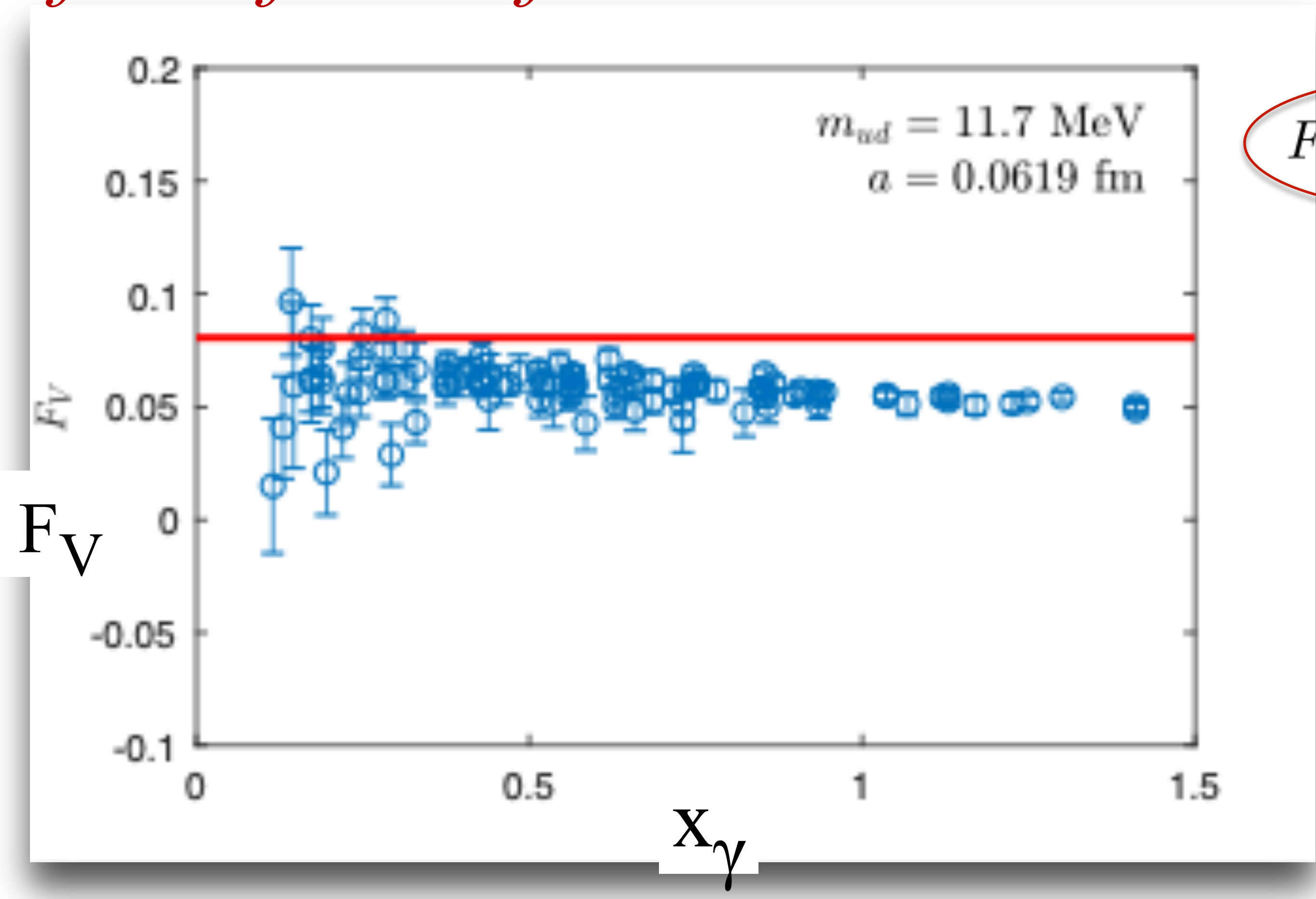
after the subtraction of the point-like axial vertex we obtain the relevant structure dependent form factor and its momentum dependence.



$$F_{(A,V)}(x_\gamma, m_{ud}, a^2, L) = c_0 + c_1\, x_\gamma + d_0\, m_{ud} + d_1 m_{ud} x_\gamma + e_0\, a^2 + e_1\, a^2 x_\gamma + \dots$$

F_V form factor for the Kaon

χPT



$$F_V = \frac{m_K}{4\pi^2 f_\pi}$$

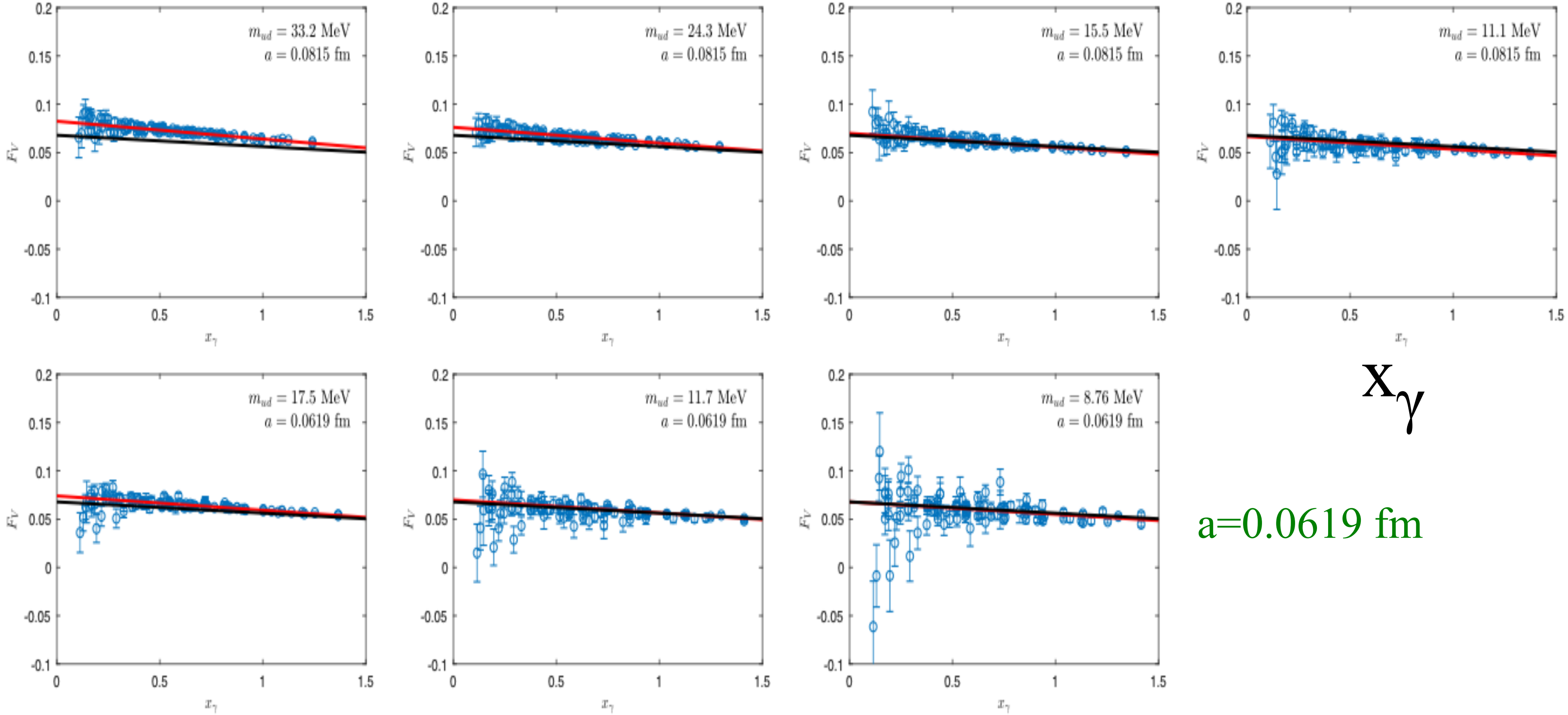
with an improved analysis we will be able to extract the momentum dependence too

Similar results were found for the other mesons π, D and D_s

F_V form factor for the Kaon

m_{ud} = 33.2 MeV -> decreasing m_{ud} @ a=0.0815 fm

m_{ud} = 11.1 MeV



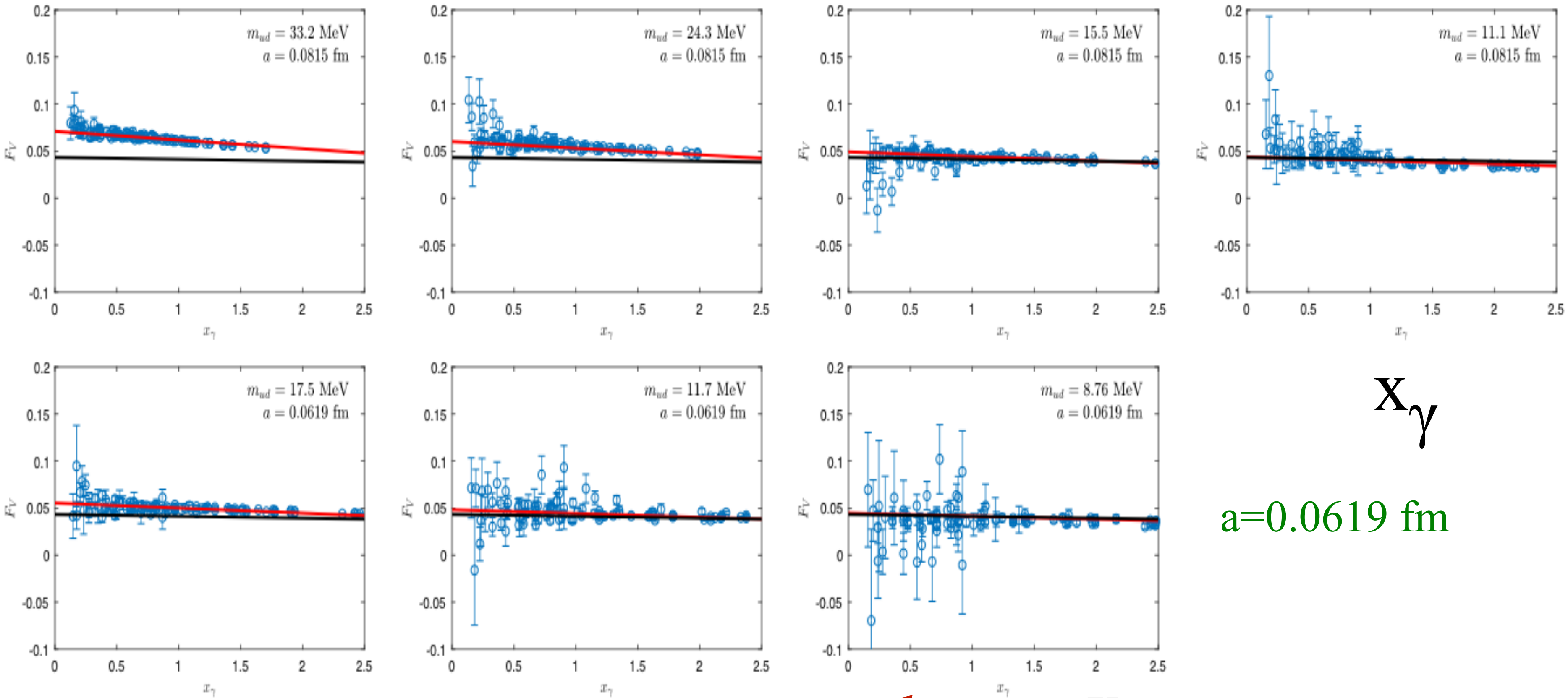
x_γ

a=0.0619 fm

x_γ

F_V form factor for the π

m_{ud} = 33.2 MeV -> decreasing m_{ud} a=0.0815 fm m_{ud} = 11.1 MeV



x_γ

a=0.0619 fm

x_γ

low mass & small lattice spacing

OUTLOOK

Present:

Full lattice calculations of radiative corrections to leptonic decays are possible

The form factors for real emissions are accessible from Euclidean correlators

Our preliminary results are very encouraging, still more work is needed and the analysis is ongoing

Future:

B-mesons are also very interesting, we expect a dynamical enhancement, but they are more difficult on current lattices

E&M corrections to $\pi^- \to \mu^- \bar{\nu}$ using infinite volume reconstruction method

Norman Christ, Xu Feng*, Luchang Jin & Chris Sachrajda

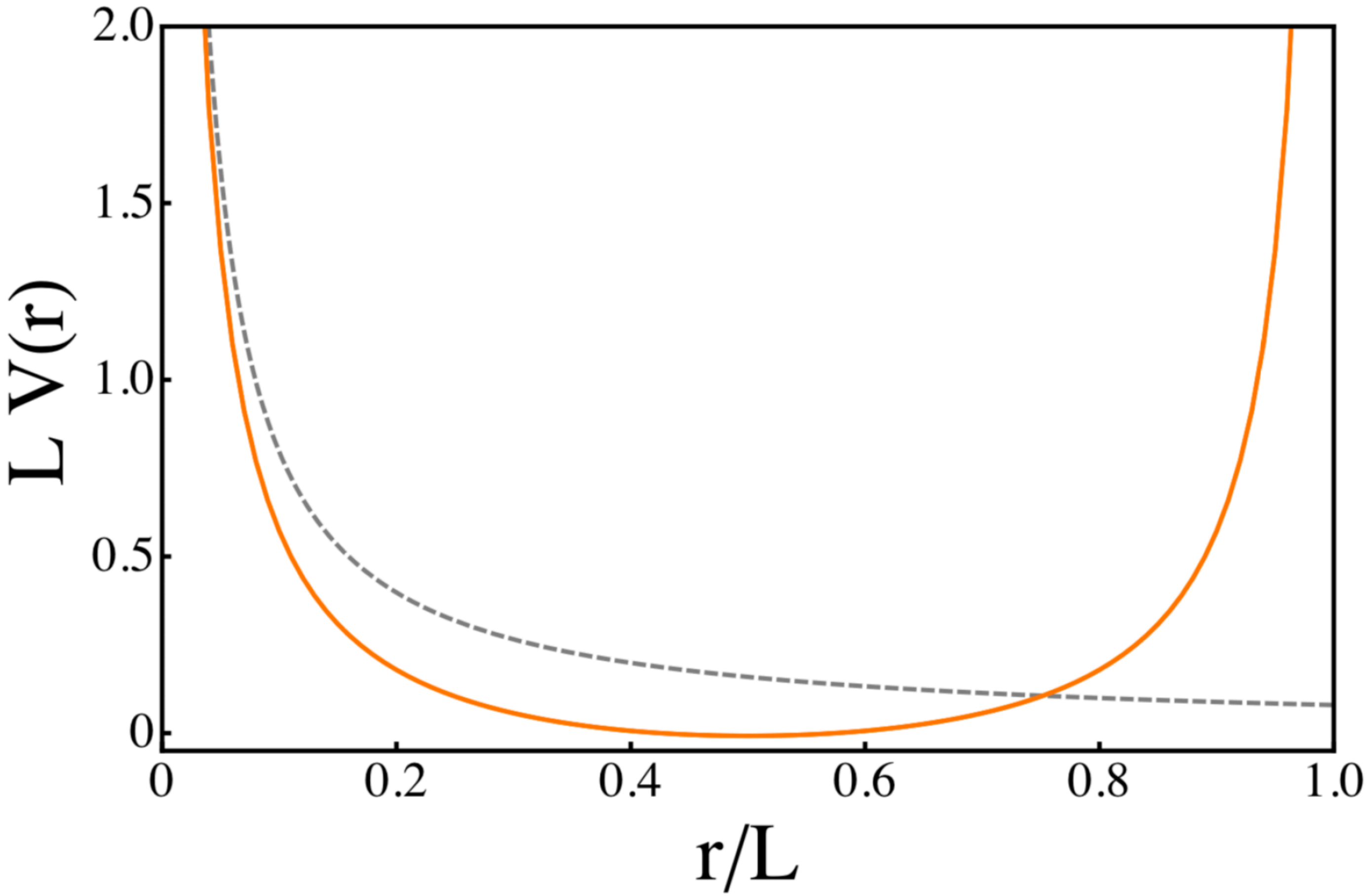
(RBC and UKQCD collaborations)

Lattice 2019 @ Wuhan, 06/19/2019

Remove zero mode - $\mathrm{QED_L}$

Infinite volume propagator $\Rightarrow$ finite-volume propagator

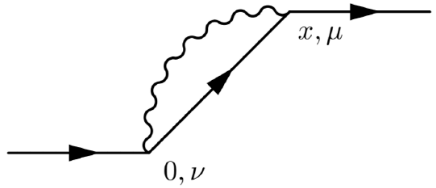
$$S_\infty(x) = \int \frac{d^4p}{(2\pi)^4}\frac{e^{ipx}}{p^2} = \frac{1}{4\pi^2x^2} \quad \Rightarrow \quad S_L(x) = \frac{1}{VT}\sum_p{}' \frac{e^{ipx}}{p^2}, \quad p = \frac{2\pi}{L}n \neq 0$$



[Davoudi, Savage, PRD90 (2014) 054503]

XF, Luchang Jin [arXiv:1812.09817]

QED self energy



- We start with infinite volume [QED$_\infty$ method, used in HVP & HLbL]

$$\mathcal{I} = \frac{1}{2} \int d^4x \, \mathcal{H}_{\mu,\nu}(x) S^{\gamma}_{\mu,\nu}(x)$$

- The hadronic part $\mathcal{H}_{\mu,\nu}(x)$ is given by

$$\mathcal{H}_{\mu,\nu}(x) = \mathcal{H}_{\mu,\nu}(t,\vec{x}) = \langle N | T[J_\mu(x) J_\nu(0)] | N \rangle$$

  - $\langle N | J_\mu(t,\vec{x}) \quad \rightarrow \quad e^{Mt}$
  - $J_\mu(t,\vec{x}) J_\nu(0) \quad \rightarrow \quad e^{-M\sqrt{t^2+\vec{x}}}$

For small $|t|$, we have exponentially suppressed FV effects:

$$\mathcal{H}_{\mu,\nu}(t,\vec{x}) \sim e^{-M\left(\sqrt{t^2+\vec{x}^2}-t\right)} \sim e^{-M|\vec{x}|} \sim e^{-ML}$$

For large $|t|$, we shall have:

$$\mathcal{H}_{\mu,\nu}(t,\vec{x}) \sim e^{-M\left(\sqrt{t^2+\vec{x}^2}-t\right)} \sim e^{-M\frac{\vec{x}^2}{2t}} \sim O(1)$$

Exponentially suppressed FV effects

Realizing at large $t > t_s$ we have ground state dominance:

- Reconstruct $\mathcal{H}_{\mu,\nu}(t,\vec{x})$ at large t using $\mathcal{H}_{\mu,\nu}(t_s,\vec{x})$ at modest t_s

$$\mathcal{H}_{\mu,\nu}(t,\vec{x}') \approx \int d^3\vec{x}\,\mathcal{H}_{\mu,\nu}(t_s,\vec{x}) \int \frac{d^3\vec{p}}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} e^{-(E_{\vec{p}}-M)(t-t_s)} e^{-i\vec{p}\cdot\vec{x}'}$$

We then split the integral $\mathcal{I}$ into two parts

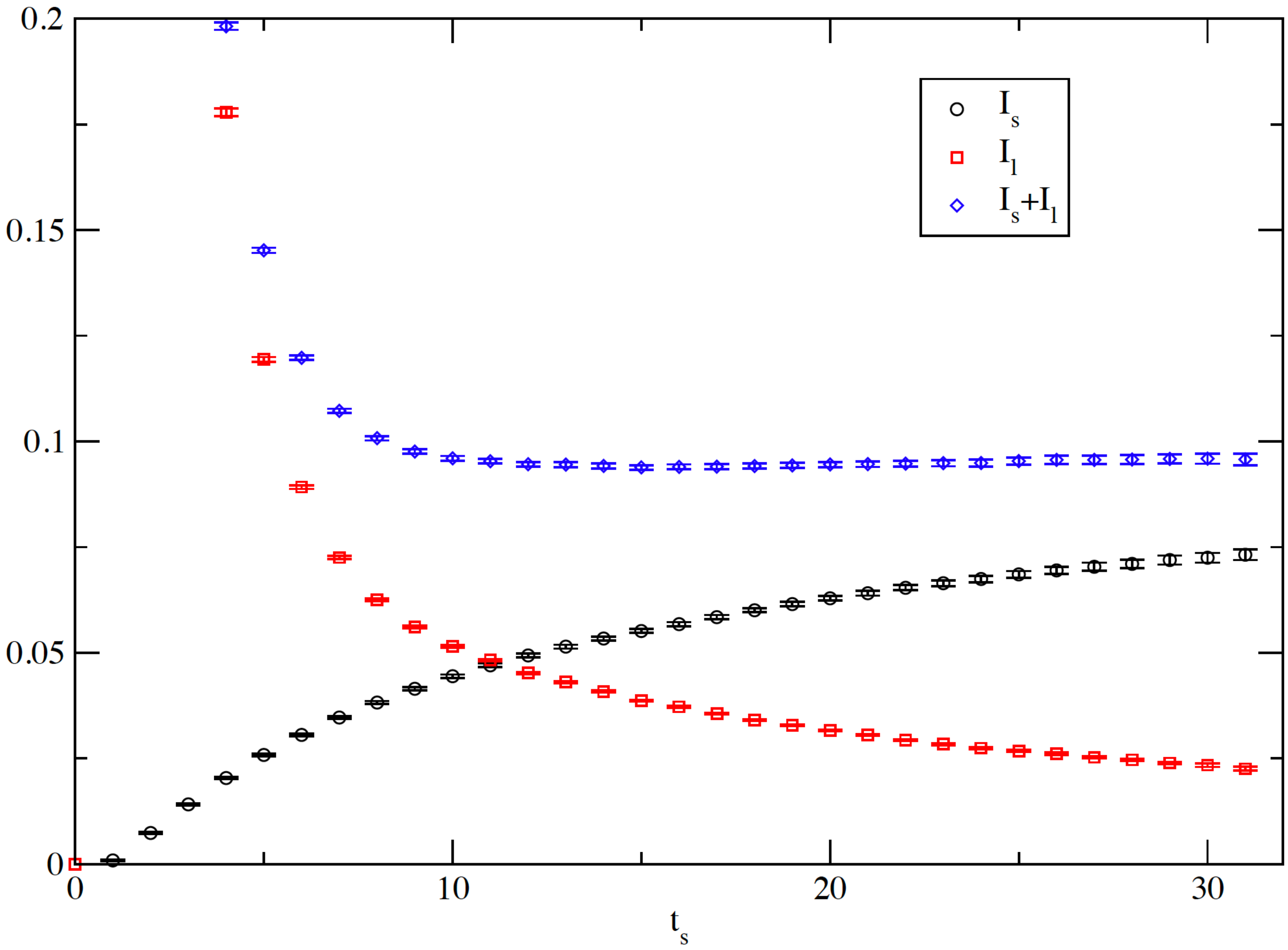
$$\begin{aligned}
\mathcal{I} &= \mathcal{I}^{(s)} + \mathcal{I}^{(l)} \\
\mathcal{I}^{(s)} &= \frac{1}{2}\int_{-t_s}^{t_s}\int d^3\vec{x}\,\mathcal{H}_{\mu,\nu}(x) S^{\gamma}_{\mu,\nu}(x) \\
\mathcal{I}^{(l)} &= \int_{t_s}^{\infty}\int d^3\vec{x}\,\mathcal{H}_{\mu,\nu}(x) S^{\gamma}_{\mu,\nu}(x) \\
&= \int d^3\vec{x}\,\mathcal{H}_{\mu,\nu}(t_s,\vec{x}) L_{\mu,\nu}(t_s,\vec{x})
\end{aligned}$$

At $t \le t_s$,

$$\mathcal{H}_{\mu,\nu}(t,\vec{x}) \quad \Rightarrow \quad \mathcal{H}^{L}_{\mu,\nu}(t,\vec{x})$$

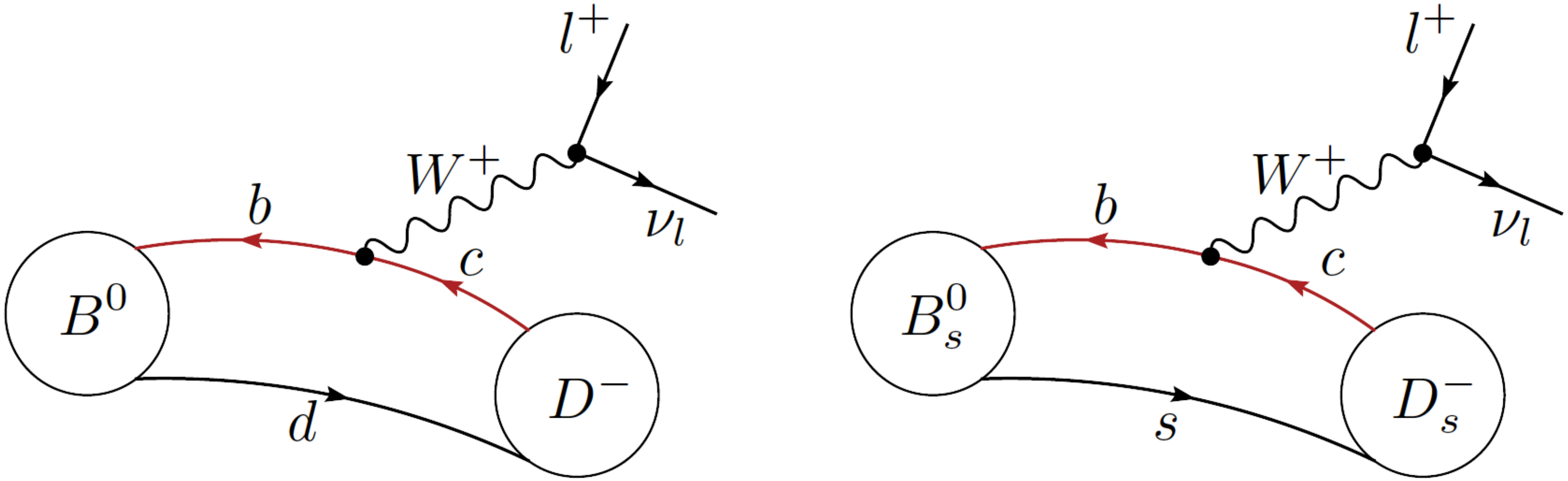
The replacement only amounts for exponentially suppressed FV effects

t_s dependence



SEMILEPTONIC DECAYS

B semileptonic decay: $|V_{cb}|$

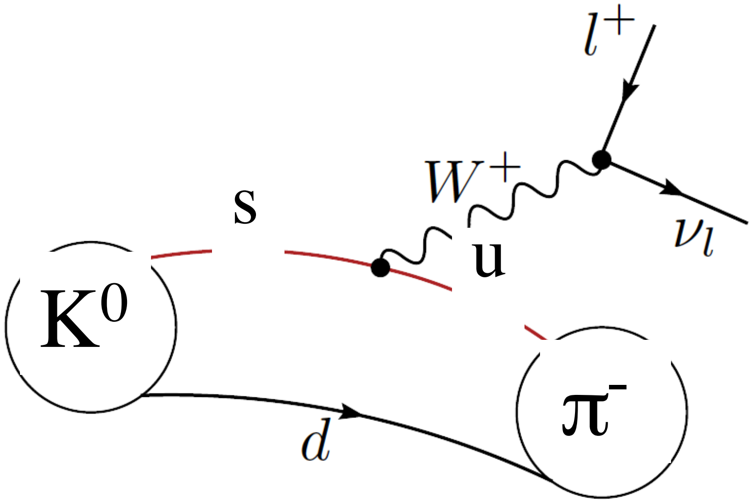


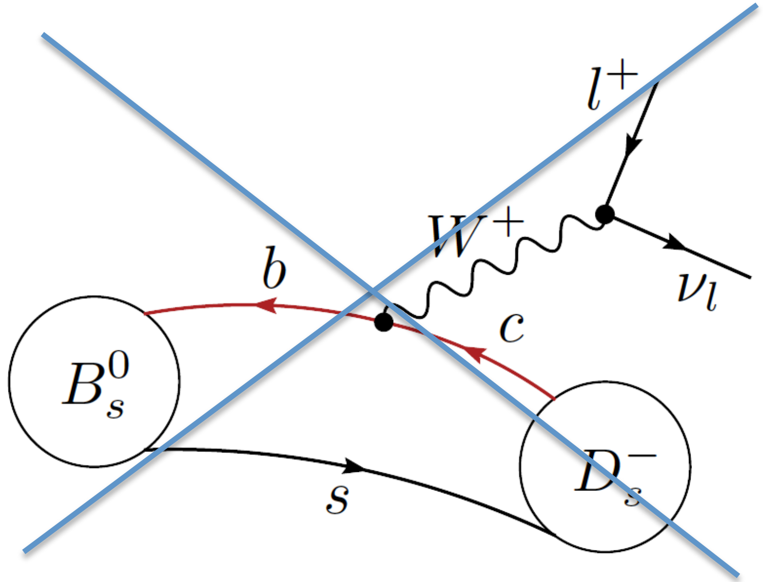
$$\frac{d\Gamma(B_{(s)} \to P l \nu)}{dq^2} = \frac{G_F^2 |V_{cb}|^2}{24\pi^3} \frac{(q^2 - m_l^2)^2 \sqrt{E_P^2 - m_P^2}}{q^4 m_{B_{(s)}}^2} \left[\left(1 + \frac{m_l^2}{2q^2}\right) m_{B_{(s)}}^2 (E_P^2 - m_P^2) |f_+(q^2)|^2 + \frac{3m_l^2}{8q^2} (m_{B_{(s)}}^2 - m_P^2)^2 |f_0(q^2)|^2 \right]$$

e, μ suppressed

uncertainties from kinematical factors / neglected h.o. OPE at the permille level

K semileptonic decay: $|V_{us}|$



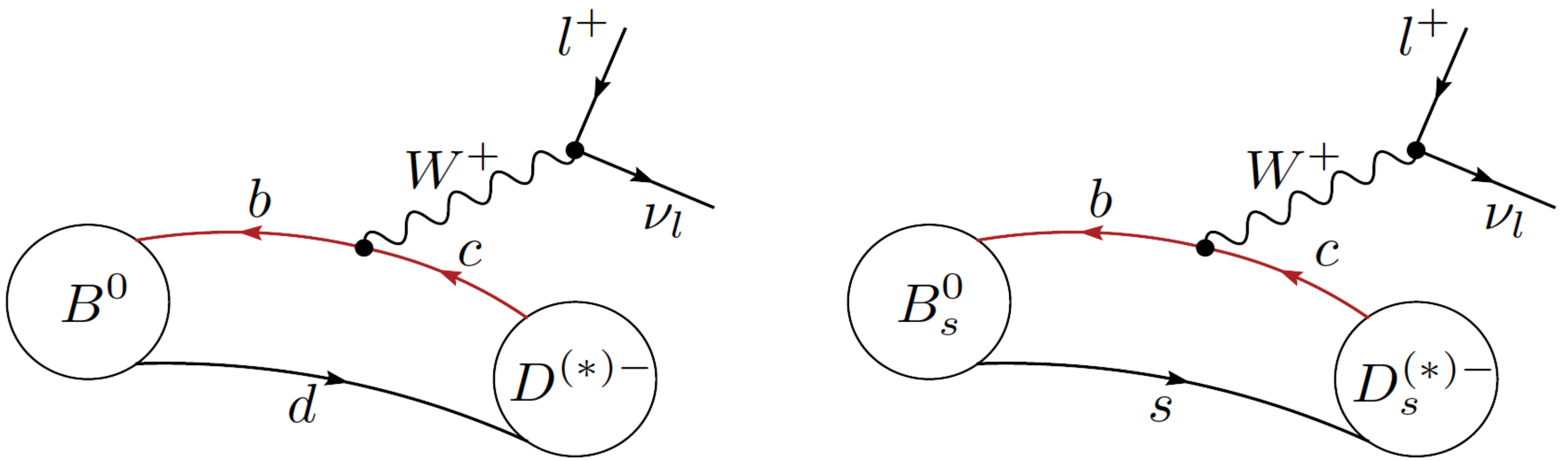


$$\frac{\mathrm{d}\Gamma(B_{(s)} \to P l \nu)}{\mathrm{d}q^2} = \frac{G_F^2 |V_{cb}|^2}{24\pi^3} \frac{(q^2 - m_l^2)^2 \sqrt{E_P^2 - m_P^2}}{q^4 m_{B_{(s)}}^2} \left[\left(1 + \frac{m_l^2}{2q^2}\right) m_{B_{(s)}}^2 (E_P^2 - m_P^2) |f_+(q^2)|^2 + \frac{3m_l^2}{8q^2} (m_{B_{(s)}}^2 - m_P^2)^2 |f_0(q^2)|^2 \right]$$

e,μ suppressed

uncertainties from kinematical factors / neglected h.o. OPE at the permille level

B semileptonic decay: $|V_{cb}|$



$$\frac{d\Gamma(B \to D l \nu_l)}{dw} = \frac{G_F^2}{48\pi^3}(m_B + m_D)^2 (w^2-1)^{3/2} |\eta_{EW}|^2 \, |V_{cb}|^2 \, |\mathcal{G}(w)|^2 \; + \; \mathcal{O}\left(\frac{m_l^2}{q^2}\right)$$

$$\frac{d\Gamma(B \to D^* l \nu_l)}{dw} = \frac{G_F^2}{4\pi^3}(m_B - m_{D^*})^2 (w^2-1)^{1/2} |\eta_{EW}|^2 \chi(w) \, |V_{cb}|^2 \, |\mathcal{F}(w)|^2 \; + \; \mathcal{O}\left(\frac{m_l^2}{q^2}\right)$$

$$w = \frac{p_B \cdot p_{D^{(*)}}}{m_B m_{D^{(*)}}} \qquad \mathcal{G}(w) = \frac{4\frac{m_D}{m_B}}{1+\frac{m_D}{m_B}} f_+(q^2) \quad \text{etc}$$

Low recoil region (w=1) accessible to lattice calculations

Basic Formulae for Semileptonic Decays

Kinematics, the importante variables are:

$$q^2 = (p_K - p_\pi)^2 = (p_\ell + p_{\nu_\ell})^2 = m_K^2 + m_\pi^2 - 2p_K \cdot p_\pi \qquad s_{\ell\pi} = (p_\ell + p_\pi)^2 = m_\ell^2 + m_\pi^2 + 2p_\ell \cdot p_\pi$$

Helicity basis:

$$\langle\, \pi(p_\pi)\,|\bar{s}\gamma_\mu u\,|\,K(p_K)\,\rangle = f_0(q^2)\,\frac{m_K^2 - m_\pi^2}{q^2} q_\mu + f_+(q^2)\,\left[(p_\pi + p_K)_\mu - \frac{m_K^2 - m_\pi^2}{q^2} q_\mu\right]$$

coupled to scalar states

coupled to vector states

$$\frac{d^2\Gamma}{dq^2\, ds_{\ell\pi}} = \frac{G_F^2}{512\pi^3 m_K^3}\left\{a_+\, f_+^2(q^2) + a_0\, f_0^2(q^2) + a_{+0} f_0 f_+\right\}$$

In the limit $m_l \to 0$ only the first term survives

A nightmare:

We write $a_+ = a_{+0} + a_{+1} m_\ell^2 + a_{+2} m_\ell^4$.

$$a_{+0} = -16\left[m_K^2(m_\pi^2 - s_{\ell\pi}) + s_{\ell\pi}(s_{\ell\pi} + q^2 - m_\pi^2)\right] \quad (36)$$

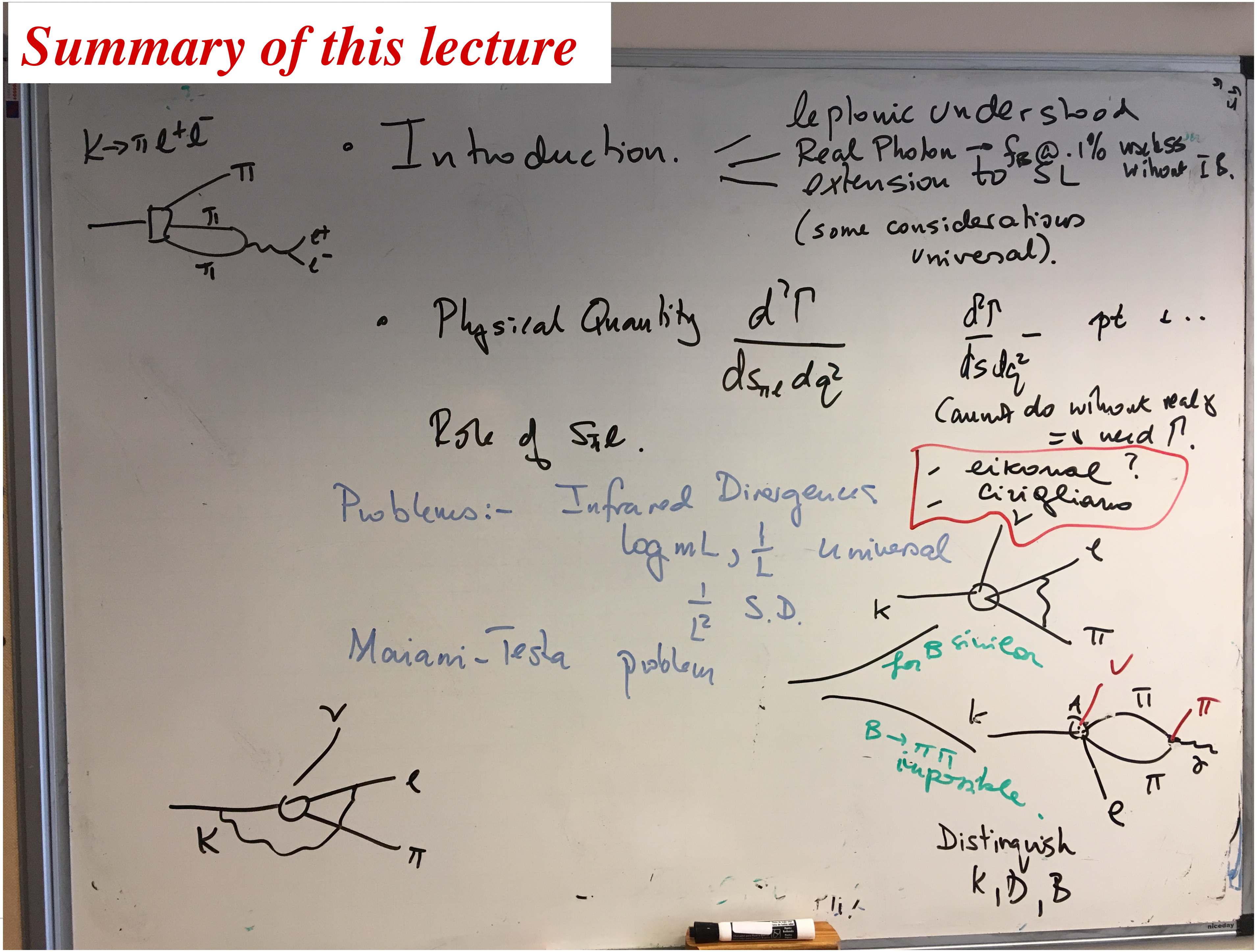
$$\begin{aligned} a_{+1} &= 4q^2 + 16 s_{\ell\pi} - 2\times 4\left(\frac{m_K^2 - m_\pi^2}{q^2}\right)(2m_K^2 - q^2 - 2s_{\ell\pi}) + 4q^2\left(\frac{m_K^2 - m_\pi^2}{q^2}\right)^2 \\ &= -\frac{4}{q^2}\left[3m_K^4 + (m_\pi^2 - q^2)(4s_{\ell\pi} + q^2 - m_\pi^2) - 2m_K^2(m_\pi^2 + q^2 + 2s_{\ell\pi})\right] \end{aligned} \quad (37)$$

$$\begin{aligned} a_{+2} &= -4 - 2\times 4\left(\frac{m_K^2 - m_\pi^2}{q^2}\right) - 4\left(\frac{m_K^2 - m_\pi^2}{q^2}\right)^2 \\ &= -4\left(\frac{m_K^2 - m_\pi^2 + q^2}{q^2}\right)^2 . \end{aligned} \quad (38)$$

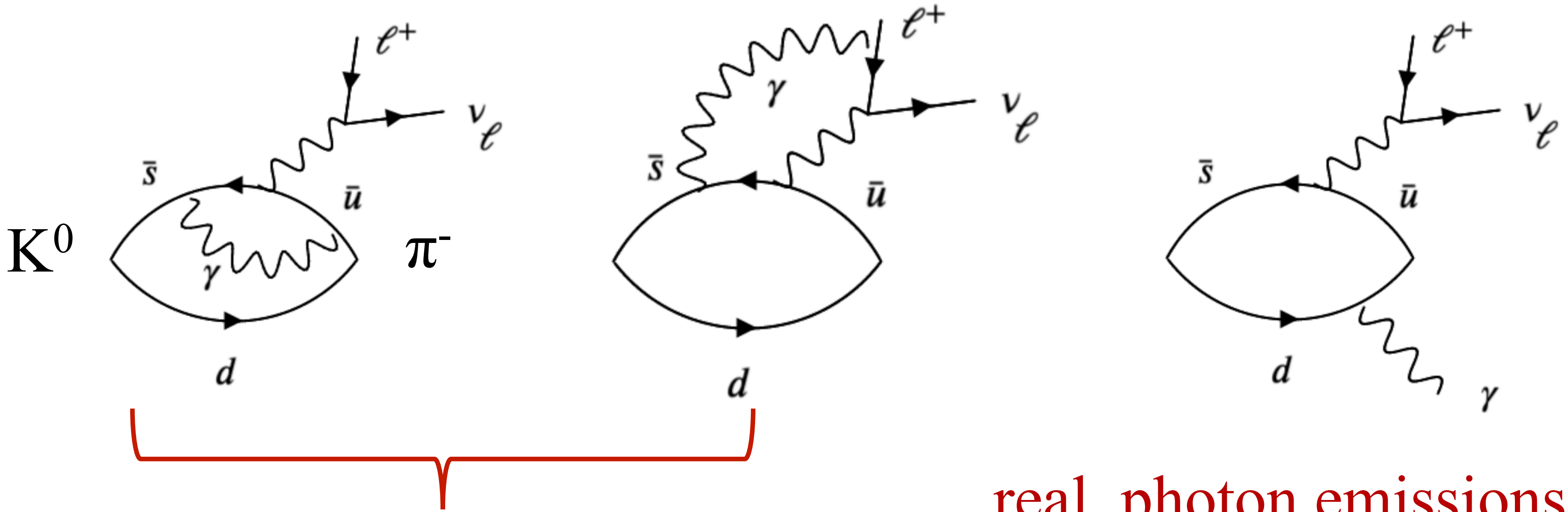
$$a_0 = N_{qq}\frac{(m_K^2 - m_\pi^2)^2}{q^4} = \frac{4m_\ell^2(q^2 - m_\ell^2)(m_K^2 - m_\pi^2)^2}{q^4} .$$

$$\begin{aligned} a_{+0} &= 2\,\frac{(m_K^2 - m_\pi^2)}{q^2}\left[N_{qC} - \frac{(m_K^2 - m_\pi^2)}{q^2}N_{qq}\right] \\ &= 8\left(\frac{m_K^2 - m_\pi^2}{q^2}\right)\left\{(m_K^2 + m_\pi^2 - q^2 - 2s_{\ell\pi})m_\ell^2 + \left(\frac{m_K^2 - m_\pi^2 + q^2}{q^2}\right)m_\ell^4\right\} \end{aligned}$$

Summary of this lecture



Feynman diagrams for radiative corrections:

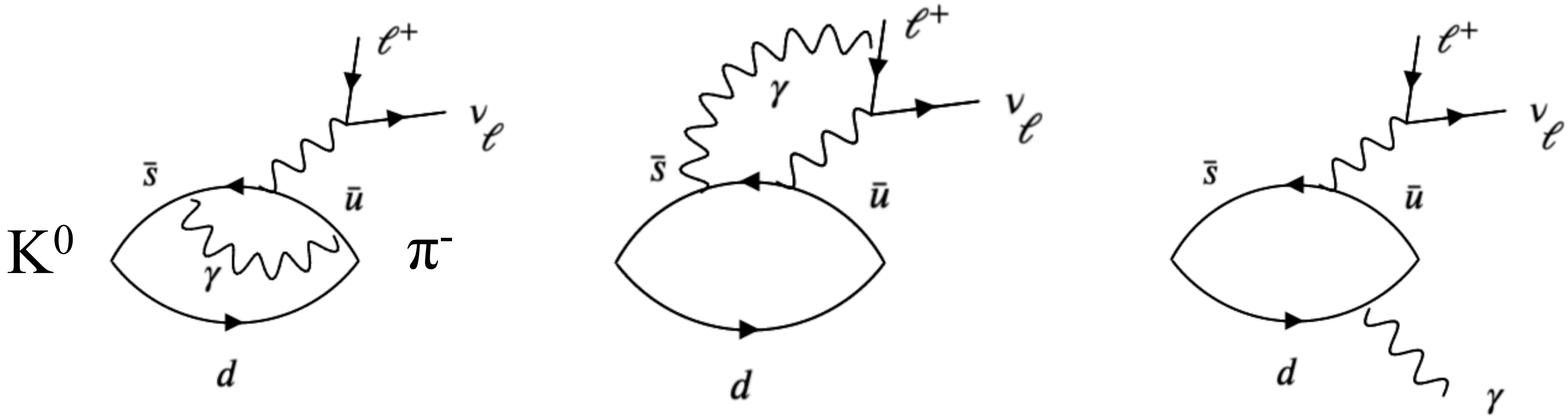


virtual photon corrections

real photon emissions

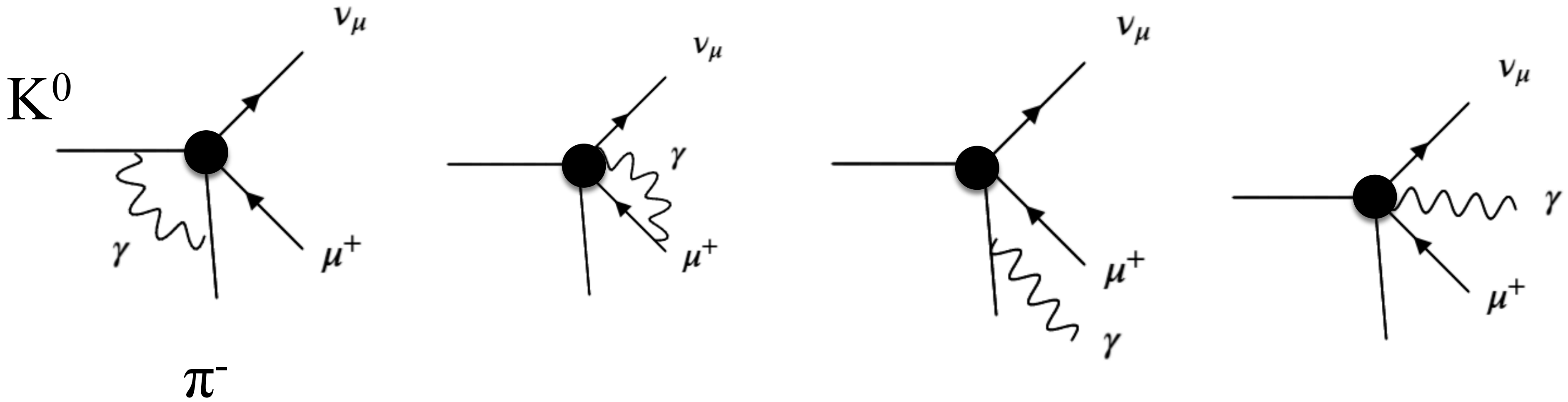
- Infrared divergences
- Continuation from Minkowsky to Euclidean
- Finite volume corrections

Feynman diagrams for radiative corrections:

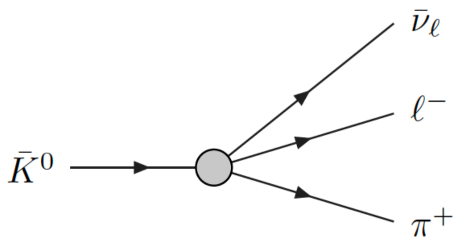


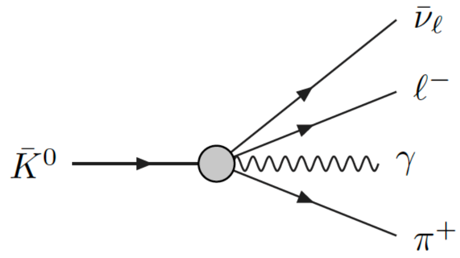
virtual photon corrections

real photon emissions



- For illustration we consider $K_{\ell 3}$ decays, but the discussion is general:





- A particularly appropriate measurable quantity to compute is

$$\frac{d^2\Gamma}{dq^2 ds_{\pi\ell}}$$

where $q^2 = (p_K - p_\pi)^2$ and $s_{\pi\ell} = (p_\pi + p_\ell)^2$.

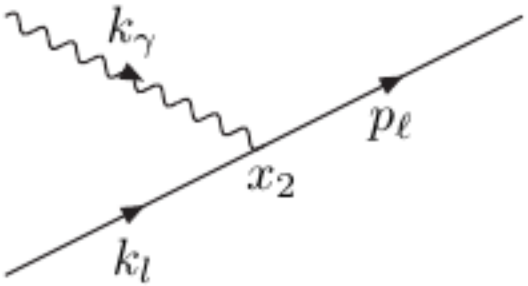
- Following the same procedure as for leptonic decays we write:

$$\frac{d^2\Gamma}{dq^2 ds_{\pi\ell}} = \lim_{V\to\infty}\left(\frac{d^2\Gamma_0}{dq^2 ds_{\pi\ell}} - \frac{d^2\Gamma_0^{\rm pt}}{dq^2 ds_{\pi\ell}}\right) + \lim_{V\to\infty}\left(\frac{d^2\Gamma_0^{\rm pt}}{dq^2 ds_{\pi\ell}} + \frac{d^2\Gamma_1(\Delta E)}{dq^2 ds_{\pi\ell}}\right)$$

  - Infrared divergences cancel separately in each of the two terms.
- If the amplitude for real photons is computed non-perturbatively, then the above formula is modified as for leptonic decays to three terms on the r.h.s..

IMPORTANT slides:
the continuation from Minkowski to Euclidean

we need to ensure that the t_2 integration up to ∞ converges in spite of the factor $e^{E_l\, t_2}$ where $E_l = \sqrt{m_l^2 + p_l^2}$ is the energy of the outgoing charged lepton



1) Momentum conservation: since we integrate over x_2 $p_l = k_l + k_\gamma$

2) The integrations over the energies k_{4l} and $k_{4\gamma}$ lead to the exponential factor $e^{-(\omega_l + \omega_\gamma - E_l)\, t_2}$ where $\omega_l = \sqrt{m_l^2 + k_l^2}$, $\omega_\gamma = \sqrt{m_\gamma^2 + k_\gamma^2}$, and m_γ is the mass of the photon introduced as an infra-red cut-off.

A few technical but non trivial IMPORTANT slides:

the continuation from Minkowski to Euclidean

3) … but $(\omega_l+\omega_\gamma) \geq \sqrt{(m_l+m_\gamma)^2 +p_l^2} > E_l = \sqrt{m_l^2 +p_l^2}$

thus the argument of the exponent $e^{-(\omega_l + \omega_\gamma - E_l)\, t_2}$ is negative for every term appearing in the sum over the intermediate states and the integral over t_2 converges

4) note that the integration over t_2 is also convergent if we set $m_\gamma=0$ but remove photon zero mode in finite volume. In this case $(\omega_l+\omega_\gamma) > E_l+[1-(p_l/E_l)]\ (k_\gamma)_{min}$

- necessity of a mass gap
- absence of a lighter intermediate state

Semileptonic Decays: New problems in the continuation from Minkowski to Euclidean

Work in progress

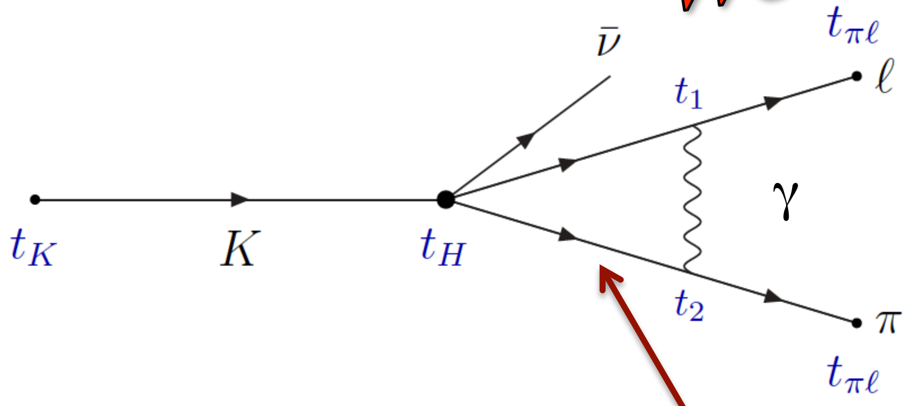


Figure 1: Diagram contributing to the $K \to \pi\ell\bar{\nu}$ decay amplitude.

A new problem may arise if there are intermediate π-l (or π-l-γ, π-γ, l-γ) states which have a smaller energy than the external π-l system.

$$G(t_H, t_{\pi\ell}) \sim e^{\Delta E\,(t_{\pi\ell} - t_H)}$$

$$\Delta E = E_{\pi\ell}^{ext} - E_{\pi\ell}^{int}$$

Remember that in QED_L the photon will always have a non-zero minimum energy

A close inspection, up to hadronic form factors, shows that

$$G(t_H, t_{\pi\ell}) = \frac{1}{8k\omega'_\pi\omega'_\ell}\left[\frac{1}{\Delta E}\left(\frac{1}{\Delta\omega_\pi - k} + \frac{1}{\Delta\omega_\ell - k}\right) + \frac{1}{\Delta E}\left(\frac{1}{\Delta\omega_\pi + k} + \frac{1}{\Delta\omega_\ell + k}\right) e^{\Delta E\,(t_{\pi\ell} - t_H)}\right.$$

$$\left. - \frac{1}{(\Delta\omega_\pi - k)(\Delta\omega_\ell + k)} e^{(\Delta\omega_\pi - k)\,(t_{\pi\ell} - t_H)} - \frac{1}{(\Delta\omega_\pi + k)(\Delta\omega_\ell - k)} e^{(\Delta\omega_\ell - k)\,(t_{\pi\ell} - t_H)}\right]$$

physical term

$$\times(\text{hadronic form factor})$$

$$\omega_{\pi,\ell} = \sqrt{m^2_{\pi,\ell} + \vec{p}^{\,2}_{\pi,\ell}} \qquad \omega'_{\pi,\ell} = \sqrt{m^2_{\pi,\ell} + (\vec{p}_{\pi,\ell} \pm \vec{k})^2} \qquad \Delta\omega_{\pi,\ell} = \omega_{\pi,\ell} - \omega'_{\pi,\ell}$$

note that the last two terms are always well behaved at large $t_{\pi l}$-t_H since, as in the leptonic case, $\Delta\omega_l$-k < 0 and $\Delta\omega_\pi$-k <0 when we remove photon zero mode in the finite volume (necessity of a mass gap).

The dangerous term is instead the term proportional to ΔE

Analytic continuation continued

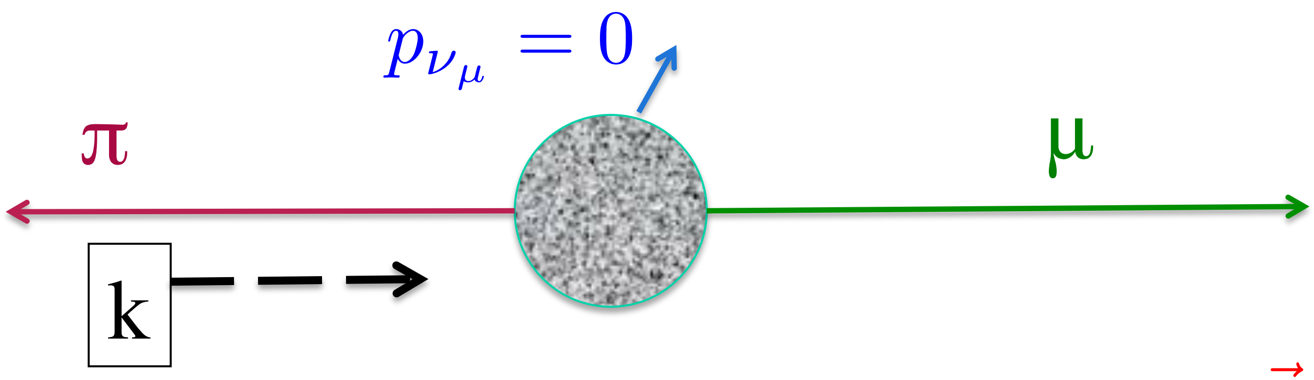
The relevant parameter is the invariant mass of the π-l pair

$$(m_\mu + m_\pi)^2 \le M_{\pi\ell}^2 \le m_K^2$$

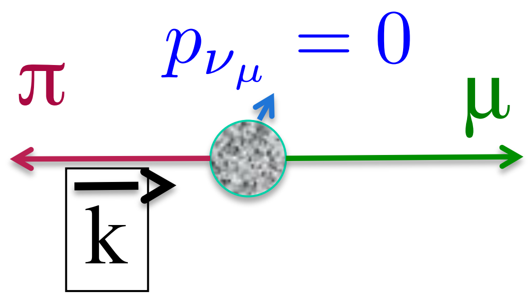
which, in turn, is determined by the neutrino momentum

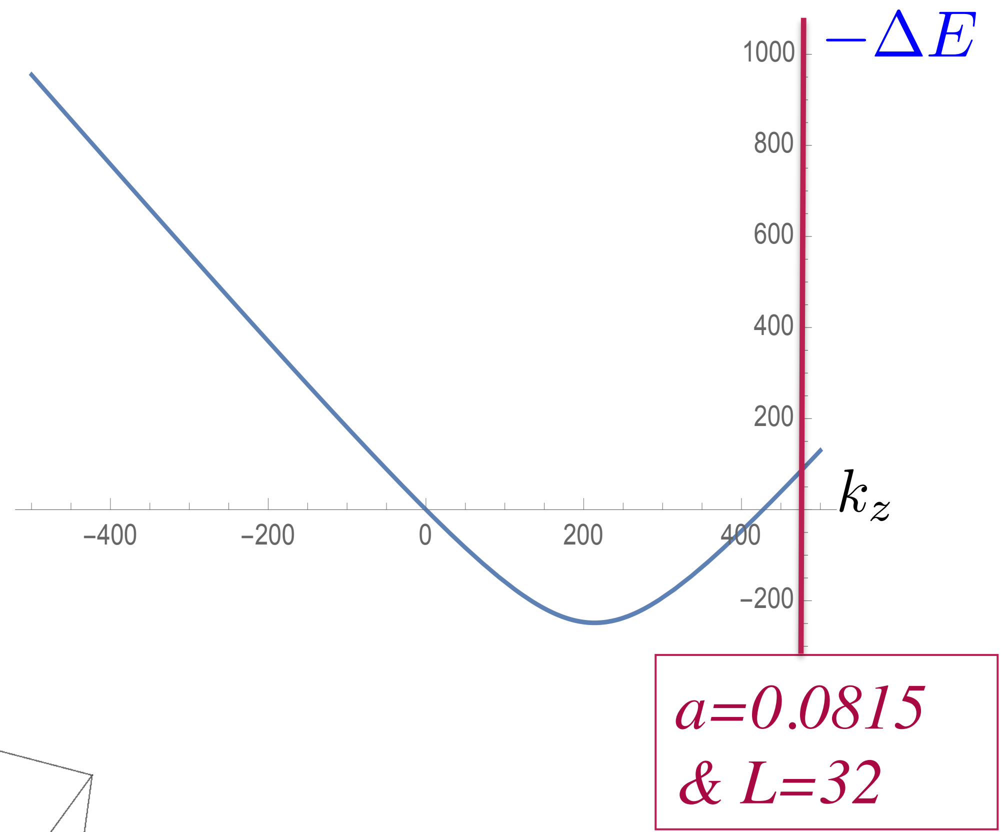
. When the invariant mass $M_{\pi l}$ is large ΔE become positive.

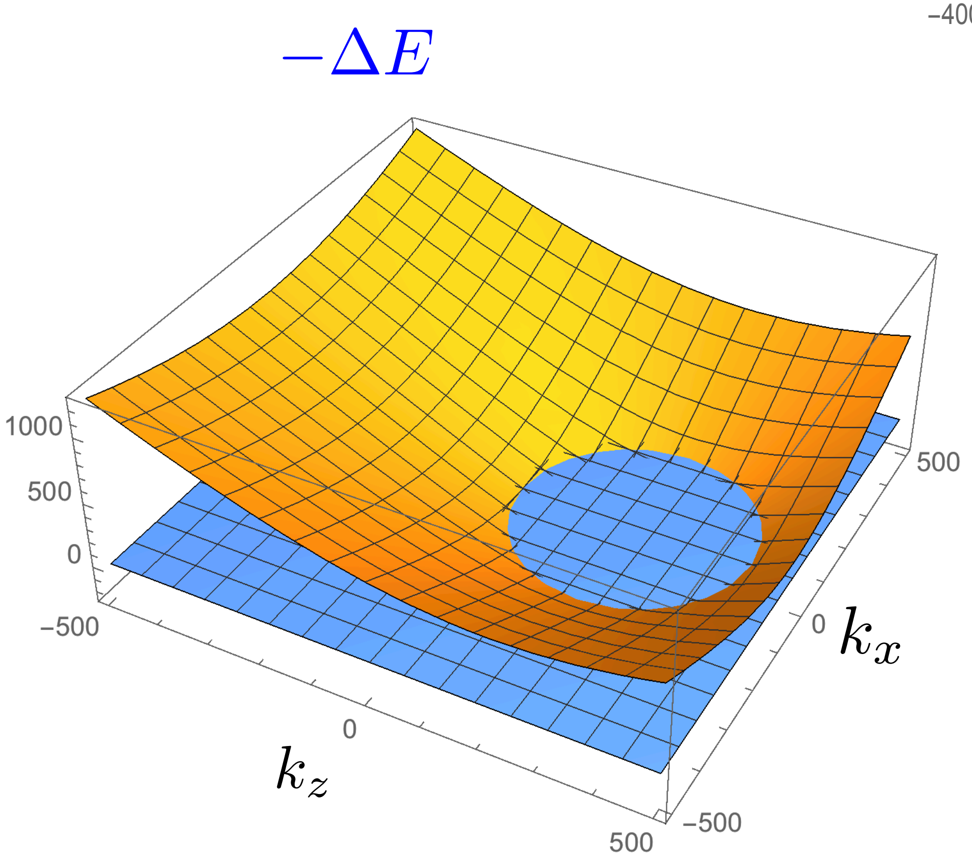
$$-\Delta E = \sqrt{m_\pi^2 + (\vec{p}_\pi + \vec{k})^2} + \sqrt{m_\ell^2 + (\vec{p}_\ell - \vec{k})^2} - \sqrt{m_\pi^2 + \vec{p}_\pi^{\,2}} - \sqrt{m_\ell^2 + \vec{p}_\ell^{\,2}}$$



for large $M_{\pi l}$ max ΔE corresponds to $\vec{k} = -\vec{p}_\pi$


$p_{\nu_\mu} = 0$
π
μ
k


$-\Delta E$
1000
800
600
400
200
−200
−400
−200
0
200
400
k_z
a=0.0815
& L=32


$-\Delta E$
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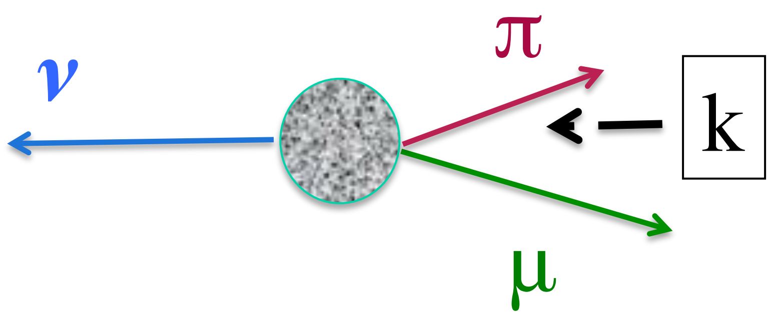
k_z
500
0
−500
k_x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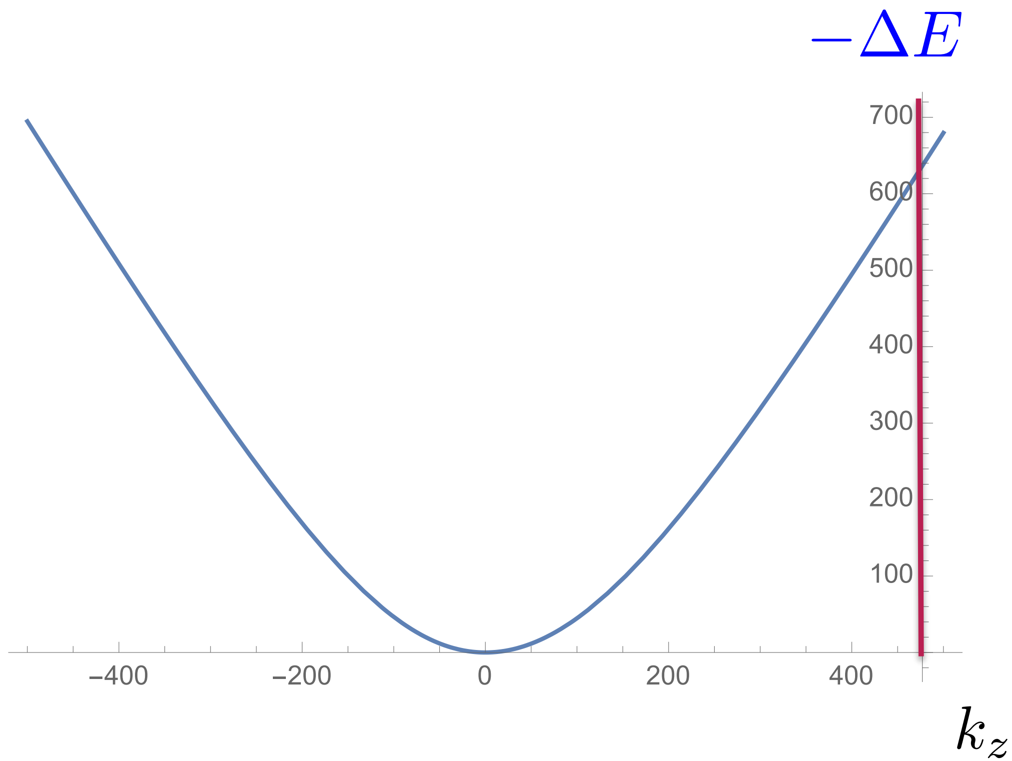
$-\Delta E$
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600
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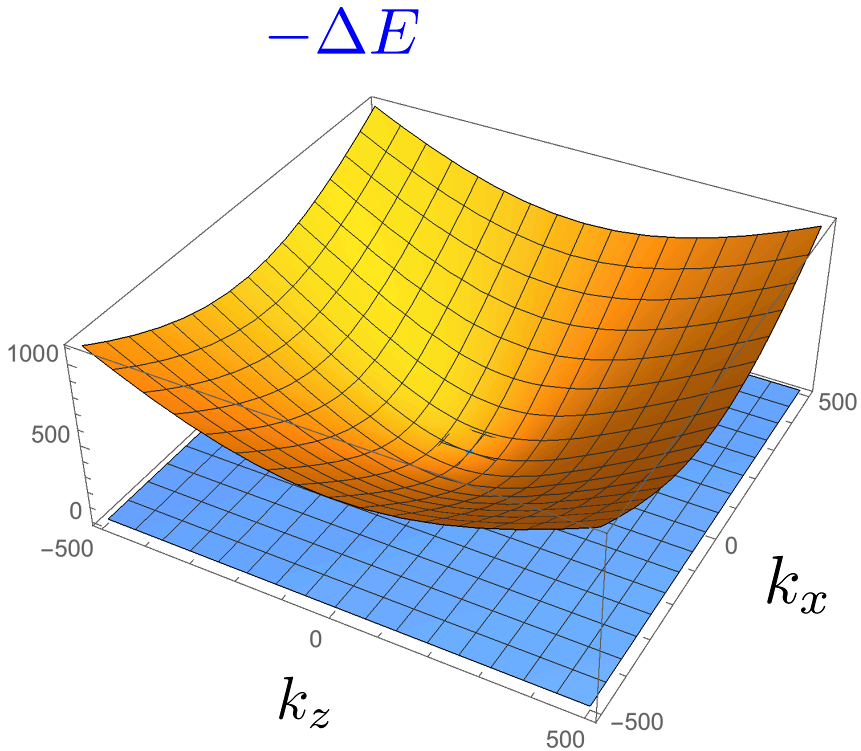
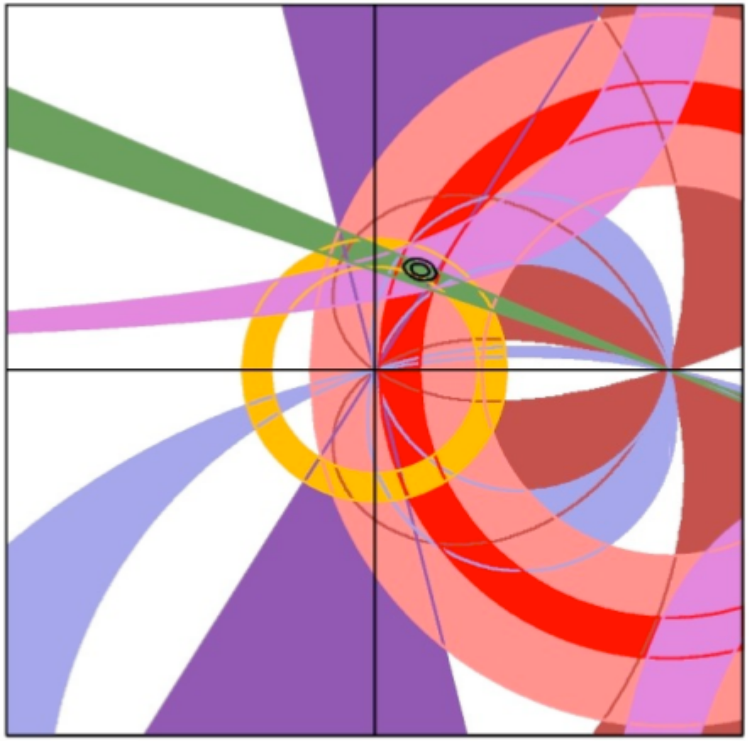
k_z


$-\Delta E$
1000
500
0
−500
0
500
k_z
500
0
−500
k_x

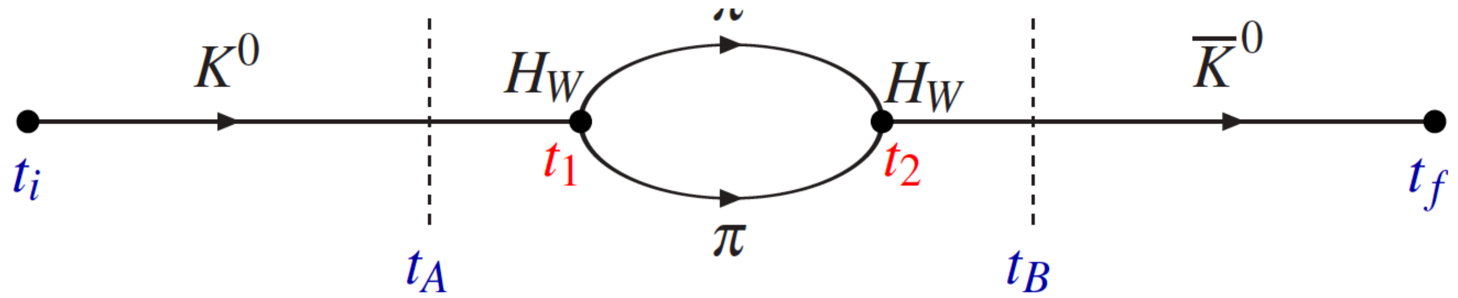
Is the presence of intermediate states with a lower energy really a problem?

Certainly is a complication
But it is not new…..



- The general formula can be written: N.H.Christ, G.Martinelli & CTS, arXiv:1401.1362 N.H.Christ, X.Feng, G.Martinelli & CTS, arXiv:1504.01170

$$\Delta m_K = \Delta m_K^{\mathrm{FV}} - 2\pi\, {}_V\langle \bar{K}^0 \,|\, H \,|\, n_0\rangle_V \, {}_V\langle n_0 \,|\, H \,|\, K^0\rangle_V \left[\cot \pi h \frac{dh}{dE}\right]_{m_K},$$



- Δm_K is given by

$$\Delta m_K \equiv m_{K_L} - m_{K_S} = 2\mathscr{P} \sum_\alpha \frac{\langle \bar{K}^0 \,|\, \mathscr{H}_W \,|\, \alpha\rangle \langle \alpha \,|\, \mathscr{H}_W \,|\, K^0\rangle}{m_K - E_\alpha} = 3.483(6) \times 10^{-12}\,\mathrm{MeV}.$$

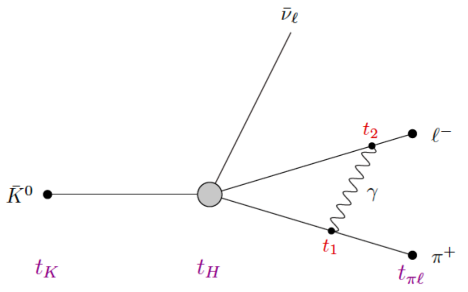
- The above correlation function gives ($T = t_B - t_A + 1$)

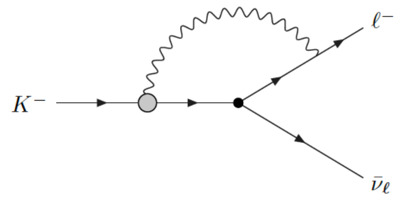
$$C_4(t_A, t_B; t_i, t_f) = |Z_K|^2 e^{-m_K(t_f - t_i)} \sum_n \frac{\langle \bar{K}^0 \,|\, \mathscr{H}_W \,|\, n\rangle \langle n \,|\, \mathscr{H}_W \,|\, K^0\rangle}{(m_K - E_n)^2} \times \left\{ e^{(M_K - E_n)T} - (m_K - E_n)T - 1 \right\}.$$

dangerous terms

- From the coefficient of T we can therefore obtain

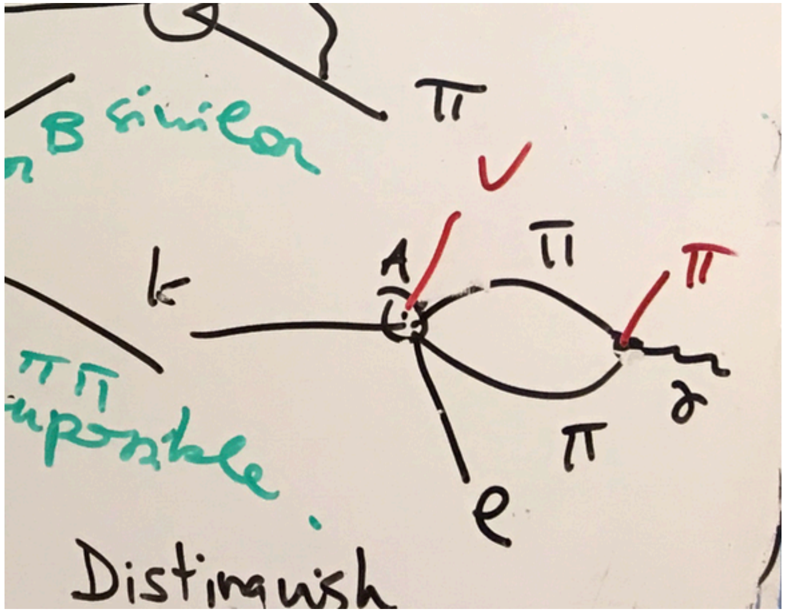
Exponentially growing terms in time (cont.)

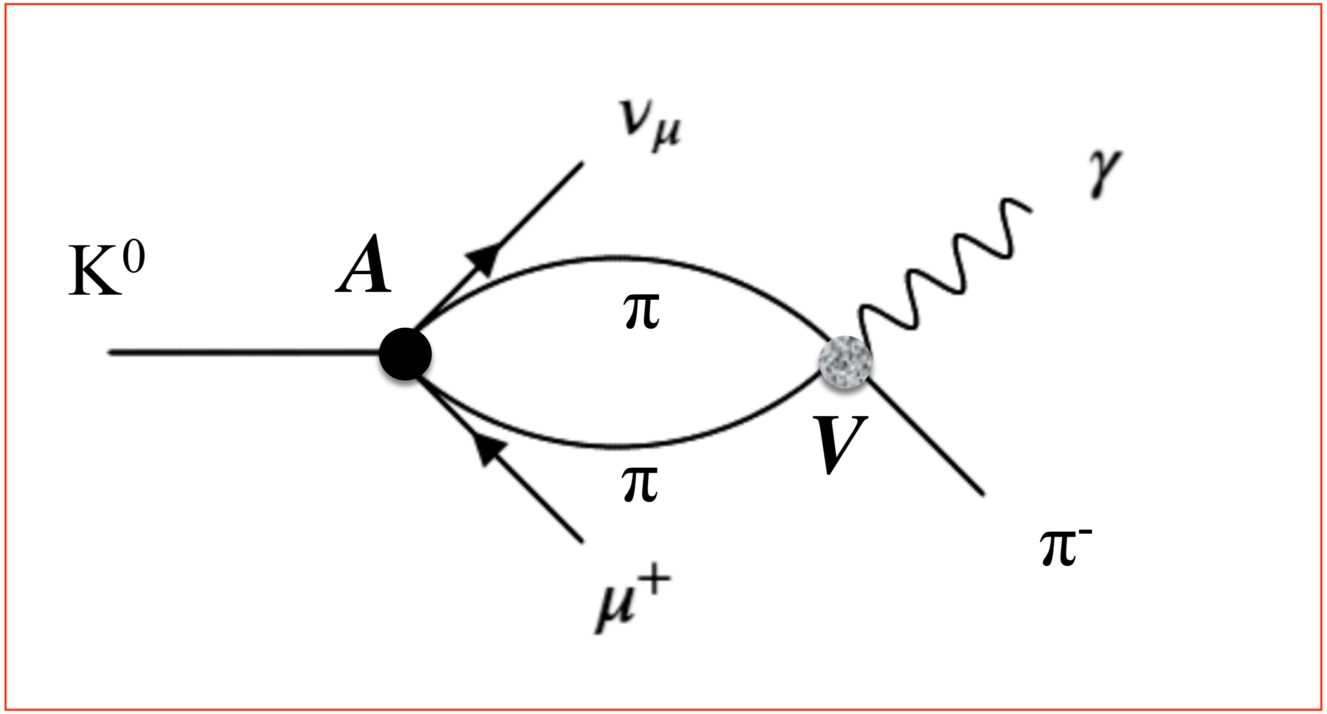




- The presence of exponentially growing terms is a generic feature in the evaluation of long-distance contributions. They must be identified and subtracted.
  - The number of such terms depends on $s_{\pi\ell}$ and on the chosen (twisted) boundary conditions.
  - For kaon decays, in some corners of phase space, there may in principle also be multi-hadron intermediate states corresponding to growing exponentials, but these are expected to be small.
    - For example $K \to \pi\pi\ell\nu \to \pi\ell\nu(\gamma)$ only contributes at high order (p^6) in ChPT and is present due to the Wess-Zumino-Witten term in the action.
    - More importantly, we can restrict the values of $s_{\pi\ell}$ to a range below the multi-hadron threshold.
- D and B decays: the large number of such terms which need to be subtracted in most of phase space, makes it *very difficult* to implement the method.
- No such exponentially growing terms are present for leptonic decays.

+ 3-pion intermediate states





Γ_0^{pt} *Universality of the logarithmically divergent term and of the 1/L correction*

$$\Gamma_0^{\rm pt}(L) = C_0(r_\ell) + \tilde{C}_0(r_\ell)\text{Log}\left[m_P L\right] + \frac{C_1(r_\ell)}{m_P L} + \dots$$

Depends on the ir regularization. The regularization dependent part does not depend on the internal structure of the hadron

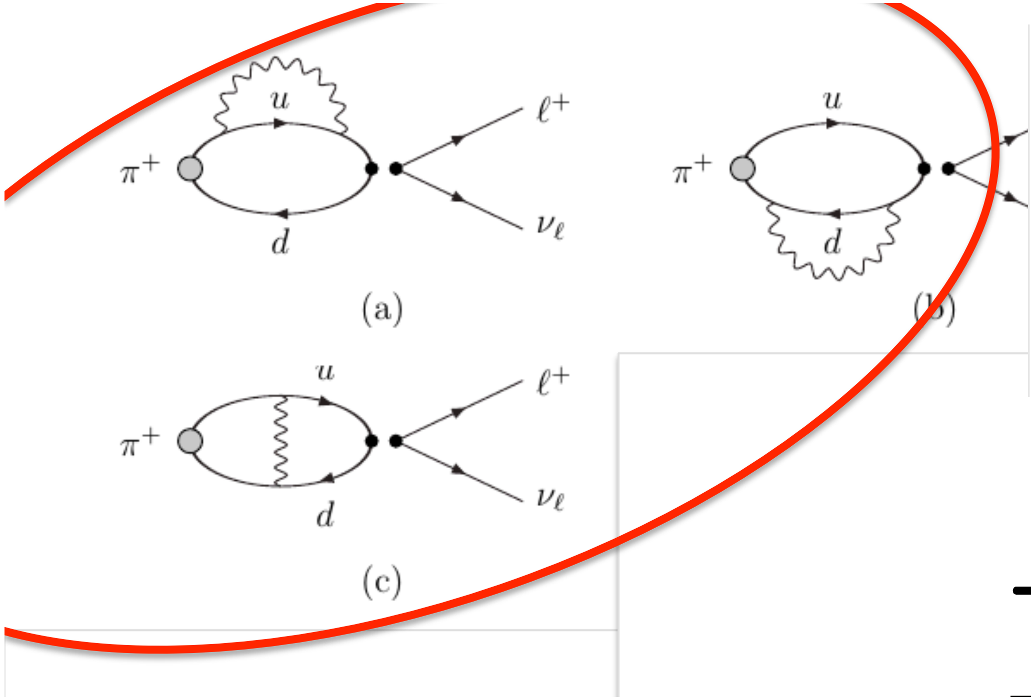
Does NOT depend on the ir regularization or on the internal structure of the hadron

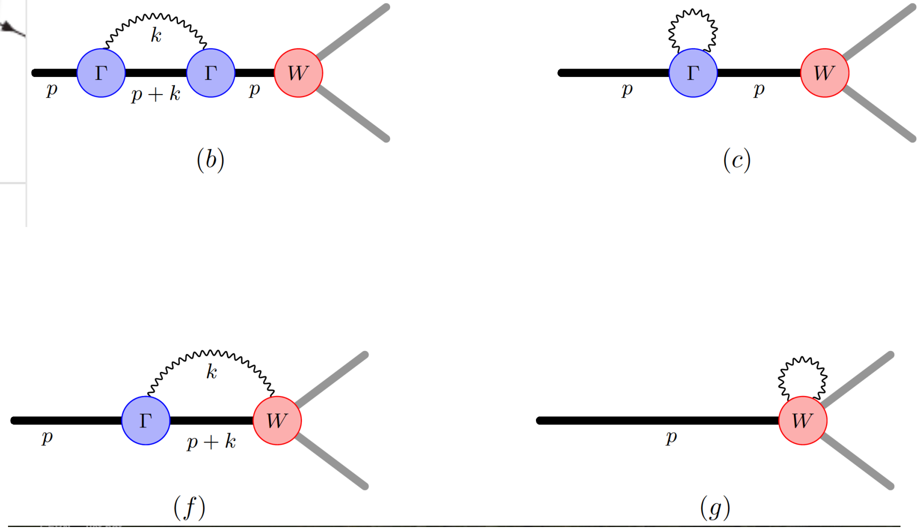
BMW, Science 347 (2015) 1452
B. Lucini et al., JHEP 1602 (2016)

Thus $\Delta\Gamma(L) = \Gamma_0 - \Gamma_0^{pt}$ = Infrared finite, independent of the regularization up to $O(1/L^2)$

Universality demonstrated via skeleton expansion or effective theory

$\Delta\Gamma(L) = \Gamma_0 - \Gamma_0^{pt}$





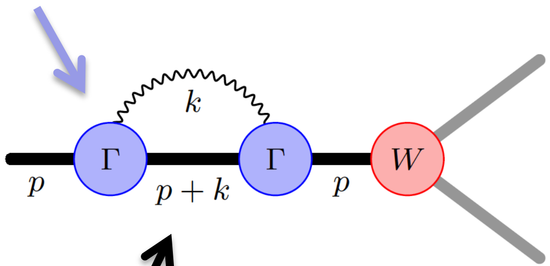
NON-UNIVERSAL TERMS ENTER WITH HIGHER POWERS OF THE PHOTON MOMENTUM i.e. LESS SINGULAR TERMS AND CORRESPOND TO HIGHER POWERS IN 1/L A practical rule summarising the relation between the power of the finite-volume corrections and the leading singularity of the integrand at k=0 is

$$\Delta\text{Loop} = \int \frac{dk_0}{2\pi} \left(\frac{1}{L^3} \sum_{\vec{k}\neq 0} - \int \frac{d^3k}{(2\pi)^3} \right) \frac{1}{(k^2)^{n/2}} = O\left(\frac{1}{L^{4-n}}\right)$$

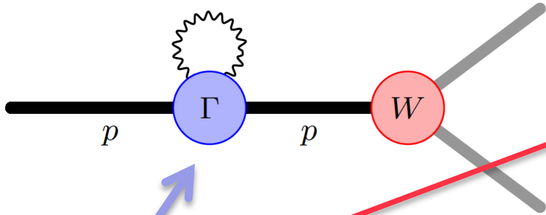
Using only gauge invariance (Ward ids) one has

$$\Gamma^\mu(p,k) = (2p+k)^\mu + 4z_1 p^\mu\, p\cdot k + 4z_1\epsilon^2 p^\mu + O(k^2,\epsilon^4,\epsilon^2 k)$$

z_1 is structure dependent



(b)



(c)

$$\Gamma^{\mu\nu}(p,k,-k) = -2\delta^{\mu\nu} - 8z_1 p^\mu p^\nu + O(k,\epsilon^2)$$

$$\Delta(p+k) = \frac{1 - 2z_1 p\cdot k - \epsilon^2 z_1 + O(k^2,\epsilon^4,\epsilon^2 k)}{\epsilon^2 + 2p\cdot k + k^2}\,.$$

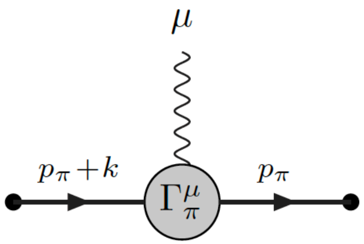
from $\quad \Delta_\gamma(k^2)\left\{\Gamma^\mu(p,k)\Delta(p+k)\Gamma^\mu(p+k,-k) + \frac{1}{2}\Gamma^{\mu\mu}(p,k,-k)\right\}$

we find the following contribution to the amplitude

$$\Delta_\gamma(k^2)\left\{\frac{4m_P^2 + O(k^2)}{(2p\cdot k + k^2)^2} - \frac{8z_1 m_P^2 + O(k)}{2p\cdot k + k^2}\right\} + O\left(\frac{1}{L^2}\right)$$

infrared logarithmic divergent term

non-universal 1/L term



- Similarly we define the amputated $\pi\gamma\pi$ vertex Γ_π^μ, by amputating the propagators and matrix elements of the interpolating operators in the correlation function:

$$C_\pi^\mu(p_\pi,k) = i\int d^4z\,d^4x\,e^{-ip_\pi\cdot z}\,e^{-ik\cdot x}\langle 0|\,T\{\phi_\pi(z)\,j^\mu(x)\,\phi_\pi^\dagger(0)\}\,|0\rangle\,.$$

- We now expand Γ_π for small k (and the ε_π).
- The key result is obtained from the Ward Identity:

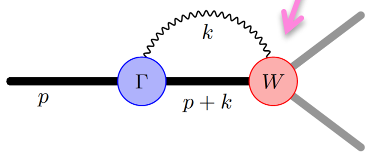
$$k_\mu \Gamma_P^\mu(p_\pi,k) = \left\{\Delta_\pi^{-1}(p_\pi+k) - \Delta_\pi^{-1}(p_\pi)\right\},$$

which relates the first-order expansion coefficients and yields

$$Z_\pi(p_\pi+k)\,\Gamma_\pi^\mu(p_\pi,k) = Q_\pi\,(2p_\pi+k)^\mu + O(k^2,\varepsilon_\pi^2)\,.$$

$\Rightarrow$ no $O(1/L)$ correction from pion propagator and $\pi\gamma\pi$ vertex.

$$W^{\mu\rho}(p,k) = f_P \delta^{\mu\rho} + 2 f_P f_1 p^\mu p^\rho + O(k)$$



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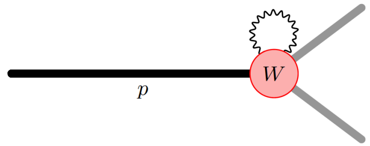
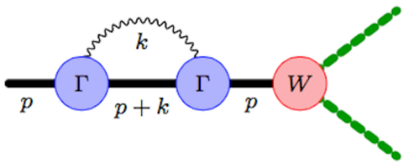


(g)

structure dependent

$$\Delta_\gamma(k^2)\,\Gamma(p,k)\,\Delta(p+k)\,W^{\mu\rho}(p+k,-k) = \Delta_\gamma(k^2)\, f_P\, p^\rho\, \frac{2 + 4 f_1 m_P^2}{2p\cdot k + k^2}$$

constrained by gauge invariance



(b)

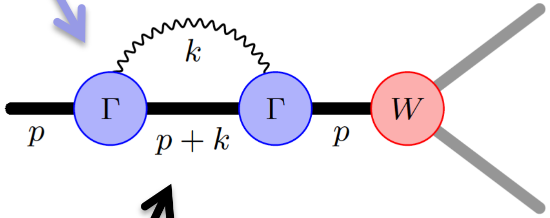
from the shift of the weak vertex

$$(-\delta m_P^2) \times f_P f_1 p^\rho = f_P f_1 \Delta_\gamma(k^2)\, p^\rho\, \frac{4 m_P^2}{2p\cdot k + k^2}\,.$$

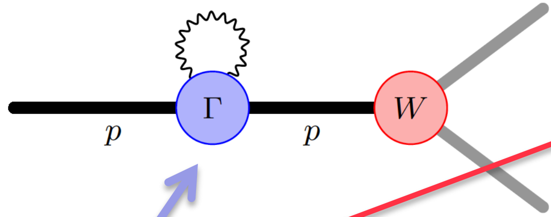
Using only gauge invariance (Ward ids) one has

$$\Gamma^\mu(p,k) = (2p+k)^\mu + 4z_1 p^\mu \, p\cdot k + 4z_1\epsilon^2 p^\mu + O(k^2,\epsilon^4,\epsilon^2 k)$$

z_1 is structure dependent



(b)



(c)

$$\Gamma^{\mu\nu}(p,k,-k) = -2\delta^{\mu\nu} - 8z_1 p^\mu p^\nu + O(k,\epsilon^2)$$

$$\Delta(p+k) = \frac{1-2z_1 p\cdot k - \epsilon^2 z_1 + O(k^2,\epsilon^4,\epsilon^2 k)}{\epsilon^2+2p\cdot k+k^2}\,.$$

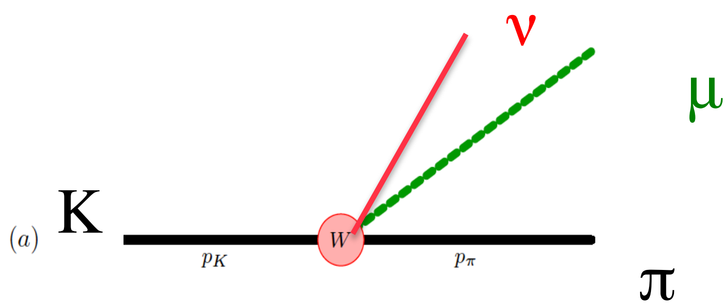
from $\quad \Delta_\gamma(k^2)\left\{\Gamma^\mu(p,k)\Delta(p+k)\Gamma^\mu(p+k,-k) + \frac{1}{2}\Gamma^{\mu\mu}(p,k,-k)\right\}$

we find the following contribution to the amplitude

$$\Delta_\gamma(k^2)\left\{\frac{4m_P^2+O(k^2)}{(2p\cdot k+k^2)^2} - \frac{8z_1 m_P^2 + O(k)}{2p\cdot k+k^2}\right\} + O\left(\frac{1}{L^2}\right)$$

infrared logarithmic divergent term

non-universal 1/L term



SEMILEPTONIC DECAYS

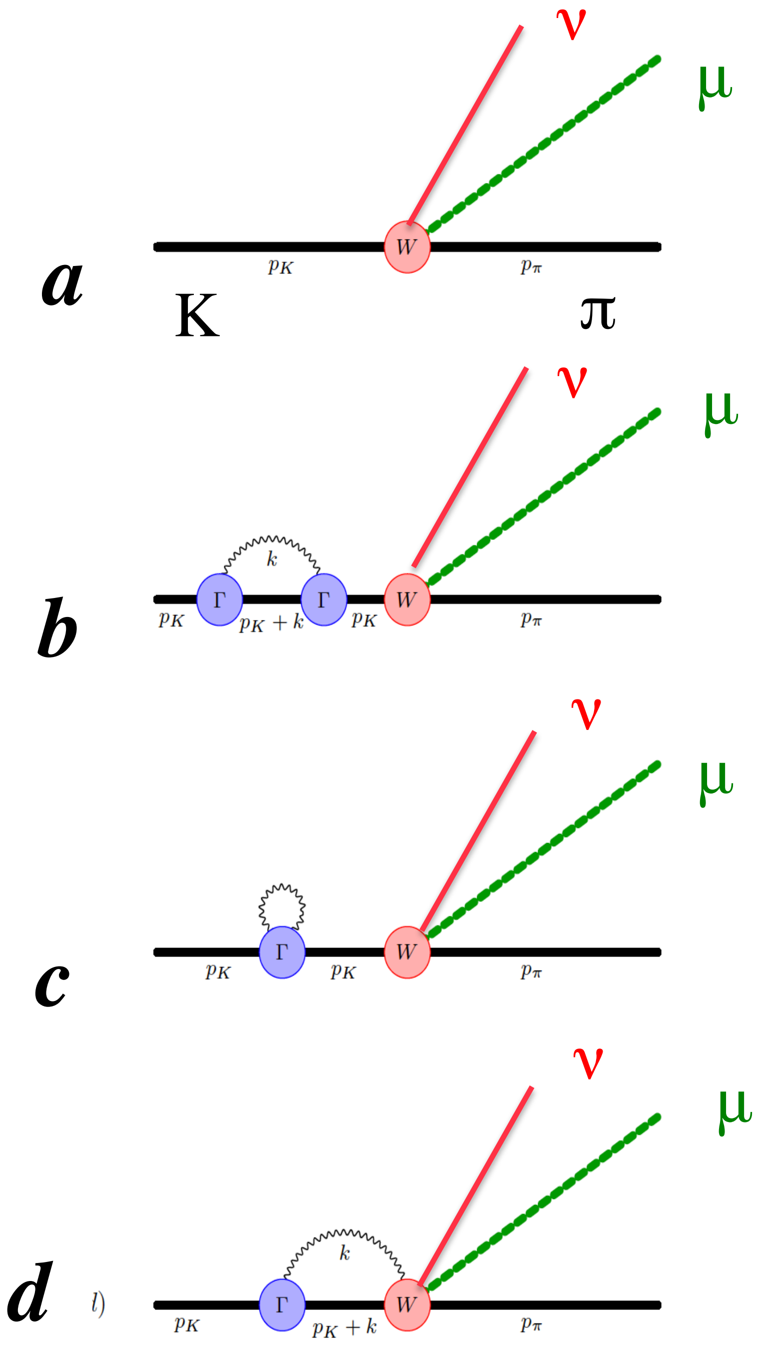
the analysis of the universality of finite volume effects is more complicated than in the leptonic case due to the dependence on q^2 (in leptonic only f_P)

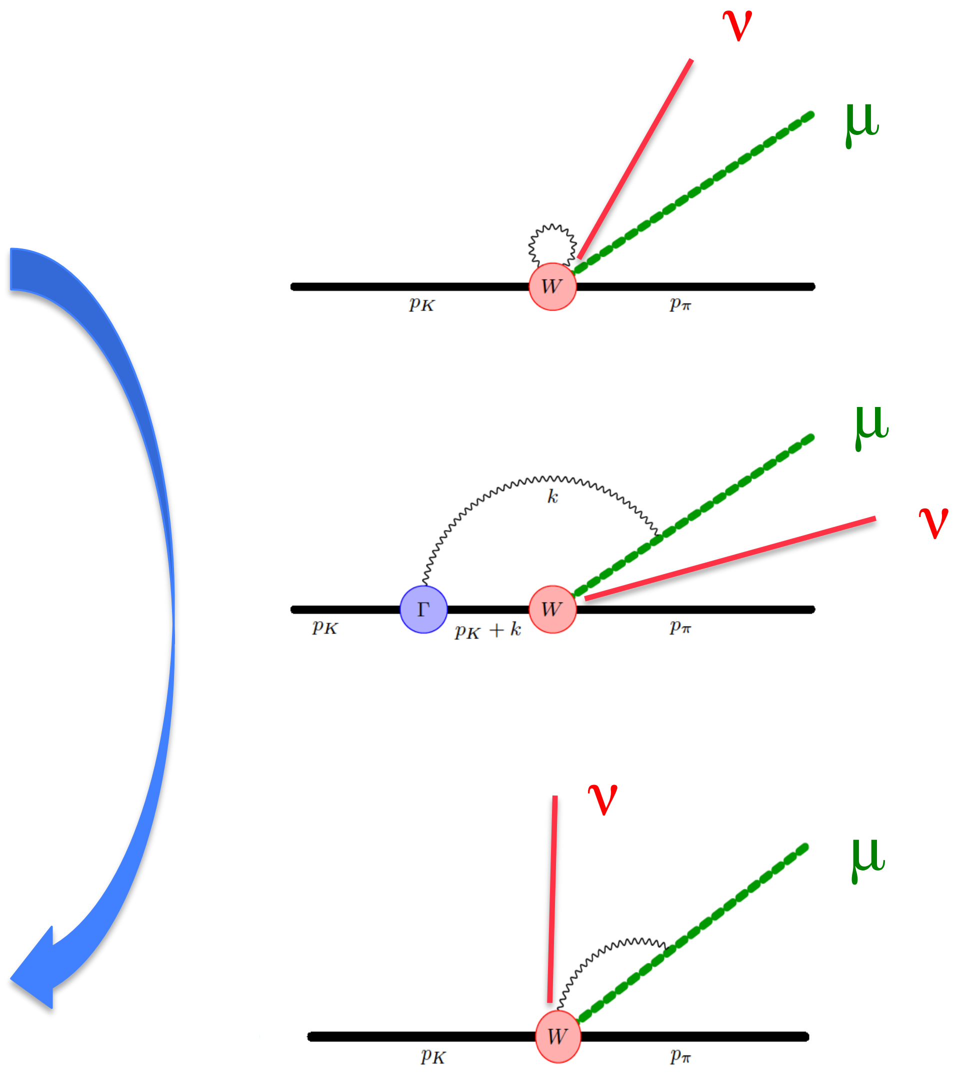
we proved that they are universal in the leptonic case

In the case of semileptonic we need a generalization of what we call *universal*

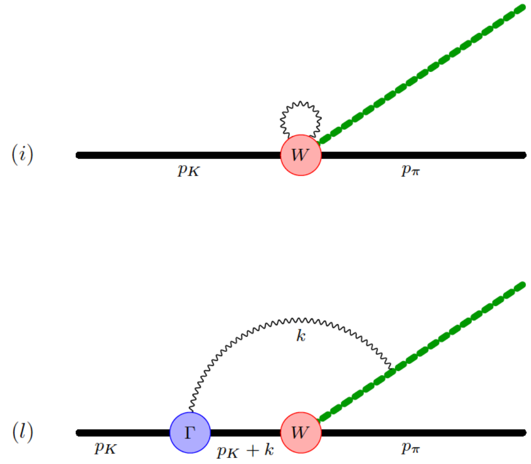
No dependence on unphysical quantities such as df/dm^2
($f = f_P, f^{\pm}$)

with the charged K at rest 1/L structure dependent terms cancel between diagrams (b) and (d) related by gauge invariance





We have seen that, as a result of the Ward Identity, we do not need the derivatives of the pion form-factors to obtain the O(1/L) corrections.



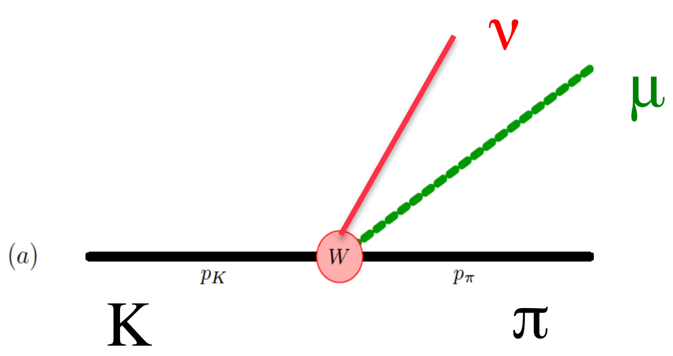
In the semileptonic case we need to expand the weak-vertex which, in QCD without QED, is a linear combination of two form-factors $f^{\pm}(q^2)$

$$\langle \pi | \bar{s}\gamma^\mu u | K \rangle = f^+(q^2)(p_K + p_\pi)^\mu + f^-(q^2)(p_K - p_\pi)^\mu \qquad q = p_K - p_\pi$$

the off-shell $F^{\pm}(p_K^2, p_\pi^2, \; 2\, p_K \cdot p_\pi)$ -> $f^{\pm}(q^2)$ on-shell.

Contrary to the leptonic case, the WI's don't lead to a complete, cancelation of O(1/L) terms. The O(1/L) corrections are found to depend on df $(q^2)/dq^2$, as well as on f(q^2) which are physical quantities. *Such derivative terms are generic (a consequence of the Low theorem).*

Absent only in particularly simple cases such as leptonic decays



SEMILEPTONIC DECAYS

the analysis of the universality of finite volume effects is more complicated than in the leptonic case due to the dependence on q^2 (in leptonic only f_P)
we proved that they are universal in the leptonic case and that for other processes and in particular for semileptonic we need a generalization of what we call *universal*

The calculation of the subleading 1/L corrections, altought conceptually straightfoward, is very complex, and the subtractions of these corrections requires the calculation of the derivatives fo the form factors $df^{\pm}(q^2)/dq^2$ at fixed hadron masses

In the ignorance of the analytic coefficients, the subtraction of the O(1/L) effects can be performed by fitting data obtained at different volumes.

Such a procedure for our numerical study of leptonic decays (where the O(1/L) corrections are known and can be subtracted) results in the doubling of the error; disappointing, but not too major a problem

C. Sachrajda @ Lattice 2019

- We are developing the framework for the computation of radiative corrections to semileptonic $K_{\ell 3}$ decays.
  - This build on our successful theoretical framework, and its successful implementation, in computations of radiative corrections to leptonic decays.
  - (We have also successfully computed the $P \to \ell\bar{\nu}\gamma$ amplitude $\Rightarrow$ makes possible to study the leptonic decays of heavy mesons.)
- Important points to note:
  1. An appropriate observable to study is $\frac{d^2\Gamma}{dq^2 ds_{\pi\ell}}$.
  2. The presence in general of unphysical exponentially growing terms in $t_{\pi\ell} - t_H$ which need to be subtracted.
  3. The universality of the $O(\frac{1}{L})$ corrections, which do however depend on the form-factors $f^{\pm}(q^2)$ and on their derivatives w.r.t. q^2.
     - Generic feature, absent only for simple processes such as leptonic decays.
- Things still to do include:
  1. To evaluate the coefficients on the $O(\frac{1}{L})$ corrections.
     - Otherwise these corrections can be fitted numerically.
     - If the $O(\frac{1}{L})$ corrections are not to be evaluated analytically, then the soft-photon approximation may be the most convenient one for the term which is added and subtracted.
  2. To test and implement the method numerically.

RADIATIVE CORRECTIONS TO MASSES AND DECAY/MIXING RATES

1)Radiative corrections are phenomenologically important (B leptonic enhanced by m_B/f_B)

2) Many ingredients to obtain physical answer (ultraviolet divergences, matching to the SM, infrared divergences, 1/L corrections, open questions)

3) For this reason is fun !

THANKS FOR YOU
PATIENCE AND ATTENTION