(Not only) Isospin breaking & QCD+QED 2

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Frontiers in Lattice QCD
Bejing 26-28/06/2019

UTfit

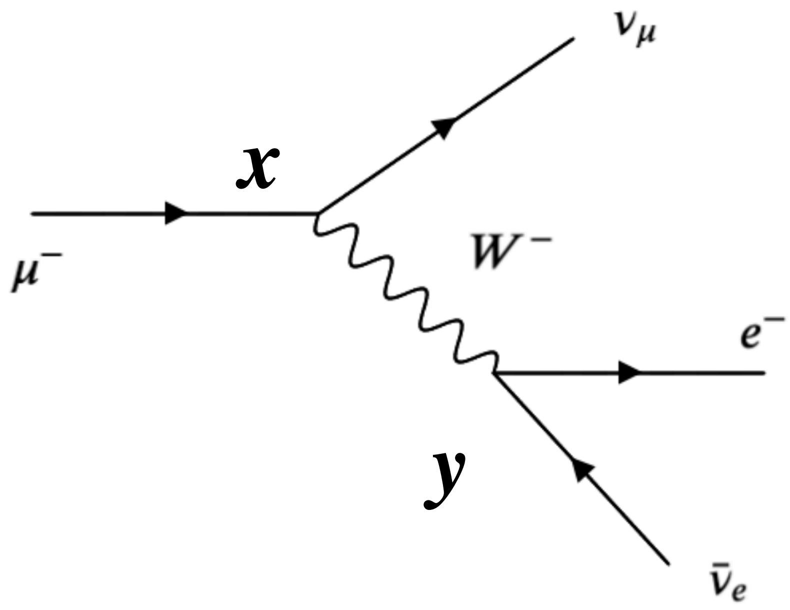
PRACE

INFN
Istituto Nazionale di Fisica Nucleare

PLAN OF THE LECTURES

- *Definition of the Fermi theory including electromagnetism*
- *Effective Hamiltonians and electromagnetism*
- *Renormalisation of the relevant operators*
- *Leptonic decays* $\pi \to \ell + \nu_\ell + (\gamma)$
- *Numerical results*

Muon decay and the definition of the Fermi constant

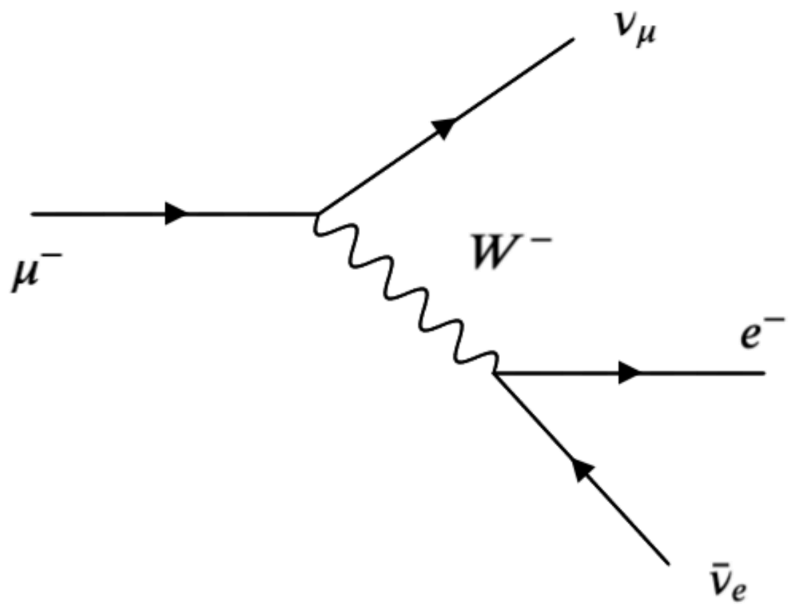


$$\begin{aligned} q &= p_\mu - k_{\nu_\mu} \\ &= p_e + k_{\nu_e} \end{aligned}$$

$$\mathcal{A} = \left(-i\frac{g_W}{\sqrt{2}}\right)^2 \bar{u}_{\nu_\mu}(k_{\nu_\mu})\frac{\gamma^\rho(1-\gamma_5)}{2}u_\mu(p_\mu)\bar{u}_e(p_e)\frac{\gamma^\sigma(1-\gamma_5)}{2}v_{\nu_e}(k_{\nu_e})\, i\,\frac{-g_{\rho\sigma}+q_\rho q_\sigma/M_W^2}{q^2-M_W^2}$$

$$\mathcal{A} \simeq i\left(\frac{g_W^2}{8M_W^2}\right)^2 \bar{u}_{\nu_\mu}(k_{\nu_\mu})\ \gamma^\rho(1-\gamma_5)\,u_\mu(p_\mu)\bar{u}_e(p_e)\gamma_\rho(1-\gamma_5)\,v_{\nu_e}(k_{\nu_e}) + O\left(\frac{m_\mu^2}{M_W^2}\right)$$

Muon decay and the definition of the Fermi constant

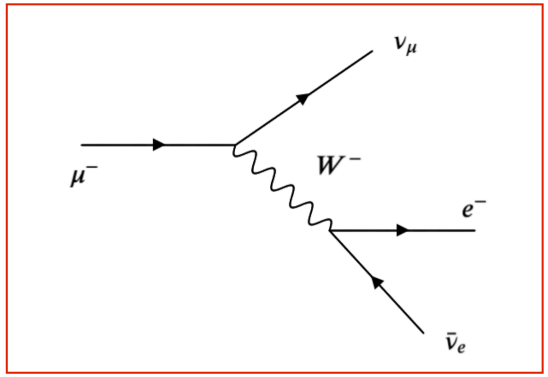


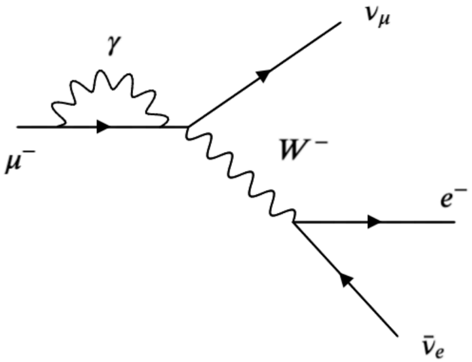
$$\frac{G_F}{\sqrt{2}} = \left(\frac{g_W^2}{8M_W^2}\right)$$

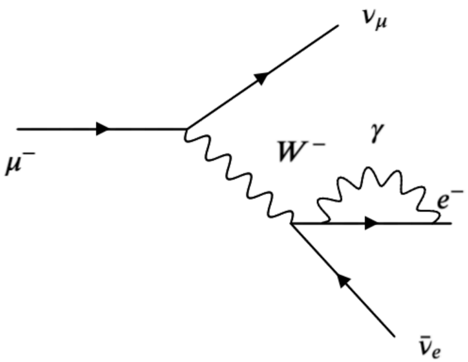
$$\mathcal{H}_F = -\frac{G_F}{\sqrt{2}}\,\bar{\psi}_{\nu_\mu}\gamma^\rho\,(1-\gamma_5)\,\psi_\mu(p_\mu)\bar{\psi}_e\gamma_\rho\,(1-\gamma_5)\,\psi_{\nu_e}$$

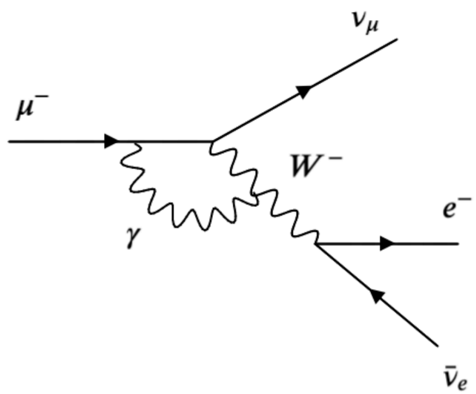
The Fermi Lagrangian is an effective (non-renormalizable) theory defined by a dimension six-operator

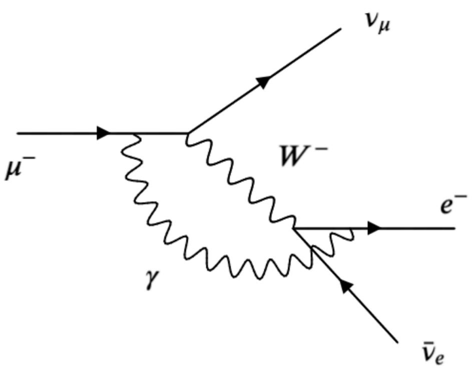
Standard Model Lagrangian and radiative corrections

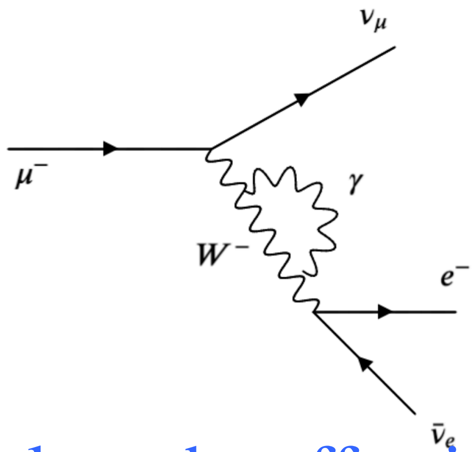


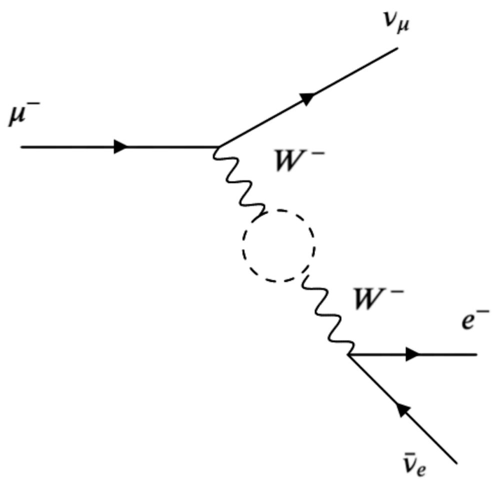






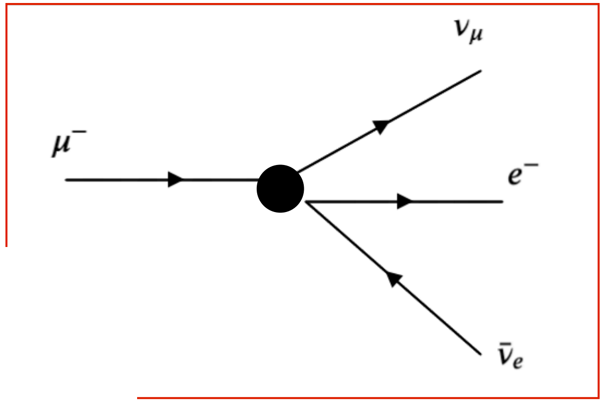


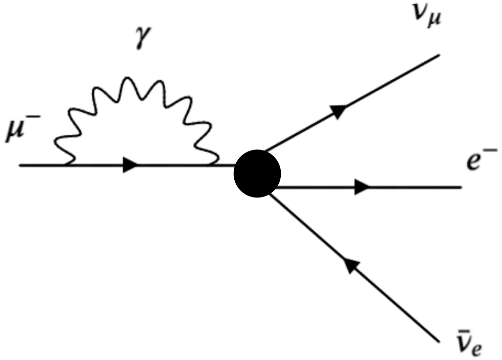


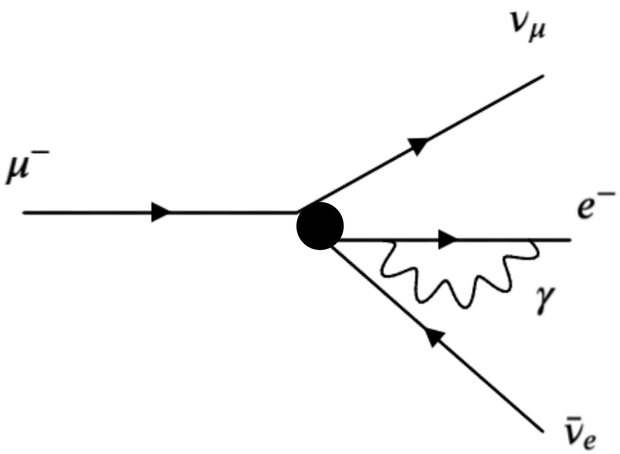


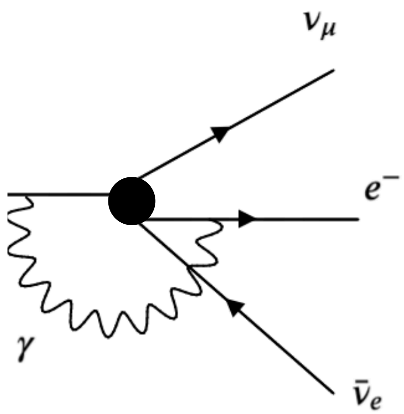
How to relate the result to the effective Fermi theory approach ?

Fermi Hamiltonian and radiative corrections





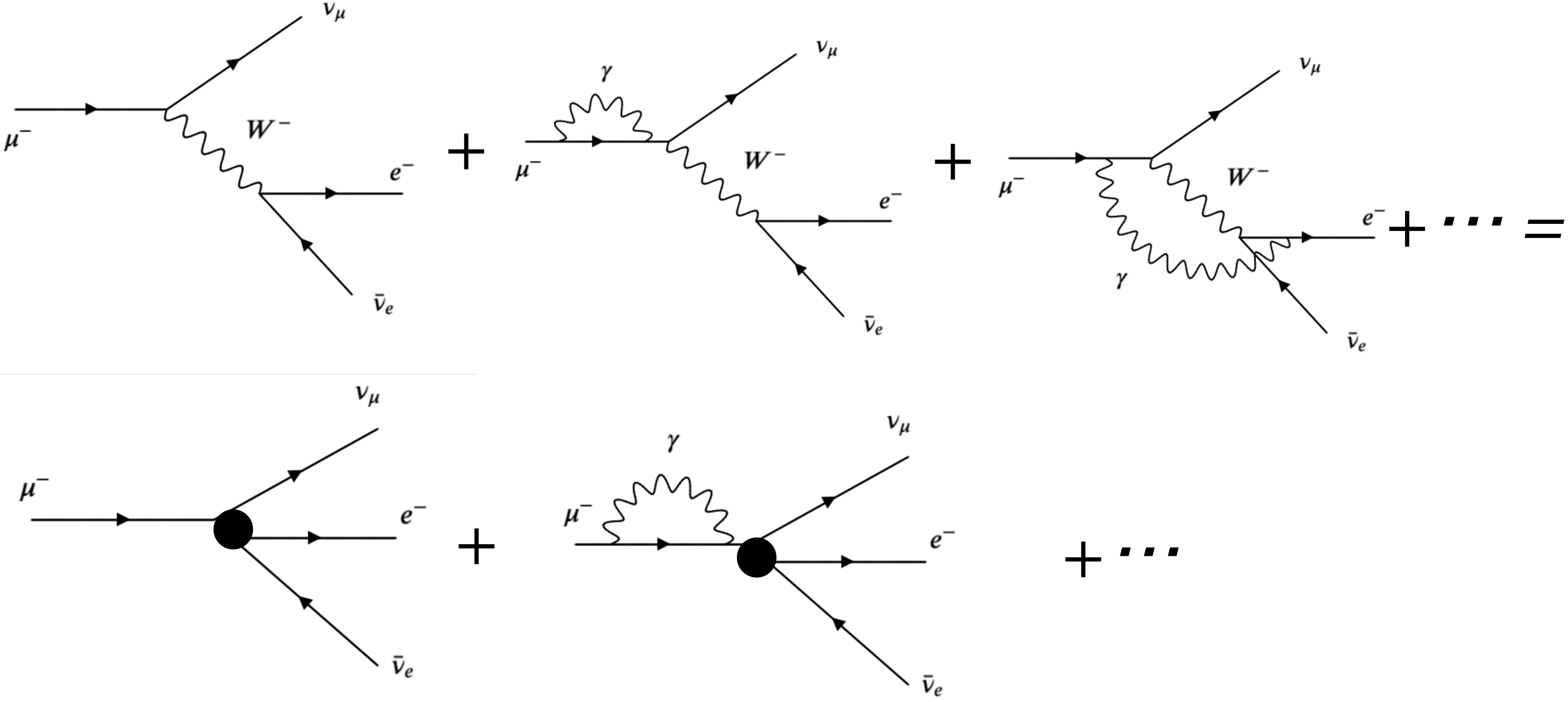




diagrams with loops involving the W boson disappear

How to relate the result to the Standard Model calculation?

The Sirlin miracle ! using the W-regularization we have:

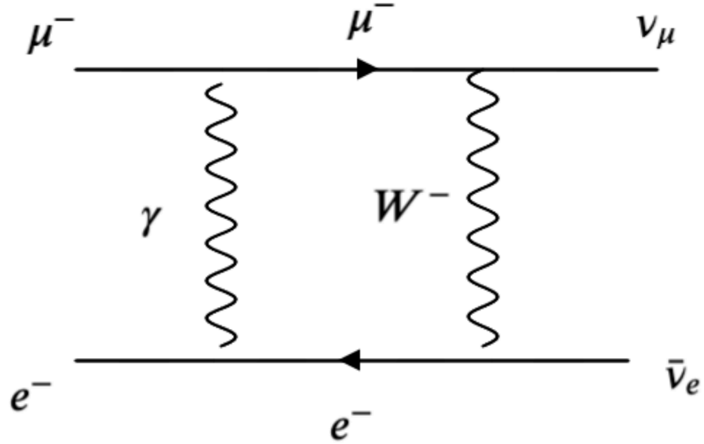


All diagrams involving the virtual W boson are taken into account by using the physical value of the Fermi constant i.e. the physical value f the renormalized g_W and W

W-regularization

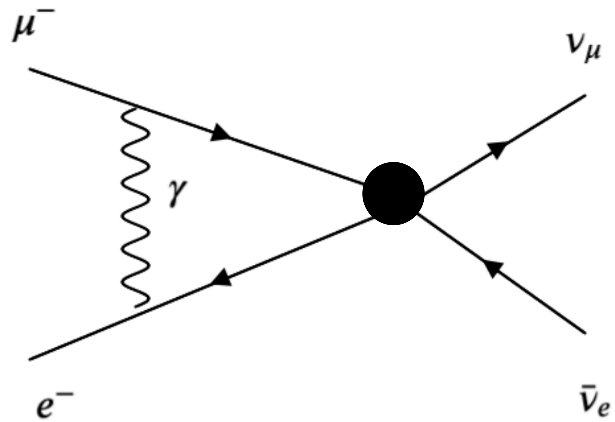
in the Standard Model we have the box diagram which is ultraviolet finite

$$A \sim \int d^4k \, \frac{1}{(k^2)^2 \not{k}^2}$$



in the Fermi theory the diagram is logarithmically divergent

$$A_F \sim \int d^4k \, \frac{1}{(k^2) \not{k}^2} \sim \log\left[\Lambda_{UV}\right] \sim \log\left[M_W\right]$$



$$\frac{1}{(k^2)} = \frac{1}{(k^2)} - \frac{1}{(k^2 - M_W^2)} + \frac{1}{(k^2 - M_W^2)} = \frac{1}{(k^2 - M_W^2)} + \frac{M_W^2}{k^2\,(M_W^2 - k^2)}$$

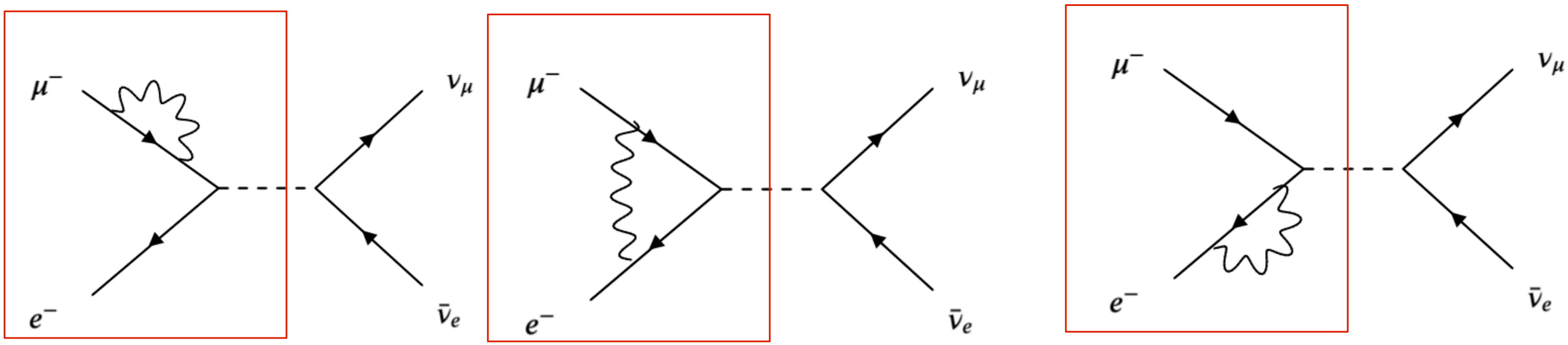
Structure of the divergences

$$A_F \sim \int d^4k \, \frac{1}{(k^2)\,\not{k}^2} \sim \log\left[\Lambda_{UV}\right] \sim \log\left[M_W\right]$$

Fierz rearrangement

$$\mathcal{H}_F = -\frac{G_F}{\sqrt{2}} \left(\bar{\nu}_\mu \gamma^\rho \left(1-\gamma_5\right) \mu\right) \left(\bar{e} \gamma_\rho \left(1-\gamma_5\right) \nu_e\right) \quad \rightarrow$$

$$\mathcal{H}_F = -\frac{G_F}{\sqrt{2}} \left(\bar{e} \gamma^\rho \left(1-\gamma_5\right) \mu\right) \left(\bar{\nu}_\mu \gamma_\rho \left(1-\gamma_5\right) \nu_e\right)$$



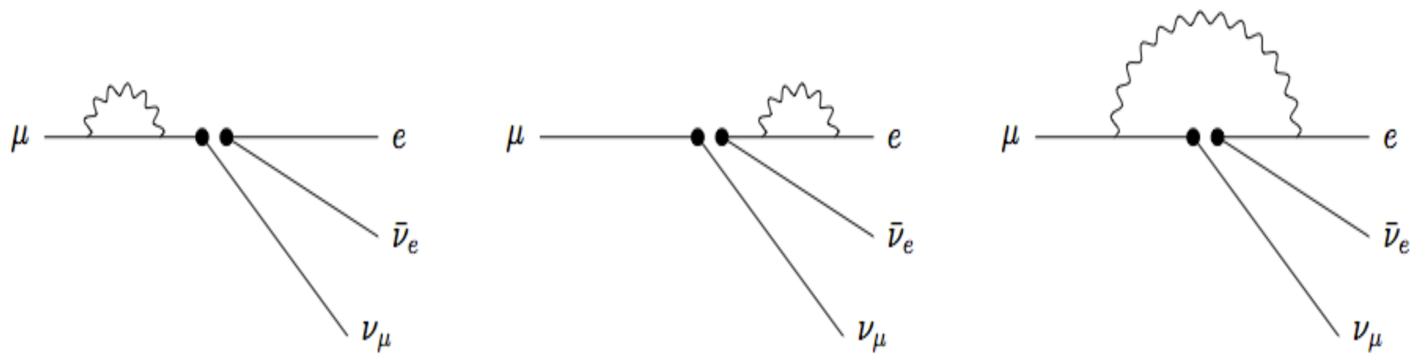
in the Fermi theory the sum of the vertex diagrams is NOT logarithmically divergent because of (quasi-) current conservation

W Regularization

The results for the widths are expressed in terms of G_F, the Fermi constant ($G_F = 1.16632(2) \times 10^{-5}\,\text{GeV}^{-2}$). This is obtained from the muon lifetime:

$$\frac{1}{\tau_\mu} = \frac{G_F^2 m_\mu^5}{192\pi^3}\left[1 - \frac{8m_e^2}{m_\mu^2}\right]\left[1 + \frac{\alpha}{2\pi}\left(\frac{25}{4} - \pi^2\right)\right]$$

- This expression can be viewed as the definition of G_F. Many EW corrections are absorbed into the definition of G_F; the explicit $O(\alpha)$ corrections come from the following diagrams in the effective theory:



together with the diagrams with a real photon.

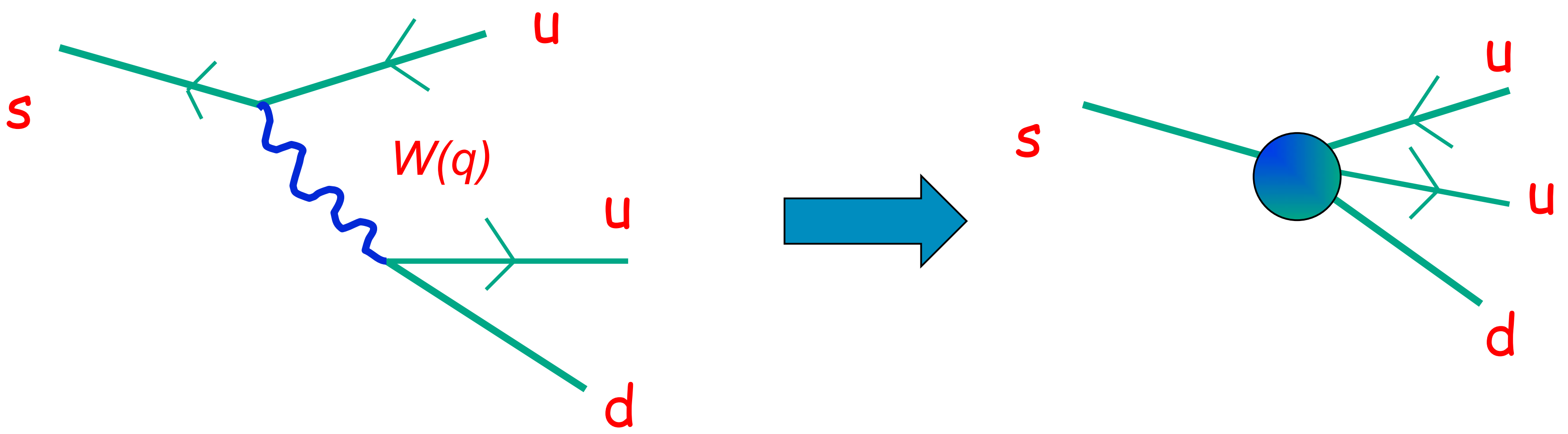
- These diagrams are evaluated in the W-regularisation in which the photon propagator is modified by:

A.Sirlin, PRD 22 (1980) 971

$$\frac{1}{k^2} \to \frac{M_W^2}{M_W^2 - k^2}\,\frac{1}{k^2}\,. \qquad \left(\frac{1}{k^2} = \frac{1}{k^2 - M_W^2} + \frac{M_W^2}{M_W^2 - k^2}\,\frac{1}{k^2}\right)$$

courtesy of C. Sachrajda

The Effective Hamiltonian

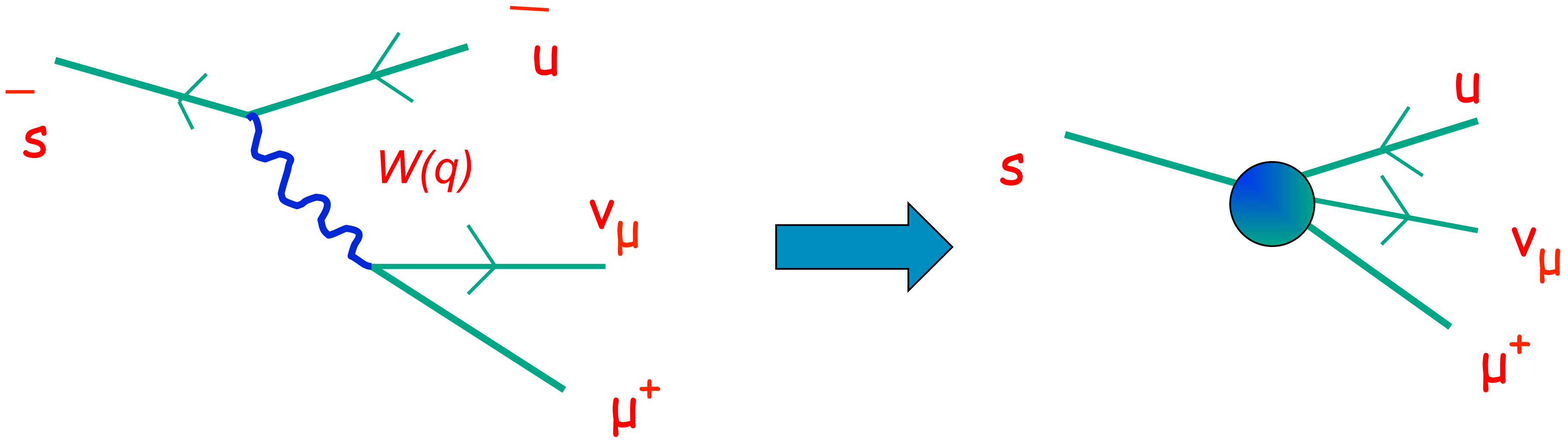


$$q \sim m_K \ll M_W$$

$$\mathcal{H}_{eff} = -\frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \; (\bar{s}\gamma_\mu(1-\gamma_5)u)\,(\bar{u}\gamma^\mu(1-\gamma_5)d)$$

Lectures by
C. Sachrajda

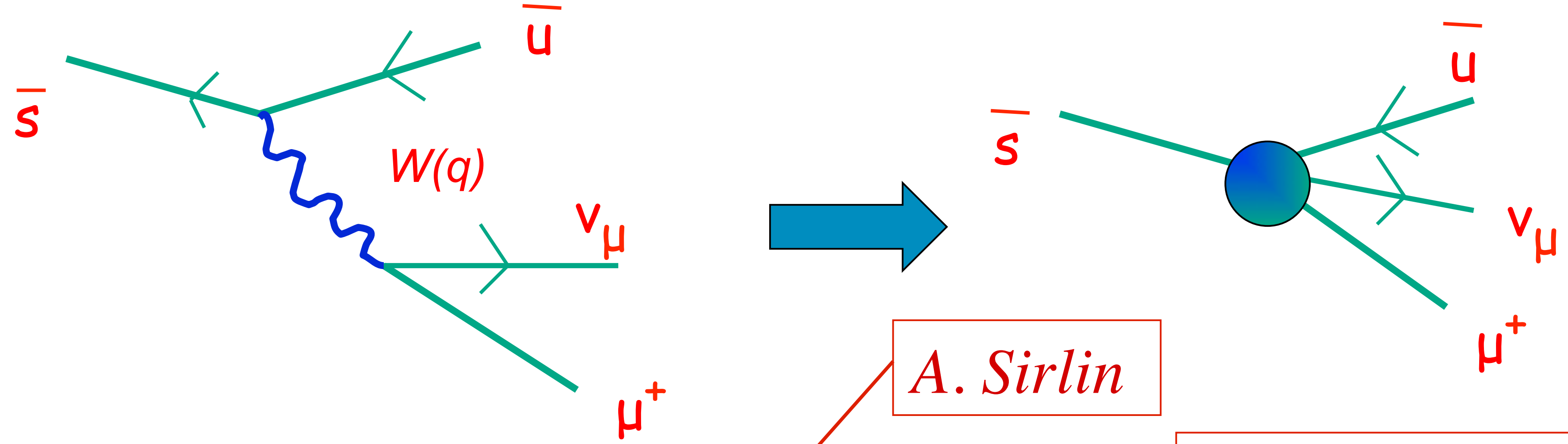
The Effective Hamiltonian



$q \sim m_K \ll M_W$

$$\mathcal{H}_F = -\frac{G_F}{\sqrt{2}} V_{us}^{*} \left(\bar{s}\gamma_\rho \left(1-\gamma_5\right) u\right) \left(\bar{\nu}_\mu \gamma^\rho \left(1-\gamma_5\right) \mu\right)$$

The Effective Hamiltonian



A. Sirlin

$$\mathcal{H}_W = \frac{G_F}{\sqrt{2}} V^*_{q_1 q_2} \left[1 + \frac{\alpha_{\rm em}}{\pi} \log\left(\frac{M_Z}{M_W}\right)\right] O_1^{\text{W-reg}}(M_W)$$

matching of the W-regularization to the SM

$$O_1 = (\bar{s}\gamma_\rho (1-\gamma_5)\, u)\, (\bar{\nu}_\mu \gamma^\rho (1-\gamma_5)\, \mu)$$

$O_1^{\text{W-reg}}(M_W)$: *is the operator O_1 renormalised in the W-scheme; its matrix elements are finite*

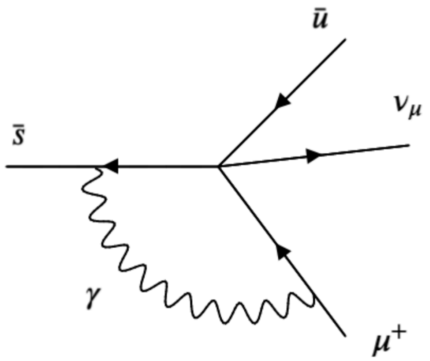
The Effective Hamiltonian

$$\mathcal{H}_W = \frac{G_F}{\sqrt{2}} V^*_{q_1 q_2} \left[1 + \frac{\alpha_{\rm em}}{\pi} \log\left(\frac{M_Z}{M_W}\right)\right] O_1^{\text{W-reg}}(M_W)\,,$$

matching of the W-regularization to the SM

$$O_1 = (\bar{s}\gamma_\rho (1-\gamma_5)\, u)\,(\bar{\nu}_\mu \gamma^\rho (1-\gamma_5)\,\mu)$$

to obtain O_1 in the W-regularisation write the remaining Feynman diagrams and compute the correction using dimensional regularisation



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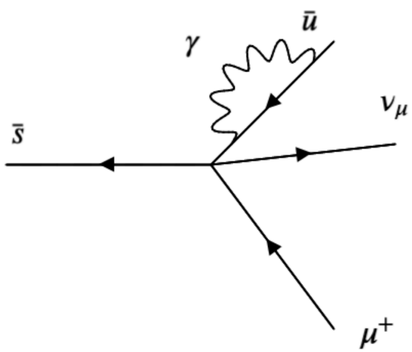

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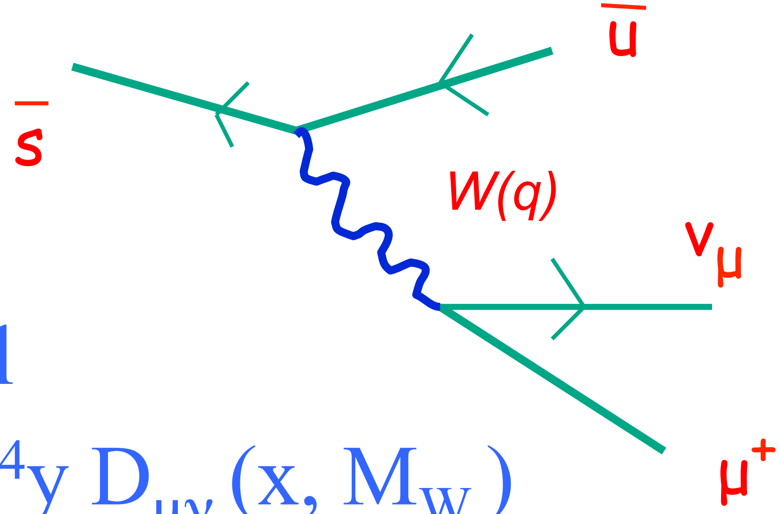
$$\frac{1}{(k^2)} = \frac{1}{(k^2)} - \frac{1}{(k^2 - M_W^2)} + \frac{1}{(k^2 - M_W^2)} = \frac{1}{(k^2 - M_W^2)} + \frac{M_W^2}{k^2(M_W^2 - k^2)}$$

GENERAL FRAMEWORK: THE OPE

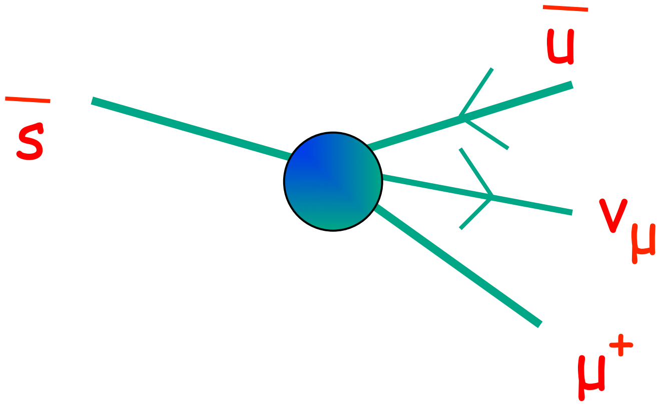


Standard Model

$$A_{FI}\,(2\pi^4)\,\delta^4\,(p_F - p_I) = \int d^4x\, d^4y\, D_{\mu\nu}(x, M_W)$$

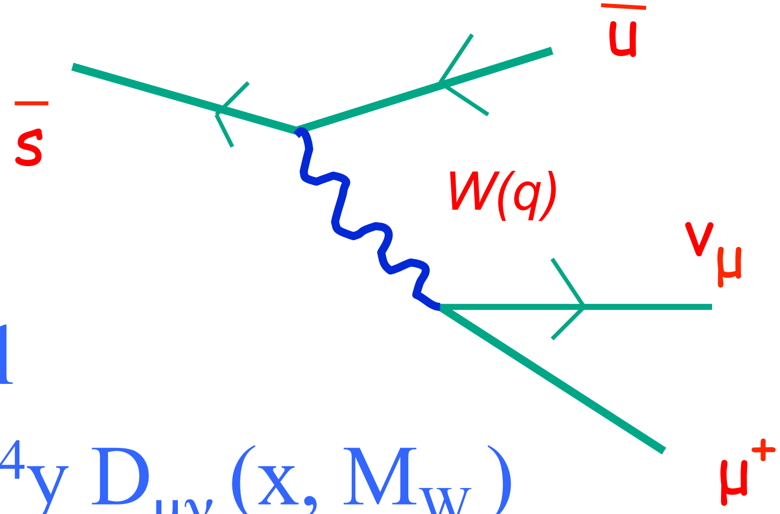
$$\langle F\,|T[\,J_\mu(y+x/2)\,J^\dagger_\nu(y-x/2)]\,|\,I\,\rangle \;\Rightarrow$$

$$\langle F\,|\,H^{\Delta S=1}\,|\,I\,\rangle = G_F/\sqrt{2}\;\; V_{us}^{*}\,\Sigma_i\, C_i(\mu)\,\frac{\langle F\,|\,Q_i(\mu)\,|\,I\,\rangle}{(M_W)^{di-6}}$$



di= dimension of the operator $Q_i(\mu)$
$C_i(\mu)$ Wilson coefficients, they depend on M_W/μ and $\alpha_s(\mu)$
$Q_i(\mu)$ local operators renormalized at the scale μ

GENERAL FRAMEWORK: THE OPE

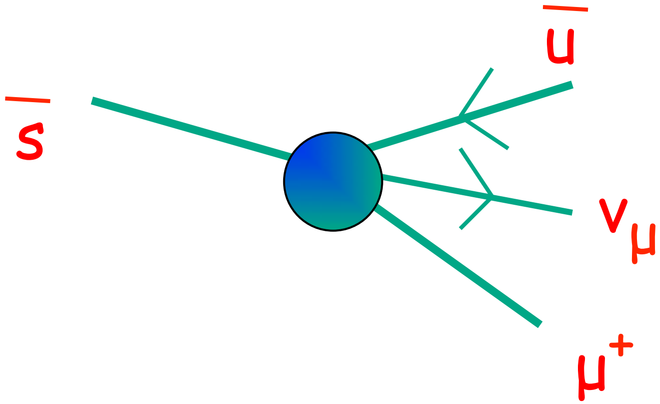


Standard Model

$$A_{FI}\ (2\pi^4)\ \delta^4\ (p_F - p_I) = \int d^4x\ d^4y\ D_{\mu\nu}(x, M_W)$$

$$\langle F\ |T[\ J_\mu (y+x/2)\ J^\dagger_\nu (y-x/2)]\ |\ I \rangle \Rightarrow$$

$$\langle F\ |\ H^{\Delta S=1}\ |\ I \rangle = G_F/\sqrt{2}\ \ V_{us}^*\ \Sigma_i\, C_i(\mu)\ \frac{\langle F\ |\ Q_i(\mu)\ |\ I \rangle}{(M_W)^{di-6}}$$



matching is an operator relations it does not depend on the external states

GENERAL FRAMEWORK

$$\mathcal{H}^{\Delta S=1} = G_F/\sqrt{2}\, \Sigma_{i=1,10} [\, (1-\tau)\, \Sigma_{i=1,2}\, z_i\, (Q_i - Q^c_i) + \tau\, (z_i + y_i)\, Q_i \,]$$

Where y_i and z_i are short distance coefficients, which are known
In perturbation theory at the NLO (Buras et al. + Ciuchini et al.)

$$\tau = -V_{ts}^* V_{td} / V_{us}^* V_{ud}$$

We have to compute $\langle F | Q_i | I \rangle$, e.g.
$\langle (\pi\pi)_{I=0,2} | Q_i | K \rangle$ with a non perturbative technique
(lattice, QCD sum rules, 1/N expansion etc.)

*The effective Hamiltonian in terms of bare lattice operators**

$$\mathcal{A}_i(\mu) = \langle F | Q_i(\mu) | I \rangle$$
$$= Z_{ik}(\mu a) \langle F | Q_k(a) | I \rangle$$

where $Q_k(a)$ is the bare lattice operator and
a the lattice spacing.

The effective Hamiltonian can then be read as:

$$\langle F | H^{\Delta S=1} | I \rangle = G_F/\sqrt{2} V_{ud} V_{us}^{*} \Sigma_i C_i(1/a) \langle F | Q_i(a) | I \rangle$$

In practice the renormalization scale (or $1/a$) are the scales which separate short and long distance dynamics

power divergences require a special treatment (R. Sommer)

The Effective Hamiltonian

$$\mathcal{H}_W = \frac{G_F}{\sqrt{2}} V^*_{q_1 q_2} \left[1 + \frac{\alpha_{\rm em}}{\pi} \log\left(\frac{M_Z}{M_W}\right)\right] O_1^{\text{W-reg}}(M_W)\,,$$

matching of the W-regularization to the SM

$$O_1 = (\bar{s}\gamma_\rho (1-\gamma_5)\, u)\, (\bar{\nu}_\mu \gamma^\rho (1-\gamma_5)\, \mu)$$

under strong interactions the quark vector and axial vector currents appearing in O_1 do not renormalise (they are conserved in the massless theory) .

with a (lattice) regularisation that breaks vector and axial vector symmetries we have to use currents renormalised by finite constants

$$O_1 = (V_\rho - A_\rho)\, (\bar{\nu}_\mu \gamma^\rho (1-\gamma_5)\, \mu) \quad \text{where} \quad V_\rho = Z_V\, \bar{s}\gamma_\rho\, u \quad A_\rho = Z_A \bar{s}\gamma_\rho\gamma_5\, u$$

W Regularization in perturbation theory

$$H_{\rm eff} = -\frac{G_F}{\sqrt{2}} V_{ud}^* \left(1 + \frac{\alpha}{\pi} \log \frac{M_Z}{M_W}\right) (\bar{d}\gamma^\mu(1-\gamma^5)u)\,(\bar{\nu}_\ell\gamma_\mu(1-\gamma^5)\ell)$$

matching the (Wilson) lattice to the W-regularization.

$$O_1^{\rm W-reg} = \left(1 + \frac{\alpha}{4\pi}\left(2\log a^2 M_W^2 - 15.539\right)\right) O_1^{\rm bare} + \frac{\alpha}{4\pi}\left(0.536\, O_2^{\rm bare} + 1.607\, O_3^{\rm bare} - 3.214\, O_4^{\rm bare} - 0.804\, O_5^{\rm bare}\right)$$

where

$$O_1 = (\bar{d}\gamma^\mu(1-\gamma^5)u)\,(\bar{\nu}_\ell\gamma_\mu(1-\gamma^5)\ell) \qquad O_2 = (\bar{d}\gamma^\mu(1+\gamma^5)u)\,(\bar{\nu}_\ell\gamma_\mu(1-\gamma^5)\ell)$$

$$O_3 = (\bar{d}(1-\gamma^5)u)\,(\bar{\nu}_\ell(1+\gamma^5)\ell) \qquad O_4 = (\bar{d}(1+\gamma^5)u)\,(\bar{\nu}_\ell(1+\gamma^5)\ell)$$

$$O_5 = (\bar{d}\sigma^{\mu\nu}(1+\gamma^5)u)\,(\bar{\nu}_\ell\sigma_{\mu\nu}(1+\gamma^5)\ell)\,.$$

GENERAL FRAMEWORK

$$\langle\, \mathcal{H}^{\Delta S=1} \,\rangle = G_F/\sqrt{2}\; V_{ud}\; V_{us}^{*} \;\ldots\; \Sigma_i\, C_i\,(\mathbf{a}) \,\langle\, Q_i(\mathbf{a}) \,\rangle$$

$M_W = 100$ GeV

perturbative regime

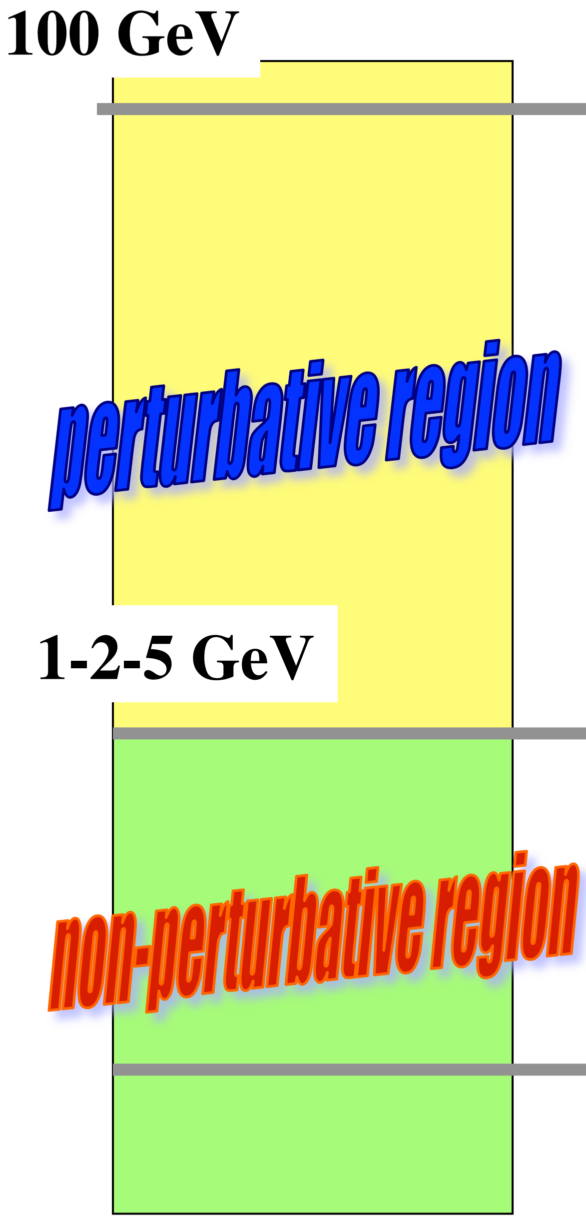
Effective Theory - quark & gluons

$a^{-1} = 2\text{-}5$ GeV

Hadronic non-perturbative region

chiral regime

$\Lambda_{QCD}\,, M_K = 0.2\text{-}0.5$ GeV



Large mass scale: heavy degrees of freedom (m_t , $M_{W,}$ M_s) are removed and their effect included in the Wilson coefficients

renormalizazion scale μ (inverse lattice spacing 1/a); this is the scale where the quark theory is matched to the effective hadronic theory

Scale of the low energy process
$\Lambda \sim M_\pi$

THE SCALE PROBLEM: Effective theories prefer low scales,
Perturbation Theory prefers large scales

if the scale μ is too low
problems from higher dimensional operators

(Cirigliano, Donoghue, Golowich)

- it is illusory to think that the problem is solved by using dimensional regularization

on the lattice this problem is called

DISCRETIZATION ERRORS

(reduced by using improved actions and/or scales $\mu > 2\text{-}4$ GeV

The Effective Hamiltonian for leptonic and semileptonic decays: Radiative corrections to the physical rates

$$\mathcal{H}_W = \frac{G_F}{\sqrt{2}} V^*_{q_1 q_2} \left[1 + \frac{\alpha_{em}}{\pi} \log\left(\frac{M_Z}{M_W}\right) \right] O_1^{\text{W-reg}}(M_W)$$

matching of the W-regularization to the SM

We have the renormalised, finite Hamiltonian expressed in terms of the lattice operators.

Now we have to compute the corrections to the physical rate.

Leptonic {
- *point-like real photon emission*
- *non-perturbative real photon emission*

Semileptonic

Electromagnetic Corrections to Decay Amplitudes:

Leptonic Decays

RM123 Collaboration:A Desidero, G de Divitiis, M Garofalo, M Hansen, R Frezzotti, N Tantalo, M di Carlo, D Giusti, V Lubicz, GM, F Mazzetti, F Sanfilippo, S Simula, C Tarantino & C Sachrajda

Physical quantities like hadron masses and decay rates are infrared finite. However infrared divergences can arise in the intermediate steps.

This is at the origin of some problems in the calculation of the radiative corrections to the decays rates

How to solve the problem of the infrared divergences discussed through an explicit example

$$\pi \to \ell + \nu_\ell + (\gamma)$$

NOTE: Chiral Perturbation Theory is NOT Used

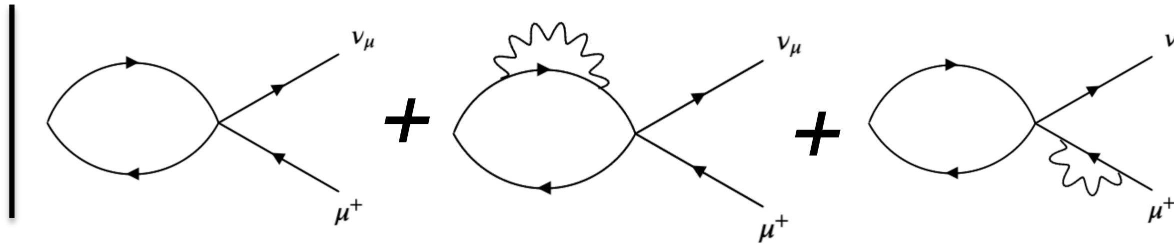
$$K \to \ell + \nu_\ell + (\gamma)$$

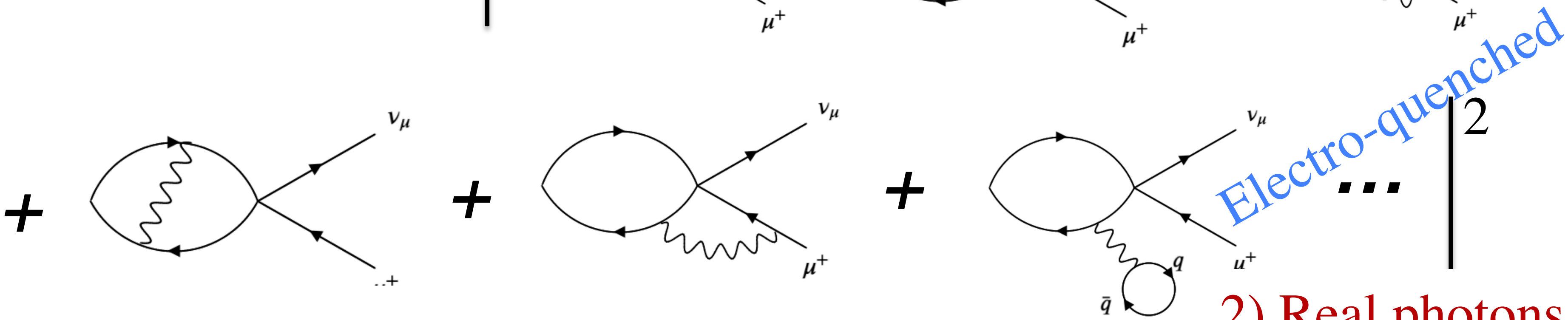
Rate at O(α)

$$\Gamma(\Delta E) = \int_0^{\Delta E} dE_\gamma \frac{d\Gamma}{dE_\gamma} = \Gamma_0 + \Gamma_1(\Delta E)$$

1) Virtual photons

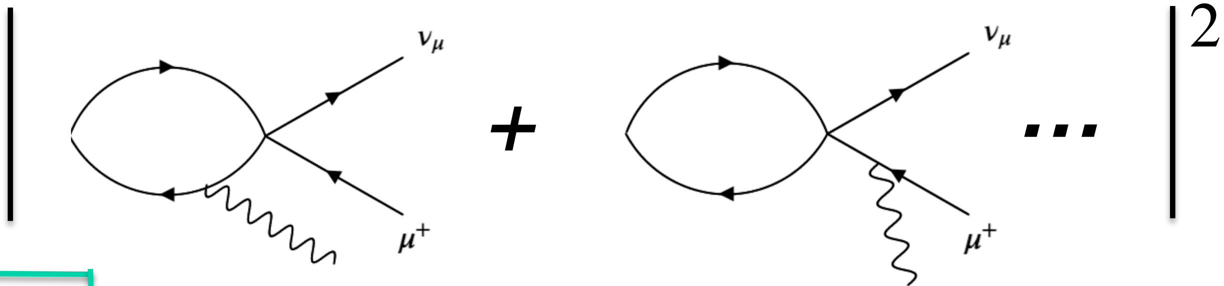
$$\Gamma(\Delta E) = \int d\Omega_2 \, \Big| \cdots \Big|^2$$



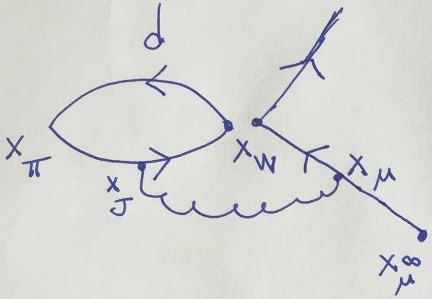


2) Real photons

$$+ \int_{E_\gamma \leq \Delta E} d\Omega_3 \, \Big| \cdots \Big|^2$$



$d\Omega_{2,3} = 2-3$ body phase $-$ space www.ictp.it



$\pi^+ \to \mu^+ \nu_\mu$

$$-\sum_{\vec{x}_\pi, x_J, x_\mu, \vec{x}_\mu^\infty, \vec{x}_\nu} \mathrm{tr}\left(\Gamma_\pi S_d(x_\pi, x_W)\gamma_L^\rho S_u(x_W, x_J)\gamma^\sigma S_u(x_J, x_\pi)\right)$$

$$\cdot\ \phi_\sigma(x_J) \quad e^{-i\vec{p}_\nu\cdot\vec{x}_\nu} S(x_\nu, x_W)\gamma_\rho^L S_\mu(x_W, x_\mu)\gamma^\lambda S(x_\mu - x_\mu^\infty)$$

$$\phi_\lambda(x_\mu) \quad e^{i\vec{p}_\mu\cdot\vec{x}_\mu^\infty}$$

$$\sum_{\vec{x}_\mu^\infty} S(x_\mu - x_\mu^\infty)\, e^{i\vec{p}_\mu\cdot\vec{x}_\mu^\infty} = \int\frac{dq_0}{(2\pi)}\int\frac{d^3q}{(2\pi)^3}\, e^{iq_0(t_\mu - t_\mu^\infty)}$$

$$e^{i\vec{q}\cdot\vec{x}_\mu} \quad \frac{e^{i(\vec{p}_\mu - \vec{q}_\mu)\vec{x}_\mu^\infty}}{q_0^2 + \vec{q}^{\,2} + m_\mu^2}$$

$$= \int\frac{dq_0}{2\pi}\, e^{i\vec{p}_\mu\cdot\vec{x}_\mu} \frac{1}{(q_0^2 + \vec{p}_\mu^{\,2} + m_\mu^2)}\, e^{iq_0(t_\mu - t_\mu^\infty)} \to e^{E_\mu t_\mu} \times e^{i\vec{p}_\mu\cdot\vec{x}_\mu} \times \left(\frac{e^{-E_\mu t_\mu^\infty}}{2E_\mu}\right)$$

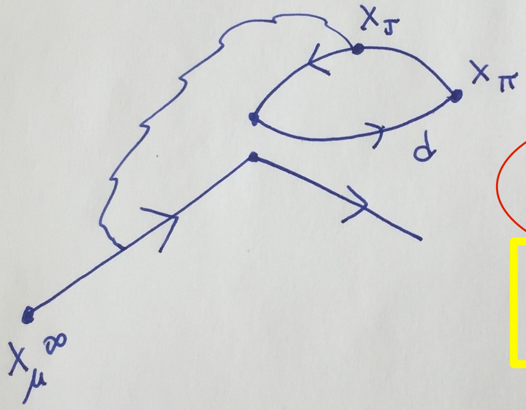
$$- \sum_{\vec{x}_\pi, x_J, x_\mu} e^{-i\vec{p}_\mu \cdot \vec{x}_w + E_\nu t_w} \; t_2 \, \Gamma_\pi S_d(x_\pi, x_w) \gamma_L^\rho S_u(x_w, x_J) \gamma^\sigma S_u(x_J, x_\pi)$$

$$\phi_\sigma(x_J) \; e^{i\vec{p}_\mu \cdot \vec{x}_\mu + E_\mu t_\mu}$$

$$\bar{u}(p_\nu = -p_\mu) \gamma_\rho^L S_\mu(x_w - x_\mu) \gamma^\lambda v(p_\mu) \phi_\lambda(x_\mu)$$

---

③



$$\mu^- \to \pi^- + \nu_\mu$$

τ

$$- \; t_2 \, \Gamma_\pi S_d(x_\pi, x_w) \gamma_L^\rho S_u(x_w, x_J) \gamma^\sigma S_u(x_J, x_\pi) \, e^{i\vec{p}_\nu \cdot (\vec{x}_\pi - \vec{x}_w)} \phi_\sigma(x_J)$$

$$\bar{u}(p_\nu) \gamma_\rho^L S_\mu(x_w - x_\mu) \gamma^\lambda u(p_\mu = p_\mu^0 = m_\mu) \, \phi_\lambda(x_\mu) \, e^{-m_\mu t_\mu}$$

Leptonic decays at tree level

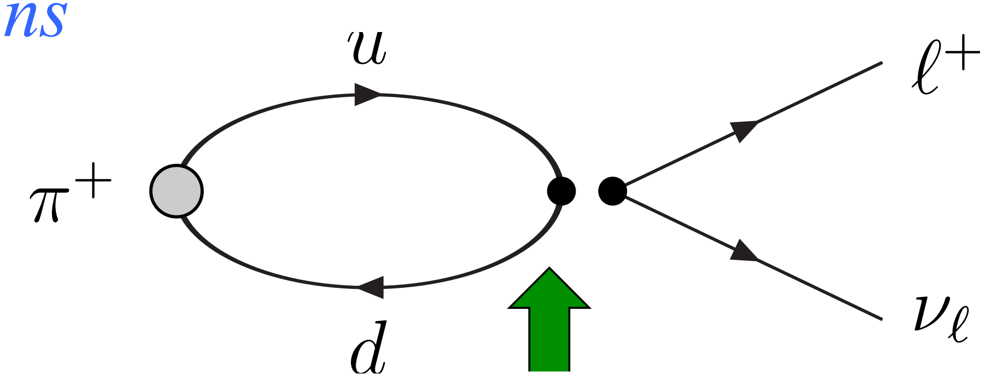
Since the mass of the pion is much lower than M_W we use the effective Hamiltonian

$$H_{\rm eff} = -\frac{G_F}{\sqrt{2}} V^*_{ud} (\bar{d}\gamma^\mu(1-\gamma^5)u)\,(\bar{\nu}_\ell\gamma_\mu(1-\gamma^5)\ell)$$

from which we compute

$$\Gamma_0^{\rm tree}(\pi^+\to\ell^+\nu_\ell) = \frac{G_F^2\,|V_{ud}|^2 f_\pi^2}{8\pi}\, m_\pi\, m_\ell^2 \left(1-\frac{m_\ell^2}{m_\pi^2}\right)^2$$

- *0 in Γ_0 means zero photons*
- *G_F is the Fermi constant defined from μ decay*
- *f_π is computed in lattice QCD*



Lectures by C. Sachrajda

Leptonic decays at O(α) – The ultraviolet matching in the ``W Regularization"

If G_F is the Fermi constant defined at O(α) from μ decay in the standard (convention dependent) way

$$\frac{1}{\tau_\mu} = \frac{G_F^2 m_\mu^5}{192\pi^3}\left[1-\frac{8m_e^2}{m_\mu^2}\right]\left[1+\frac{\alpha}{2\pi}\left(\frac{25}{4}-\pi^2\right)\right]$$

S.M.Berman, PR 112 (1958) 267; T.Kinoshita and A.Sirlin, PR 113 (1959) 1652

then the effective Hamiltonian in the W-regularization is given by (Sirlin PRD 22 (80) 971)

$$H_{\rm eff} = -\frac{G_F}{\sqrt{2}}\,V_{ud}^*\left(1+\frac{\alpha}{\pi}\log\frac{M_Z}{M_W}\right)(\bar d\gamma^\mu(1-\gamma^5)u)\,(\bar\nu_\ell\gamma_\mu(1-\gamma^5)\ell)$$

matching the (Wilson) lattice to the W-regularization.

$$O_1^{\rm W-reg} = \left(1+\frac{\alpha}{4\pi}\left(2\log a^2M_W^2-15.539\right)\right)O_1^{\rm bare}+\frac{\alpha}{4\pi}\left(0.536\,O_2^{\rm bare}+1.607\,O_3^{\rm bare}-3.214\,O_4^{\rm bare}-0.804\,O_5^{\rm bare}\right)$$

Rate at O(α) $$\Gamma(\Delta E) = \Gamma_0 + \Gamma_1(\Delta E)$$

$|V_{ud}|$

where $$\Gamma(\Delta E) = \int_0^{\Delta E} dE_\gamma \frac{d\Gamma}{dE_\gamma}$$

contrary to the hadron masses

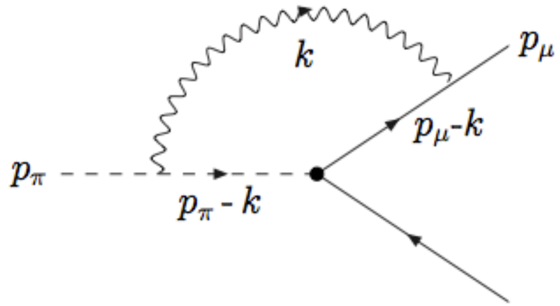
at O(α) both Γ_0 and $\Gamma_1(\Delta E)$ are INFRARED DIVERGENT

although the divergence cancel in the sum

F. Bloch, A. Nordsieck Phys.Rev. 52 (1937) T.D. Lee, M. Nauenberg Phys.Rev. 133 (1964)

and the infinite volume limit cannot be separately taken

Courtesy of C. Sachrajda



$$I \sim \int_{\text{small } k} d^4k \, \frac{1}{(k^2+i\epsilon)((p_\mu-k)^2-m_\mu^2+i\epsilon)((p_\pi-k)^2-m_\pi^2+i\epsilon)}$$
$$\sim \int_{\text{small } k} d^4k \, \frac{1}{k^2(-2p_\mu\cdot k)(-2p_\pi\cdot k)}$$
$$\sim \int_{\text{small } k} d^4k \, \frac{1}{k^4} \quad \Rightarrow \text{infrared divergence}\,.$$

- This leads to a contribution to Γ_0 of

$$\Gamma_0^{\pi\mu} = \Gamma_0^{\text{tree}} \frac{\alpha}{4\pi}\left(\frac{2(1+r_\mu^2)}{1-r_\mu^2}\log r_\mu^2 \log\left(\frac{m_\pi^2}{m_\gamma^2}\right)+\cdots\right),$$

where the photon mass, m_γ, is introduced to regulate the infrared divergences and $r_\mu = m_\mu/m_\pi$.

In a first paper it was proposed to compute $\Gamma_1(\Delta E)$ in perturbation theory @ values of ΔE corresponding to photons which are sufficiently soft for the point-like approximation of the pion to be valid

$(\Delta E << \Lambda_{QCD} \approx 4\pi f_\pi)$

but hard enough with respect to the experimental resolution.

A value of O(10-20 MeV) seems to be appropriate both theoretically and experimentally.

F. Ambrosino et al., KLOE Collaboration,
PLB 632 (2006) 76; EPJC 64 (2009) 627; 65 (2010) 703(E);
J. Bijnens, G. Ecker, J. Gasser, NPB 396 (1993) 81; V.Cirigliano, I.Rosell, JHEP 0710 (2007) 005

In the future, as techniques and resources improved, it will possible (and certainly appropriate for heavy mesons) to compute $\Gamma_1(\Delta E)$ nonperturbatively over a larger range of photon energies *(about the analytical continuation to the Euclidean see later). See last lecture*

NOTE: we do not use chiral perturbation theory !!

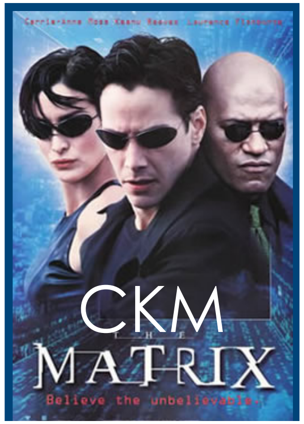
MASTER FORMULA for the rate at O(α)

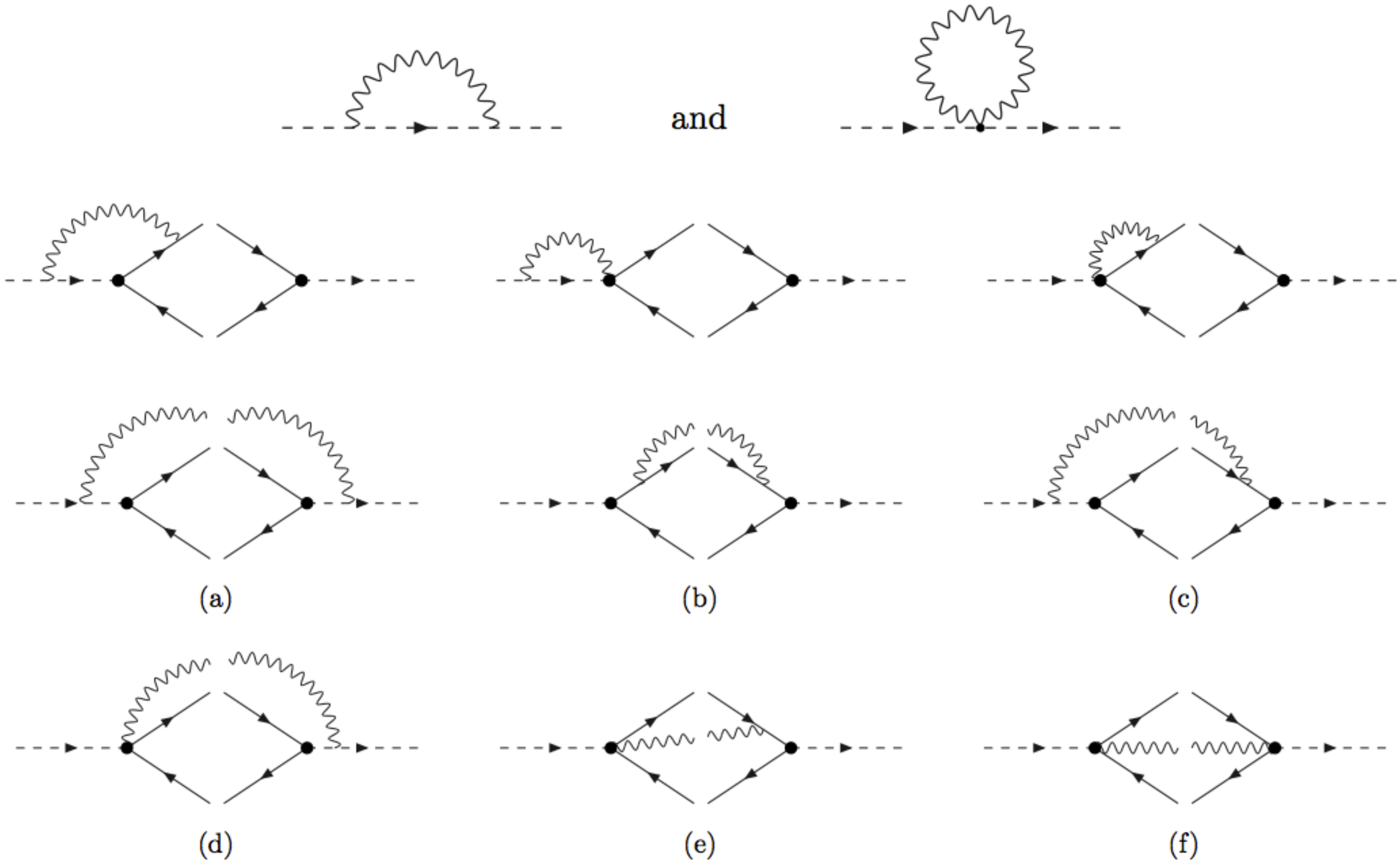
$$\Gamma(\Delta E) = \lim_{V\to\infty} (\Gamma_0 - \Gamma_0^{pt}) + \lim_{V\to\infty} (\Gamma_0^{pt} + \Gamma_1(\Delta E))$$

pt = point-like & perturbative

- the infrared divergences in Γ_0 and Γ_0^{pt} are exactly the same and cancel in the difference
- $\Gamma'(\Delta E) = \Gamma_0^{pt} + \Gamma_1(\Delta E)$ is infrared finite since is a physical, well defined quantity *F. Bloch, A. Nordsieck Phys.Rev. 52 (1937) T.D. Lee, M. Nauenberg Phys.Rev. 133 (1964)*
- the infrared divergences in $\Delta\Gamma_0(L) = \Gamma_0 - \Gamma_0^{pt}$ and $\Gamma(\Delta E) = \Gamma_0^{pt} + \Gamma_1(\Delta E)$ cancel separately hence they can be regulated with different infrared cutoff
- Γ_0 and Γ_0^{pt} are also ultraviolet finite

We now discuss the two terms, $\Delta\Gamma_0(L)$ and $\Gamma'(\Delta E)$




and
(a)
(b)
(c)
(d)
(e)
(f)

Leptonic decays at O(α) – Perturbative Calculation of $\Gamma(\Delta E) = \Gamma_0^{pt} + \Gamma_1(\Delta E)$

U.V. & Infrared finite but contains $\log(M_W)$ & $\log(\Delta E)$

$$\Gamma(\Delta E) = \Gamma_0^{\rm tree} \times \left(1 + \frac{\alpha}{4\pi}\left\{3\log\left(\frac{m_\pi^2}{M_W^2}\right) + \log\left(r_\ell^2\right) - 4\log(r_E^2) + \frac{2-10r_\ell^2}{1-r_\ell^2}\log(r_\ell^2)\right.\right.$$
$$-2\frac{1+r_\ell^2}{1-r_\ell^2}\log(r_E^2)\log(r_\ell^2) - 4\frac{1+r_\ell^2}{1-r_\ell^2}\,{\rm Li}_2(1-r_\ell^2) - 3$$
$$+\left[\frac{3+r_E^2-6r_\ell^2+4r_E(-1+r_\ell^2)}{(1-r_\ell^2)^2}\log(1-r_E) + \frac{r_E(4-r_E-4r_\ell^2)}{(1-r_\ell^2)^2}\log(r_\ell^2)\right.$$
$$\left.\left.\left.-\frac{r_E(-22+3r_E+28r_\ell^2)}{2(1-r_\ell^2)^2} - 4\frac{1+r_\ell^2}{1-r_\ell^2}\,{\rm Li}_2(r_E)\right]\right\}\right)$$

$$r_E = \frac{2\Delta E}{m_\pi} \qquad r_\ell = \frac{m_\ell}{m_\pi}$$

We think that this is a new result;

$\Gamma(\Delta E_1)$ *T.Kinoshita, PRL 2 (1959) 477*

Leptonic decays at O(α) – Perturbative Calculation of $\Gamma(\Delta E) = \Gamma_0^{pt} + \Gamma_1(\Delta E)$

- The total rate is readily computed by setting r_E to its maximum value, namely $r_E = 1 - r_\ell^2$, giving

$$\Gamma^{pt} = \Gamma_0^{tree} \times \left\{ 1 + \frac{\alpha}{4\pi} \left(3 \log\left(\frac{m_\pi^2}{M_W^2}\right) - 8 \log(1-r_\ell^2) - \frac{3r_\ell^4}{(1-r_\ell^2)^2} \log(r_\ell^2) \right.\right.$$
$$\left.\left. -8 \frac{1+r_\ell^2}{1-r_\ell^2} \mathrm{Li}_2(1-r_\ell^2) + \frac{13-19r_\ell^2}{2(1-r_\ell^2)} + \frac{6-14r_\ell^2-4(1+r_\ell^2)\log(1-r_\ell^2)}{1-r_\ell^2} \log(r_\ell^2) \right) \right\} .$$

- This result agrees with the well known results in literature providing an important check of our calculation.

Structure dependent contributions to the O(α) perturbative calculation of $\Gamma_1(\Delta E)$

1) For sufficiently small values of $\Delta E(/\Lambda_{QCD})$ the structure dependent contributions to $\Gamma_1(\Delta E)$ can be neglected

2) How big are they for experimentally accessible values of ΔE ? We can have an estimate from chiral perturbation theory (although not all LEC are available)

J.Bijnens, G.Ecker and J.Gasser, hep-ph/9209261, J.Bijnens, G.Colangelo, G.Ecker and J.Gasser, hep-ph/9411311. V. Cirigliano and I. Rosell, arXiv:0707.3439 [hep-ph]], L. Ametller, J. Bijnens, A. Bramon and F. Cornet, hep-ph/9302219.

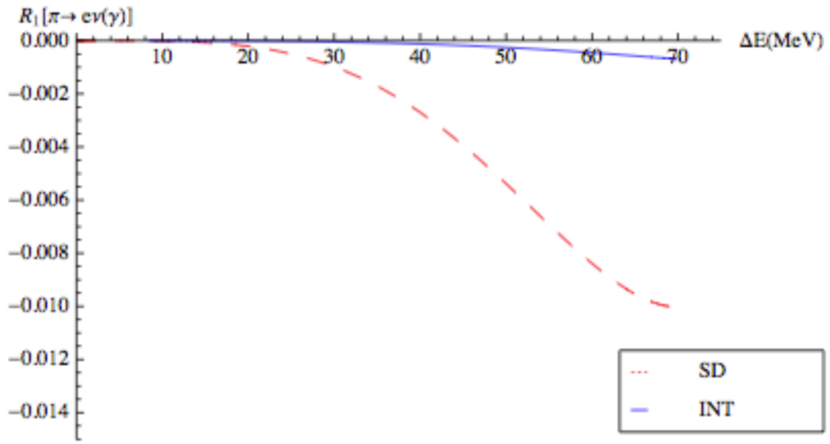
$$R_1^A(\Delta E) = \frac{\Gamma_1^A(\Delta E)}{\Gamma_0^{\alpha,\mathrm{pt}} + \Gamma_1^{\mathrm{pt}}(\Delta E)}\,, \qquad A = \{\mathrm{SD,INT}\}\,,$$

The structure dependent contributions to perturbative calculation of $\Gamma_1(\Delta E)$: the decay into an electron is the worse case !

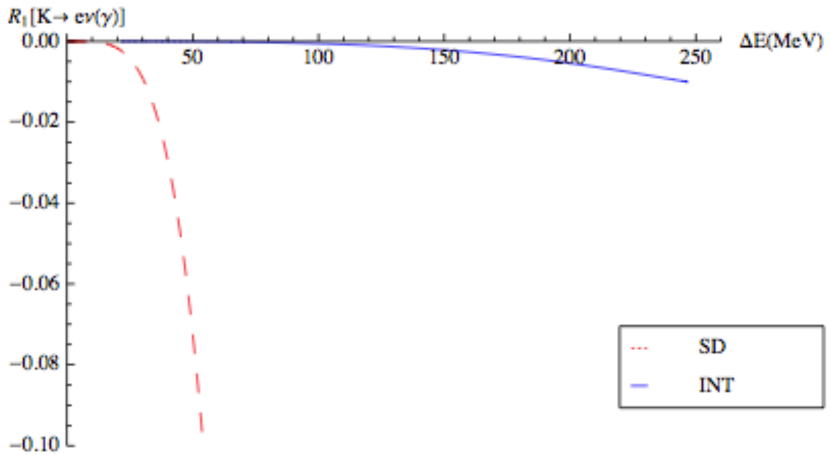
In the case of the decay in a muon the effect is of the $O(10^{-3}\text{-}10^{-7})$

In the case of B mesons, due to the small scale represented by $m_{B*} - m_B$, it is likely that it will be necessary to perform a full non-perturbative calculation of the real emission

Pion



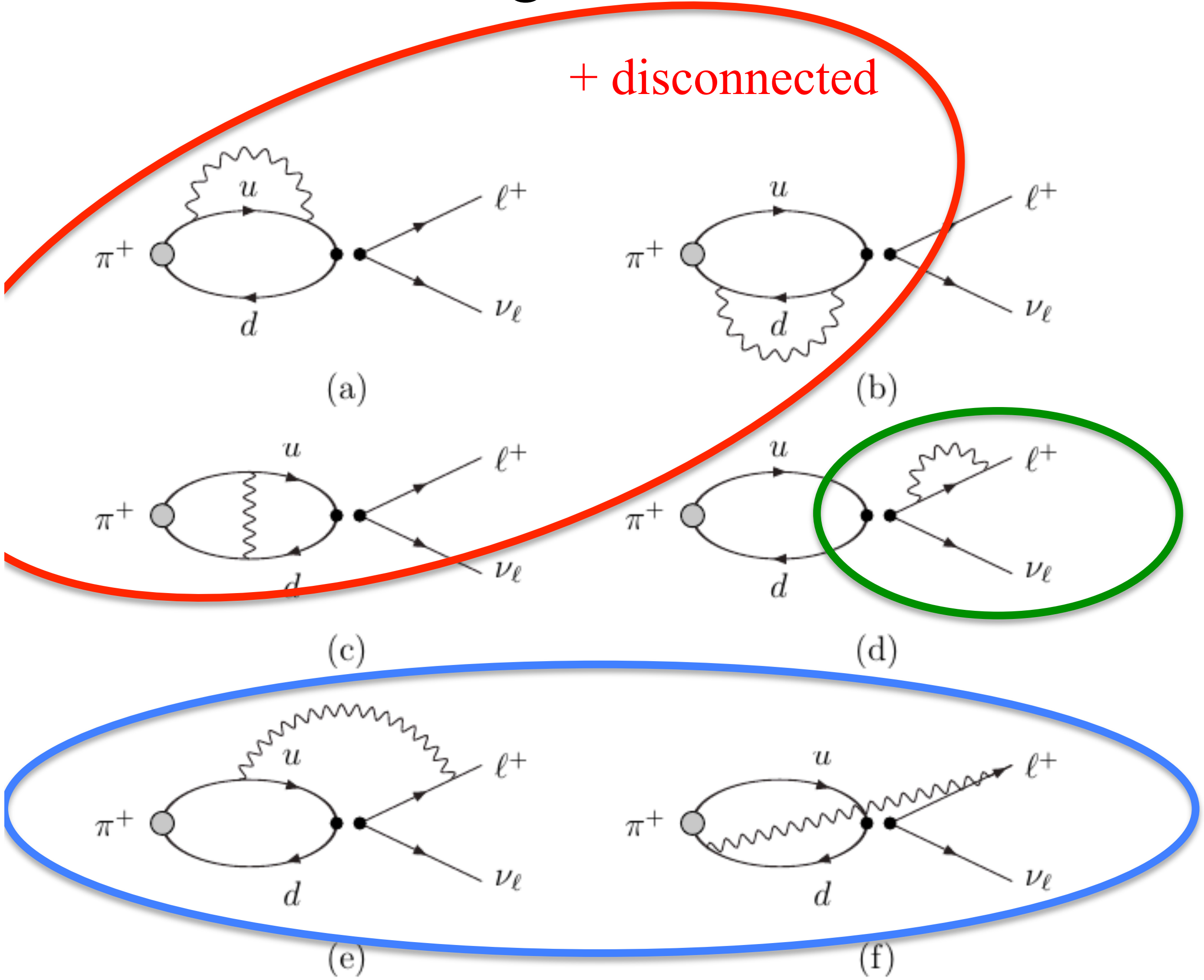
Kaon



D. Becirevic, B. Haas and E. Kou, arXiv:0907.1845 [hep-ph]

Leptonic decays at O(α) – The first term of the Master Formula $\Delta\Gamma(L) = \Gamma_0 - \Gamma_0^{pt}$

- Each of the two terms is U.V. finite but contains $\log(M_W)$
- Infrared divergences cancel in the difference

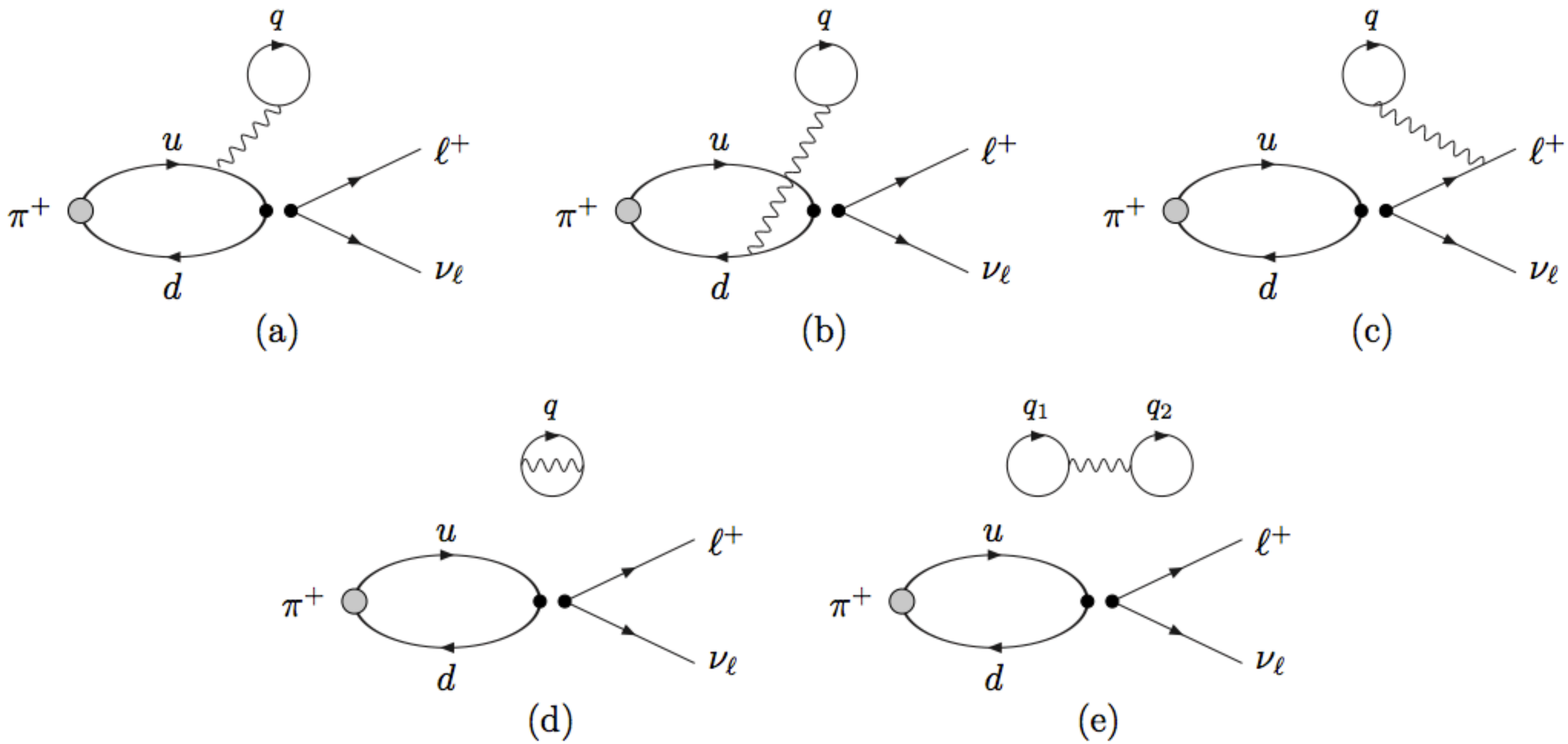


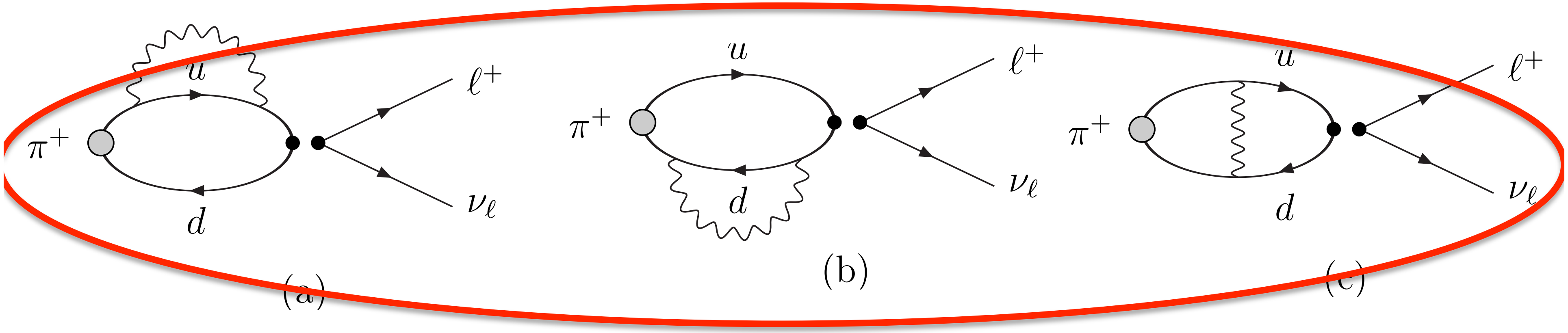
at this order we can take the difference of the amplitudes

Can be computed as discussed in arXiv: 1303.4896, Phys.Rev. D87(2013)

NOT by including the electromagnetic field in the action

DISCONNECTED DIAGRAMS





The relevant correlation function is (the lepton leg is trivial)

$$C_1(t) = \frac{1}{2}\int d^3\mathbf{x}\, d^4x_1\, d^4x_2\ \langle 0|T\{J_W^\nu(0)\, j^\mu(x_1) j_\mu(x_2)\phi^\dagger(\mathbf{x},t)\}\,|0\rangle\ \Delta(x_1,x_2)$$

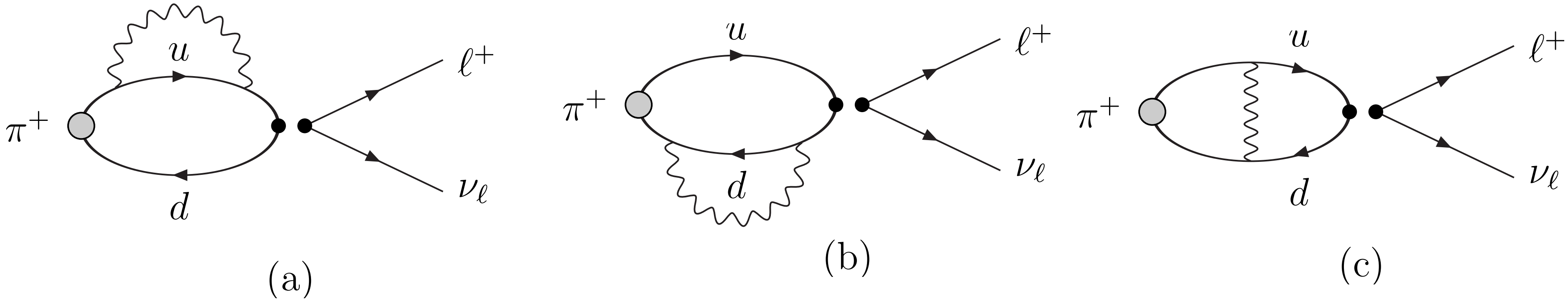
weak V-A current

electromagnetic current

$$j_\mu(x) = \sum_f Q_f\, \bar{f}(x)\gamma_\mu f(x)$$

this is the same set of diagrams used to compute the electromagnetic corrections to the pion (hadron) mass

(the lepton leg is completely irrelevant)



Combining $C_1(t)$ with the lowest order correlator

$$C_0(t) + C_1(t) \simeq \frac{e^{-m_\pi t}}{2m_\pi} Z^\phi \langle 0 | J_W^0(0) | \pi^+ \rangle$$

where the $O(\alpha)$ corrections are included; by writing

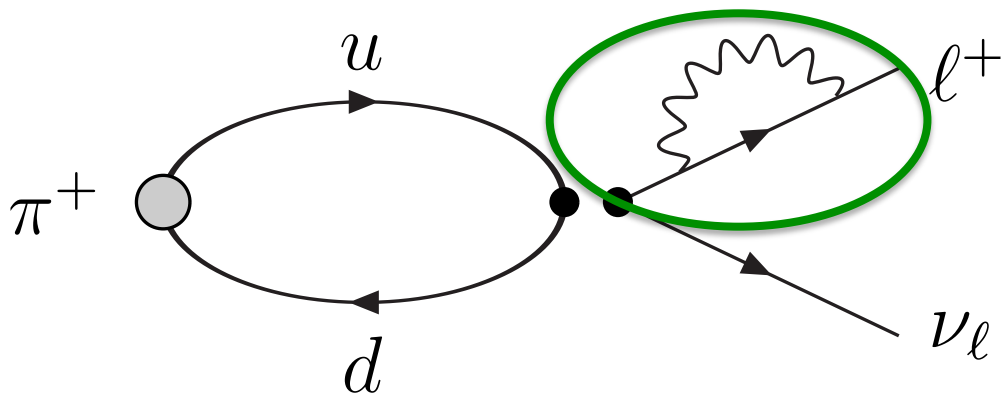
$$e^{-m_\pi t} \simeq e^{-m_\pi^0 t} (1 - \delta m_\pi t)$$

δm_π is infrared finite and gauge invariant

Z^ϕ and the matrix element of the axial current *however* are infrared divergent and cannot be interpreted as a correction to f_π

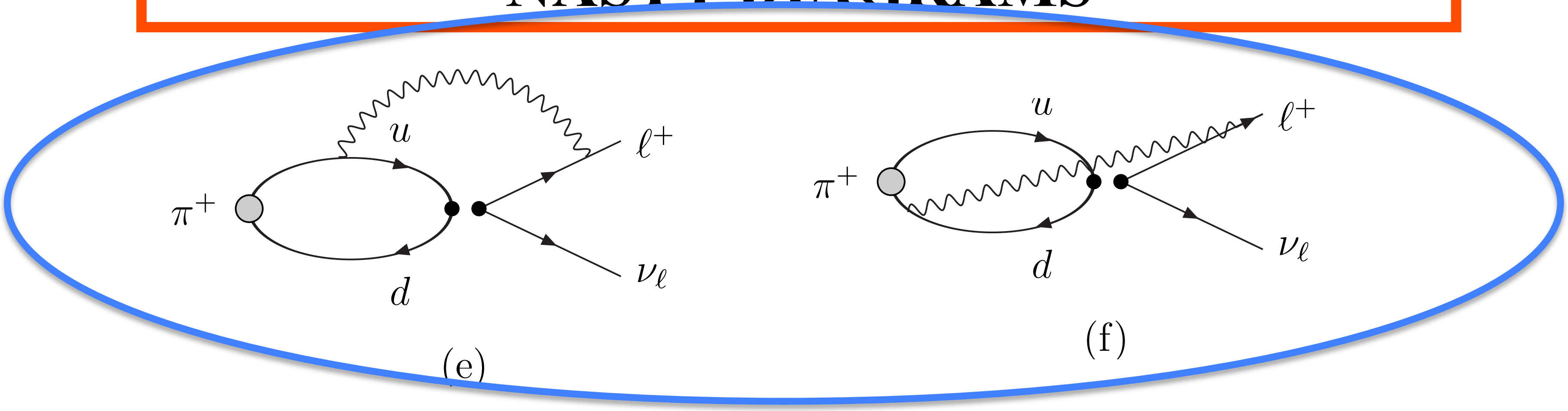
This diagram is an easy case: its contribution to $\Delta\Gamma(L) = \Gamma_0 - \Gamma_0^{pt}$ can be readily obtained in perturbation theory.

The recipe is simply to redefine the operator $O_1^{W\text{-}reg}$ and compute f_π in the numerical simulation



(d)

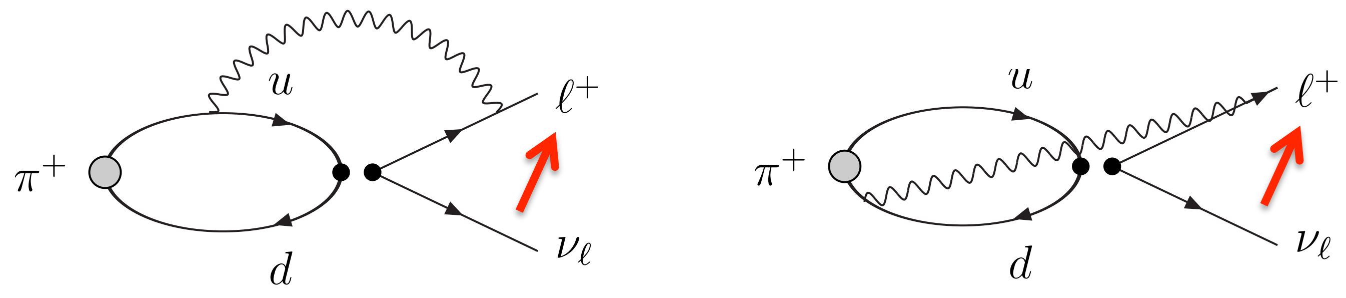
NASTY DIAGRAMS



- *Certainly these diagrams are not simply a generalization of the evaluation of f_π ; they are also infrared divergent)*
- *We have to isolate the finite volume ground state (necessity of a mass gap – Minkowski $\leftrightarrow$ Euclidean continuation* J. Gasser and G.R.S. Zarnauskas Phys. Lett. B 693 (2010) 122 *)*
- *Finite volume effects,* expected of the $O(1/L\,\Lambda_{QCD})$ after the cancellation of the infrared divergence, *should be investigated in a numerical simulation.*

This diagram does not contribute to the mass renormalisation

Calculation of the `nasty' diagrams in a lattice simulation



(e) (f)

The starting point is the Minkowski Green function

$$\int d^4x_1 d^4x_2 < 0|\, T(j_\mu(x_1) J_W^\nu(0))\, |\pi > \, iD_F(x_1-x_2)\{\bar{u}(p_{\nu_\ell})\gamma^\nu(1-\gamma^5)(iS_F(x_2))\gamma^\mu v(p_\ell)\} e^{ip_\ell\cdot x_2}$$

from which we can compute the on-shell amplitude

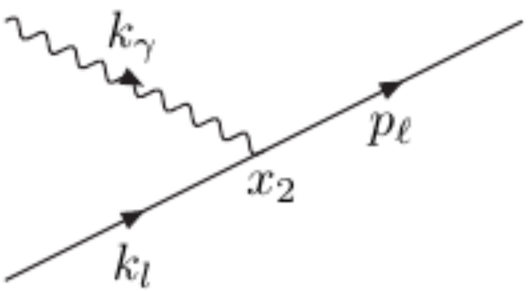
$$\bar{u}_\alpha(p_{\nu_\ell})(\bar{M}_1)_{\alpha\beta} v_\beta(p_\ell) = \quad -i \lim_{k_0\to m_\pi} ({k_0}^2 - m_\pi^2) \int d^4x_1 d^4x_2\, d^4x\, e^{-ik^0y^0} < 0|T(j_\mu(x_1)J_W^\nu(0)\pi(x))|0 >$$
$$\times iD_F(x_1-x_2)\{\bar{u}(p_{\nu_\ell})\gamma_\nu(1-\gamma^5)(iS_F(x_2))\gamma^\mu v(p_\ell)\} e^{ip_\ell\cdot x_2}$$

which in the Euclidean simulation becomes

$$\bar{C}_1(t)_{\alpha\beta} = \quad \int d^3\mathbf{x}\, d^4x_1\, d^4x_2\, \langle 0|T\{J_W^\nu(0)\, j_\mu(x_1)\phi^\dagger(\mathbf{x},t)\}|\,0\rangle\, \Delta(x_1-x_2)$$
$$\times \left(\gamma_\nu(1-\gamma^5)S(x_2)\gamma^\mu\right)_{\alpha\beta} e^{E_\ell\, t_2} e^{-i\mathbf{p}_\ell\cdot\mathbf{x}_2}$$

A few technical but non trivial IMPORTANT slides:
the continuation from Minkowski to Euclidean

we need to ensure that the t_2 integration up to ∞ converges in spite of the factor $e^{E_l t_2}$ where $E_l = \sqrt{m_l^2 + p_l^2}$ is the energy of the outgoing charged lepton



1) Momentum conservation: since we integrate over x_2
$p_l = k_l + k_\gamma$

2) The integrations over the energies k_{4l} and $k_{4\gamma}$ lead to the exponential factor $e^{-(\omega_l + \omega_\gamma - E_l)\, t_2}$ where $\omega_l = \sqrt{m_l^2 + k_l^2}$, $\omega_\gamma = \sqrt{m_\gamma^2 + k_\gamma^2}$, and m_γ is the mass of the photon introduced as an infra-red cut-off.

A few technical but non trivial IMPORTANT slides:
the continuation from Minkowski to Euclidean

3) … but $(\omega_l+\omega_\gamma) \geq \sqrt{(m_l+m_\gamma)^2 + p_l^2} > E_l = \sqrt{m_l^2 + p_l^2}$

thus the argument of the exponent $e^{-(\omega_l + \omega_\gamma - E_l)\, t_2}$ is negative for every term appearing in the sum over the intermediate states and the integral over t_2 converges

4) note that the integration over t_2 is also convergent if we set $m_\gamma=0$ but remove photon zero mode in finite volume. In this case $(\omega_l+\omega_\gamma) > E_l+[1-(p_l/E_l)]\ (k_\gamma)_{min}$

- necessity of a mass gap
- absence of a lighter intermediate state

under these conditions

$$\bar{C}_1(t)_{\alpha\beta} \simeq Z_0^{\phi}\,\frac{e^{-m_\pi^0|t|}}{2m_\pi^0}\,(\bar{M}_1)_{\alpha\beta}$$

and the contribution to the amplitude from these diagrams is given by

$$\bar{u}_\alpha(p_{\nu_\ell})(\bar{M}_1)_{\alpha\beta}v_\beta(p_\ell)$$

FINITE VOLUME CORRECTIONS

$$\Gamma(\Delta E) = \lim_{V\to\infty}\left(\Gamma_0 - \Gamma_0^{pt}\right) + \lim_{V\to\infty}\left(\Gamma_0^{pt} + \Gamma_1^{pt}(\Delta E)\right)$$

- $\Gamma_0^{pt}(L)$ is calculated in perturbation theory with a pointlike pion

- UV divergences are regularized with the W-regularization
- IR divergences are regularized by the finite volume (same of $\Gamma_0(L)$)

$$\Gamma_0^{pt}(L) = C_0(r_\ell) + \tilde{C}_0(r_\ell)\log(m_P L) + \frac{C_1(r_\ell)}{m_P L} + \dots$$

where $r_l = m_l/m_P$ and m_P and m_l are the masses of the pseudoscalar meson and the lepton respectively

Γ_0^{pt} Universality of the logarithmically divergent term and of the 1/L correction (Tantalo at Lattice 2016)

$$\Gamma_0^{\rm pt}(L) = C_0(r_\ell) + \tilde{C}_0(r_\ell)\mathrm{Log}\,[m_P L] + \frac{C_1(r_\ell)}{m_P L} + \ldots$$

Depends on the ir regularization. The regularization dependent part does not depend on the internal structure of the hadron

Does NOT depend on the ir regularization or on the internal structure of the hadron

BMW, Science 347 (2015) 1452
B. Lucini et al., JHEP 1602 (2016)

Thus $\Delta\Gamma(L) = \Gamma_0 - \Gamma_0^{pt}$ = Infrared finite, independent of the regularization up to $O(1/L^2)$

$$\Gamma_0^{\mathrm{pt}}(L) = C_0(r_\ell) + \tilde{C}_0(r_\ell)\log(m_P L) + \frac{C_1(r_\ell)}{m_P L} + \ldots$$

1) *The coefficients* $C_0(r_l)$, $\hat{C}_0(r_l)$ *and* $C_1(r_l)$ *are universal, although* $C_0(r_l)$ *and* $C_1(r_l)$ *depend on the infrared regulator (Ward Ids – highly non trivial)*
2) $\hat{C}_0(r_l)$ *is universal and does not depend on the regularisation.*
3) $C_0(r_l)$, $\hat{C}_0(r_l)$ *and* $C_1(r_l)$ *cancel the corresponding terms contained in* $\Gamma_0(L)$*. In this way* $\Gamma_0(L)-\Gamma^{pt}_0(L)$ *is infrared finite and independent of the infrared regularisation up to terms of* $O(1/L^2)$

for the generalization to semileptonic or other cases see later

$$\Gamma_0^{\mathrm{pt}}(L) = \Gamma_0^{\mathrm{tree}} \left\{ 1 + 2\frac{\alpha}{4\pi} Y(L) \right\}$$

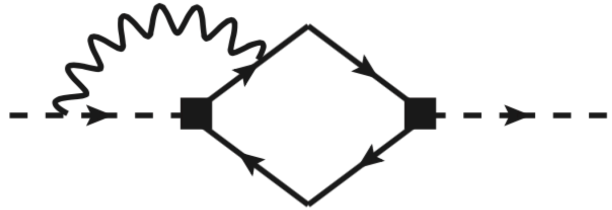
$$\begin{aligned} Y(L) \;=\; & \left(1 + r_\ell^2\right) \left[2(K_{31} + K_{32}) + \frac{\left(\gamma_E + \log\left[\frac{L^2 m_P^2}{4\pi}\right]\right) \log\left[r_\ell^2\right]}{(1 - r_\ell^2)} + \frac{\log^2\left[r_\ell^2\right]}{2\left(1 - r_\ell^2\right)} \right] + \\ & + \frac{(1 - 3\,r_\ell^2)\,\log\left[r_\ell^2\right]}{(1 - r_\ell^2)} - \log\left[\frac{M_W^2}{m_P^2}\right] + \log[m_P^2 L^2] - \frac{1}{2} K_P + \frac{1}{12} + \\ & + \frac{1}{m_P L} \left(\frac{2 r_\ell^2}{1 - r_\ell^2} \left(K_{21} + K_{22} - 2\pi \left(\frac{1}{1 + r_\ell^2} + \frac{1}{r_\ell} \right) \right) - \frac{\pi(1 + r_\ell^2)}{(1 - r_\ell^2)} (K_{11} + K_{12} - 3) \right) \end{aligned}$$

K_{ij} are suitable constants that can be easily computed numerically

Γ_0^{pt} ***The nasty diagram***

sum vs integral under study

A NEW STRATEGY



$$\Gamma = \int \frac{dq_0}{2\pi} \frac{1}{L^d} \sum_{\vec{q}=2\pi/L(n1,n2,\ldots,n_d)}' \frac{1}{(q^2+\Delta^2)\left((p-q)^2+m^2+\Delta^2\right)\left((p_\mu-q)^2+m_\mu^2+\Delta^2\right)}$$

$$\Gamma = \int \frac{dq_0}{2\pi} \frac{1}{L^d} \sum_{\vec{q}=2\pi/L(n1,n2,\ldots,n_d)} \int_0^1 dydx\, x \int_0^\infty d\lambda \lambda^2 e^{-\lambda[(1-x)\,q^2+x(1-y)\left((p-q)^2+m^2\right)+xy\left((p_\mu-q)^2+m_\mu^2\right)+\Delta^2]}$$

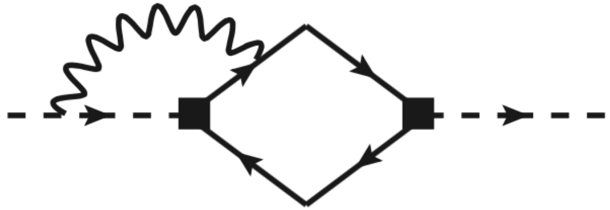
$$\Gamma_2 = -\frac{1}{L^d} \int \frac{dq_0}{2\pi} \int_0^1 dydx\, x \int_0^\infty d\lambda \lambda^2\, e^{-\lambda[(1-x)\,q^2+x(1-y)\left((p-q)^2+m^2\right)+xy\left((p_\mu-q)^2+m_\mu^2\right)+\Delta^2]}|_{\vec{q}=0} =$$

$$\boxed{-\frac{1}{L^d} \int \frac{dV_0}{2\pi}\, 2 \int_0^1 dydx\, x \frac{1}{[V_0^2+M^2x^2+\vec{p}_\mu^2x^2y^2+\Delta^2]^3}}$$

$$M^2 = m^2(1-y)+m_\mu^2 y$$

$$\Gamma_2 = -\frac{(E_\mu+m)\left(E_\mu^2+m^2\right)}{32E_\mu^4 L^3 m^4} + \frac{1}{8E_\mu L^3 m\Delta^3}$$

Γ_0^{pt} *The nasty diagram*



$$\Gamma = \int \frac{dq_0}{2\pi}\,\frac{1}{L^d} \sum_{\vec{q}=2\pi/L(n1,n2,\dots,n_d)} \int_0^1 dydx\,x \int_0^\infty d\lambda\lambda^2 e^{-\lambda[(1-x)\,q^2+x(1-y)\left((p-q)^2+m^2\right)+xy\left((p_\mu-q)^2+m_\mu^2\right)+\Delta^2]}$$

using the Poisson formula

$$\Gamma_{IV} + \sum_{\vec{k}=\cdots-2,-1,+1,+2,\dots}{}' \int_0^1 dydx\,x \int_0^\infty d\lambda\lambda^2 \int \frac{dq_0}{2\pi}\,\frac{dq^d}{(2\pi)^d} e^{iL\vec{k}\cdot\vec{q}-\lambda[(1-x)\,q^2+x(1-y)\left((p-q)^2+m^2\right)+xy\left((p_\mu-q)^2+m_\mu^2\right.}$$

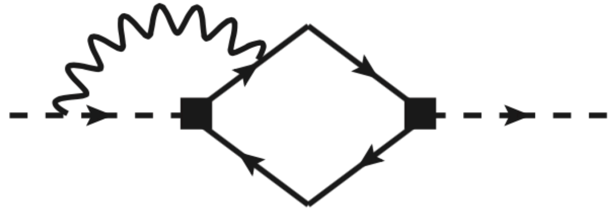
You may shift and integrate on the loop momenta

$$\delta\Gamma = \frac{1}{(4\pi)^{(d+1)/2}}\left(\frac{L^2}{4\pi}\right)^{\alpha-(d+1)/2} \sum_{\vec{k}=\dots}{}' \int_0^1 dydx\,x \int_0^\infty dt\,t^{\alpha-(d+1)/2-1} e^{ixyL\vec{k}\cdot\vec{p}_\mu-tL^2/(4\pi)M^2x^2}\, e^{-\pi\vec{k}^2/t-tL^2\Delta^2/(4\pi)}$$

$$\propto \int_0^1 dydx\,x \int_0^\infty dt\,t^{\alpha-(d+1)/2-1} e^{-tL^2/(4\pi)M^2x^2}\, e^{-tL^2\Delta^2/(4\pi)} \left(\prod_{j=x,y,z}\theta\left(\frac{Lxy\,p_\mu^j}{2\pi},\frac{i}{t}\right)-1\right)$$

$$\Gamma_0^{pt}$$ **The nasty diagram**

the HL trick !!



$$\delta\Gamma = \frac{1}{(4\pi)^{(d+1)/2}} \left(\frac{L^2}{4\pi}\right)^{\alpha-(d+1)/2} \times \int_0^1 dy dx\, x$$

$$\left(\int_0^1 dt\, t^{\alpha-(d+1)/2-1} e^{-tL^2/(4\pi)M^2x^2}\, e^{-tL^2\Delta^2/(4\pi)} \left(\prod_{j=x,y,z} \theta\left(\frac{Lxy\, p_\mu^j}{2\pi}, \frac{i}{t}\right) - 1 \right) + \right.$$

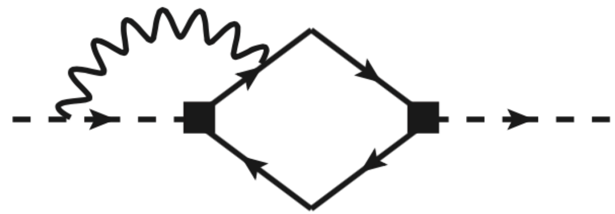
$$\int_0^1 dt\, t^{-\alpha+1/2-1} e^{-L^2/(4\pi\, t)(M^2+y^2\vec{p}_\mu^{\,2})x^2}\, e^{-L^2\Delta^2/(4\pi\, t)} \left(\prod_{j=x,y,z} \theta\left(\frac{-iLxy\, p_\mu^j}{2\pi\, t}, \frac{i}{t}\right) - 1 \right) +$$

$$\int_1^\infty dt\, t^{\alpha-1/2-1} e^{-L^2\, t/(4\pi)(M^2+y^2\vec{p}_\mu^{\,2})x^2}\, e^{-L^2\Delta^2\, t/(4\pi)} -$$

$$\left. \int_1^\infty dt\, t^{\alpha-(d+1)/2-1} e^{-L^2 t/(4\pi)M^2x^2}\, e^{-tL^2\Delta^2/(4\pi)} \right)$$

$$x \to z/L \qquad \int_0^1 dx \to 1/L \int_0^\infty dz$$

Γ_0^{pt} *The nasty diagram*

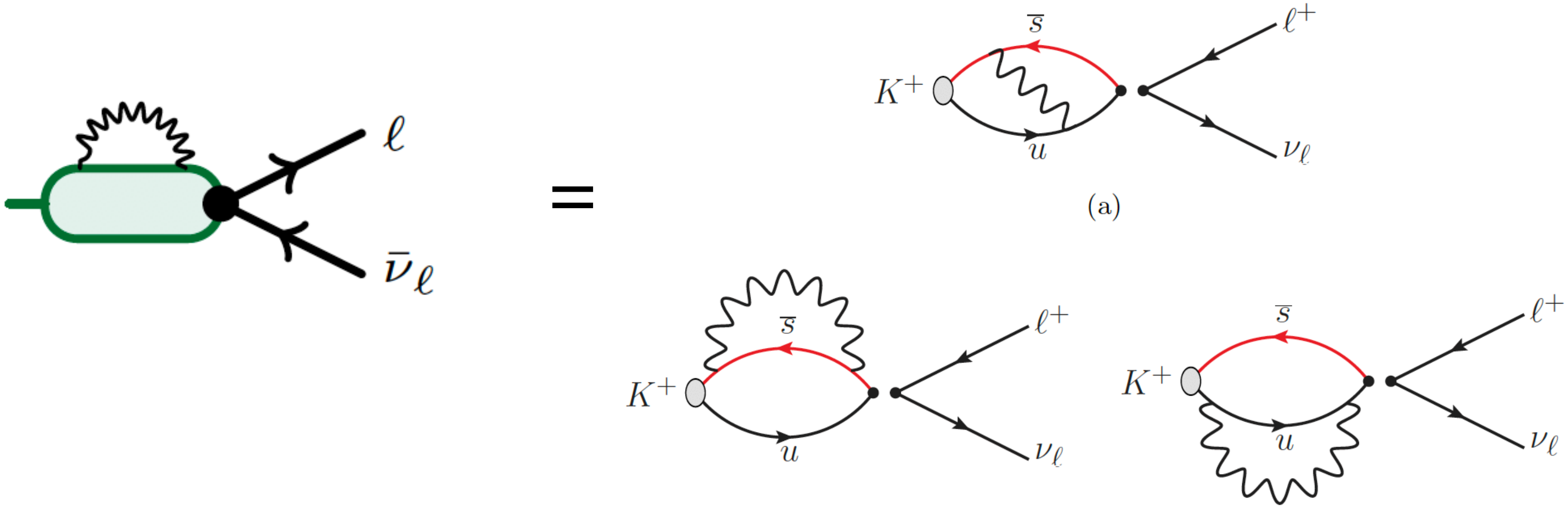


$$\Gamma_L = \frac{1}{16\pi^2}\left(1+r_\mu^2\right)\left[-\frac{1}{m^3L^3}\frac{\pi^2\left(\epsilon_\mu+1\right)\left(\epsilon_\mu^2+1\right)}{\epsilon_\mu^4}+\frac{2}{3\epsilon_\mu}+X+\frac{\left(\gamma_E+\mathrm{Log}\left[\frac{L^2m^2}{4\pi}\right]\right)\mathrm{Log}\left[r_\mu^2\right]}{\left(1-r_\mu^2\right)}+\frac{\mathrm{Log}\left[r_\mu^2\right]^2}{2\left(1-r_\mu^2\right)}\right]+$$

$$-\frac{2\,\mathrm{Log}\left[r_\mu^2\right]}{\left(1-r_\mu^2\right)}+1+\mathrm{Log}\left[\frac{M_W^2}{m^2}\right]+\ldots\Bigg]$$

$$\delta Z_\pi = \frac{\alpha}{4\pi}\left[2\,\mathrm{Log}[m_W^2/m_\pi^2]-3/2+2\,\mathrm{Log}[L^2m_\pi^2]-8\pi^2\times 0.0621547\right]$$

Leptonic Decays: Numerical Results



$$C_P^{(0)}(t) = \sum_{\vec{x}} \langle 0 | T\left\{ J_W^\rho(0)\phi_P^\dagger(\vec{x}, -t)\right\} |0\rangle \frac{p_P^\rho}{M_P}$$

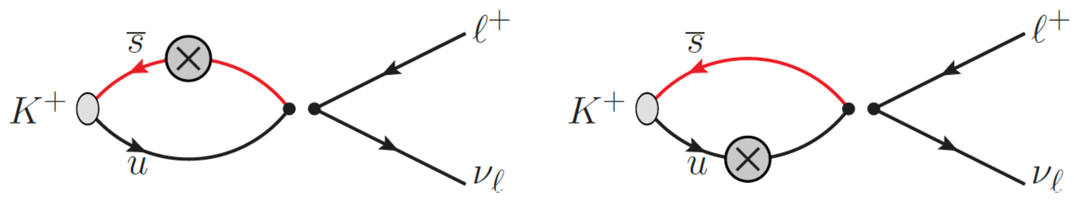
weak quark current

pseudoscalar meson source

$$\delta C_P^J(t) = 4\pi\alpha_{\rm em}\frac{1}{2}\sum_{\vec{x},y_1,y_2} \langle 0 | T\left\{ J_W^\rho(0)\; \underline{j_\mu^{\rm em}(y_1) j_\nu^{\rm em}(y_2)}\; \phi_P^\dagger(\vec{x}, -t)\right\} |0\rangle \Delta_{\mu\nu}^{\rm em}(y_1, y_2)\frac{p_P^\rho}{M_P}$$

Leptonic Decays: Numerical Results

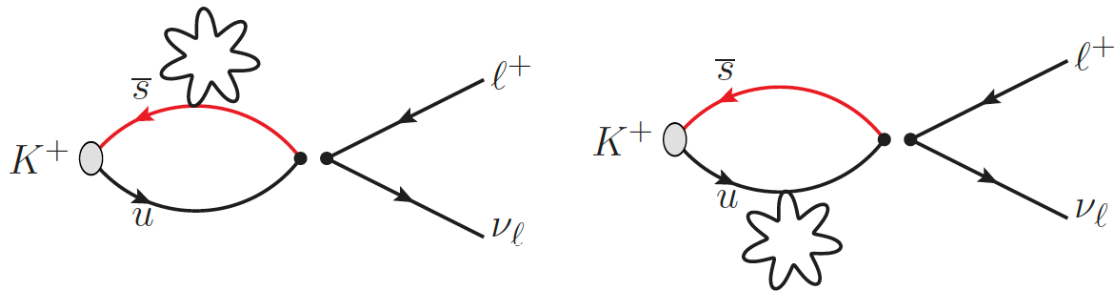
Strong Isospin Breaking



(a) (b)

$$\delta C_P^S(t) = -4\pi\alpha_{\rm em} \sum_{f=f_1,f_2} m_f \frac{\mathcal{Z}_m^f}{Z_m^{(0)}} \cdot \sum_{\vec{x},y} \langle 0|T\left\{ J_W^\rho(0)\ \left[\overline{q}_f(y) q_f(y)\right]\ \phi_P^\dagger(\vec{x},-t)\right\}|0\rangle \frac{p_P^\rho}{M_P}$$

Tadpole



(d) (e)

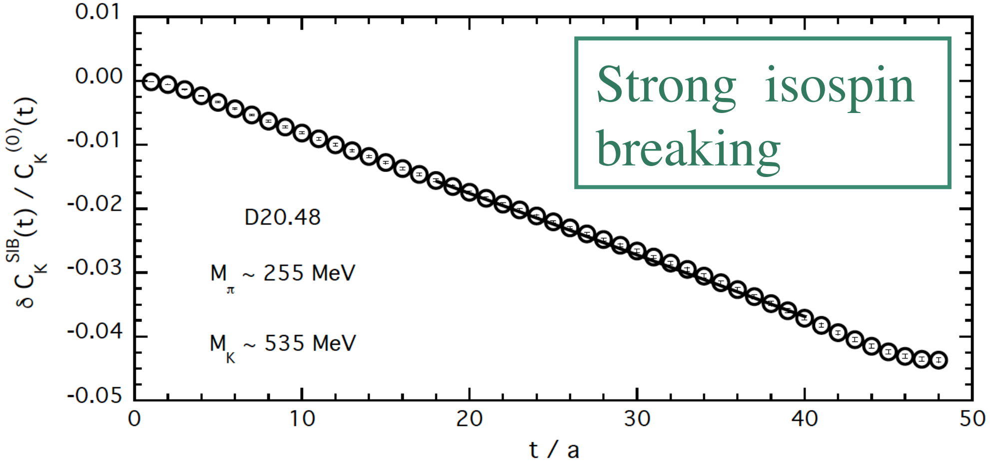
$$\delta C_P^T(t) = 4\pi\alpha_{\rm em} \sum_{\vec{x},y} \langle 0|T\left\{ J_W^\rho(0)\ T_\mu^{\rm em}(y)\ \phi_P^\dagger(\vec{x},-t)\right\}|0\rangle \Delta_{\mu\mu}^{\rm em}(y,y) \frac{p_P^\rho}{M_P}\ ,$$

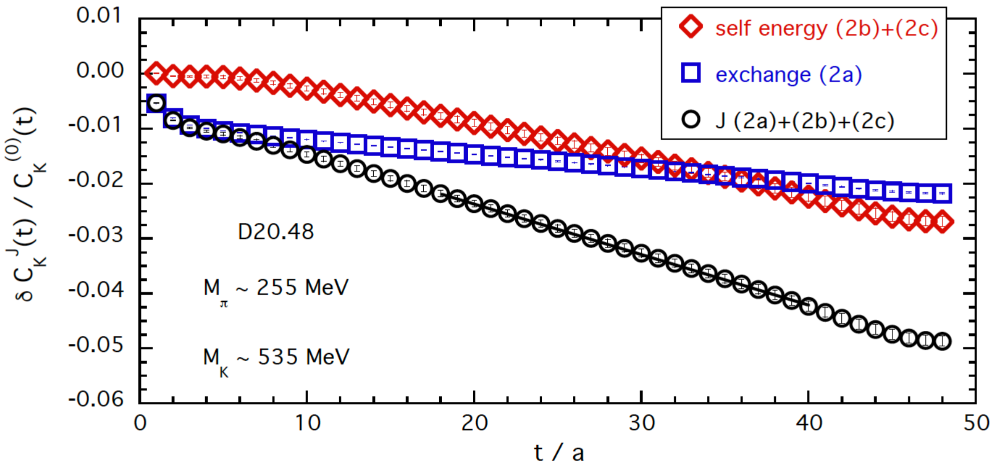
$$\frac{\delta C_P^i(t)}{C_P^{(0)}(t)} \xrightarrow[t\gg a,(T-t)\gg a]{} \frac{\delta[G_P^i A_P^i]}{G_P^{(0)} A_P^{(0)}} + \frac{\delta M_P^i}{M_P^{(0)}}\left[M_P^{(0)}\left(\frac{T}{2}-t\right)\frac{e^{-M_P^{(0)}t}+e^{-M_P^{(0)}(T-t)}}{e^{-M_P^{(0)}t}-e^{-M_P^{(0)}(T-t)}} - 1 - M_P^{(0)}\frac{T}{2}\right]$$

O(α_{em}) correction to the amplitude
* source matrix element

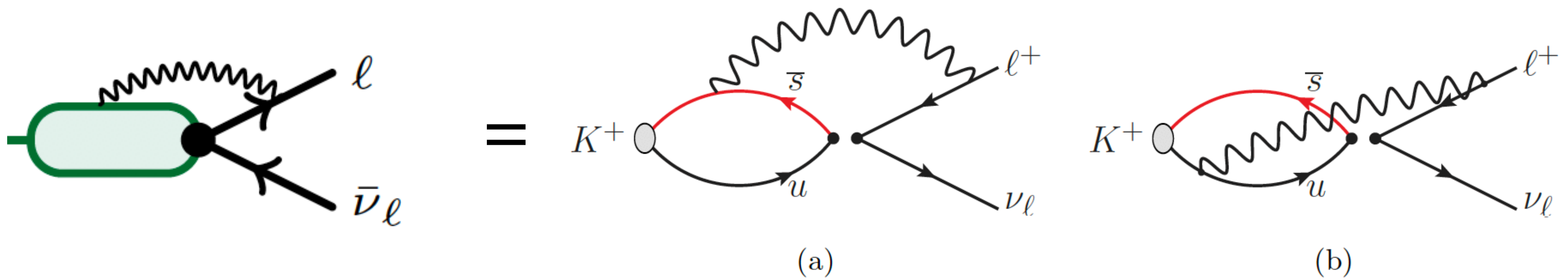
up to finite time effects $\approx$ - δM_P t

Strong isospin breaking





Leptonic Decays: Nasty Diagrams

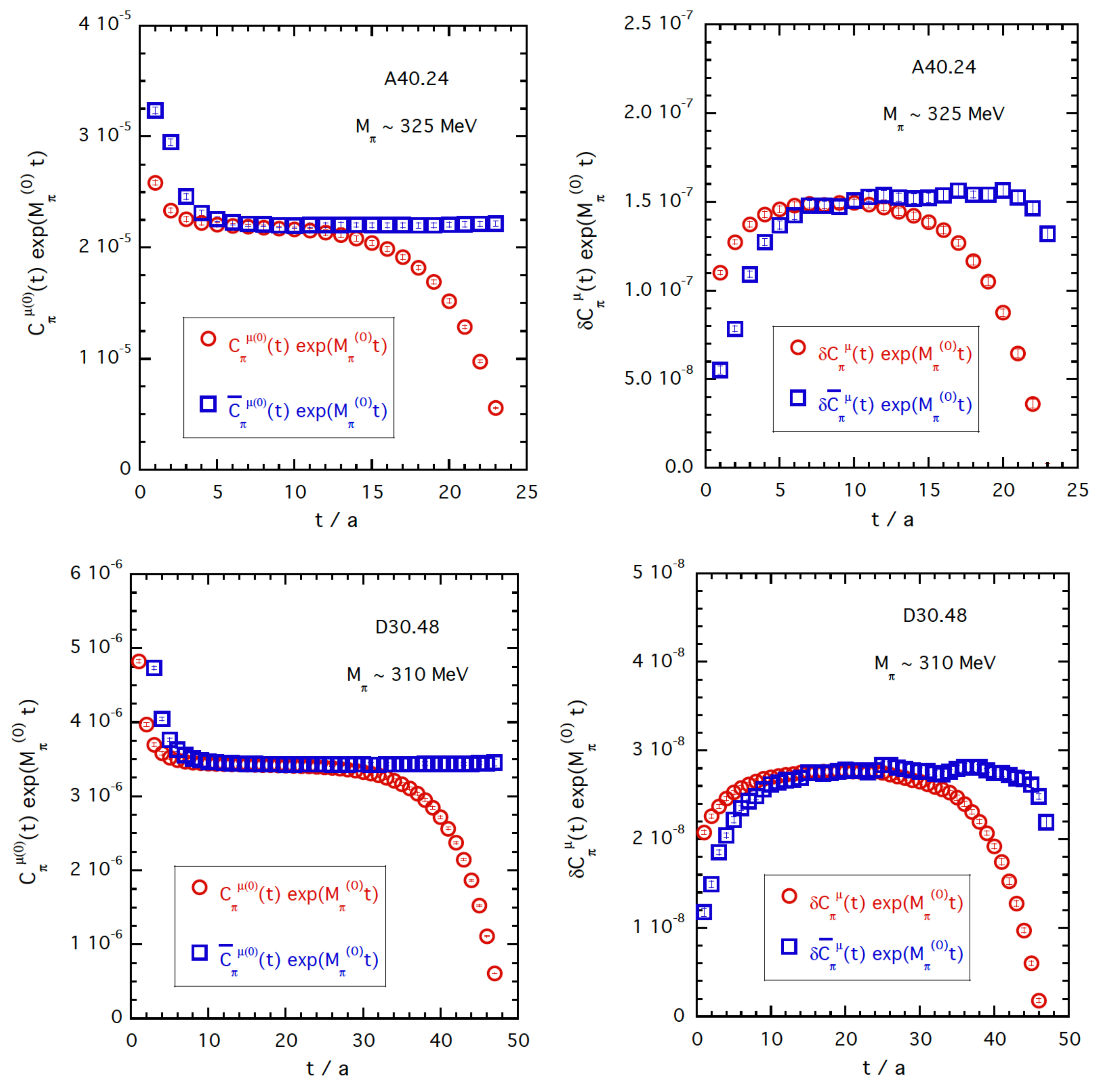


No mass renormalisation = No term linear in t ; to remove the backward signal:

$$\delta\overline{C}^{\ell}_{P}(t) \equiv \frac{1}{2}\left\{\delta C^{\ell}_{P}(t) + \frac{\delta C^{\ell}_{P}(t-1) - \delta C^{\ell}_{P}(t+1)}{e^{M_P^{(0)}} - e^{-M_P^{(0)}}}\right\} \xrightarrow[t\gg a,(T-t)\gg a]{} \delta A^{\ell}_{P}\, X^{\ell,0}_{P}\,\frac{G_P^{(0)}}{2M_P^{(0)}}e^{-M_P^{(0)}t}\,,$$

$$\overline{C}^{\ell(0)}_{P}(t) \equiv \frac{1}{2}\left\{C^{\ell(0)}_{P}(t) + \frac{C^{\ell(0)}_{P}(t-1) - C^{\ell(0)}_{P}(t+1)}{e^{M_P^{(0)}} - e^{-M_P^{(0)}}}\right\} \xrightarrow[t\gg a,(T-t)\gg a]{} A^{(0)}_{P}\, X^{\ell,0}_{P}\,\frac{G_P^{(0)}}{2M_P^{(0)}}e^{-M_P^{(0)}t}\,,$$

$$\frac{\delta\overline{C}^{\ell}_{P}(t)}{\overline{C}^{\ell(0)}_{P}(t)} \xrightarrow[t\gg a,(T-t)\gg a]{} \frac{\delta A^{\ell}_{P}}{A^{(0)}_{P}}\,.$$



A40.24
M_π ~ 325 MeV
C_π^μ(0)(t) exp(M_π^(0) t)
C̄_π^μ(0)(t) exp(M_π^(0) t)
δC_π^μ(t) exp(M_π^(0) t)
δC̄_π^μ(t) exp(M_π^(0) t)
D30.48
M_π ~ 310 MeV
t / a

Leptonic Decays

It is time to put all the ingredients together

$$\delta R_P = \frac{\alpha_{em}}{\pi}\log\left(\frac{M_Z^2}{M_W^2}\right) + 2\frac{\delta A_P}{A_P^{(0)}} - 2\frac{\delta M_P}{M_P^{(0)}} + \delta\Gamma_P^{(pt)}(\Delta E_\gamma)$$

Matching W-regularisation to SM

shift in the lowest order amplitude due to the mass-shift

Point-like $O(\alpha_{em})$ correction

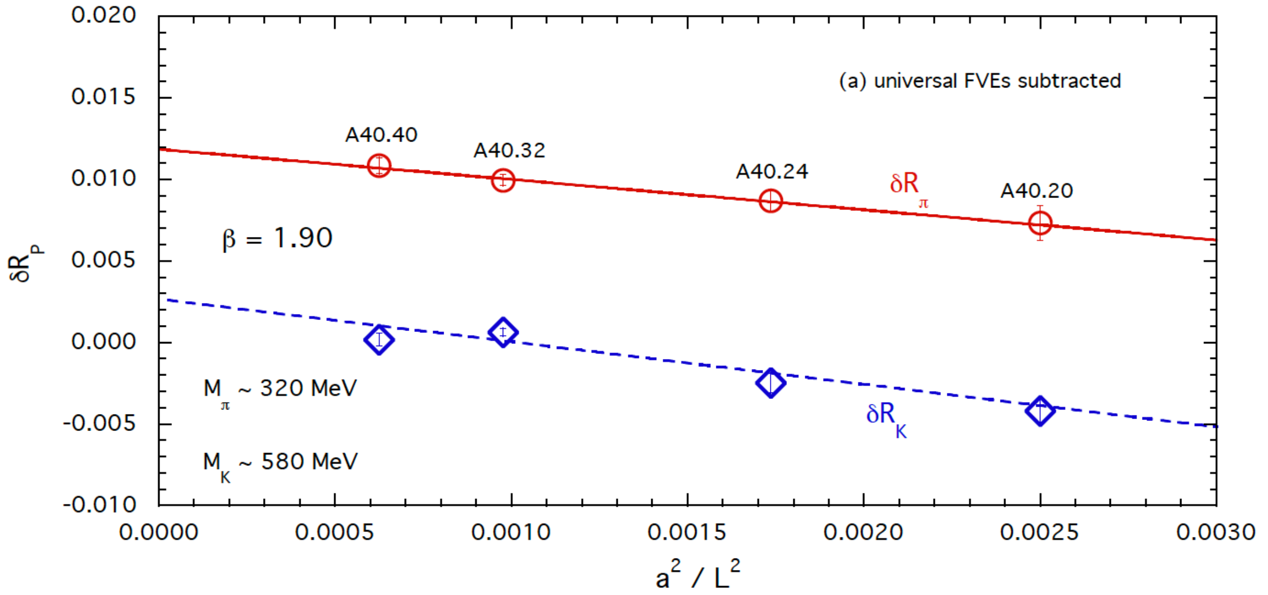
$$\delta A_P = \delta A_P^W + \delta A_P^{SIB} + \sum_{i=J,T,P,S} \delta A_P^i + \delta A_P^\ell - Y_P^\ell(L)\, A_P^{(0)}$$

matching lattice operator to the W-regularisation

nasty

finite volume corrections

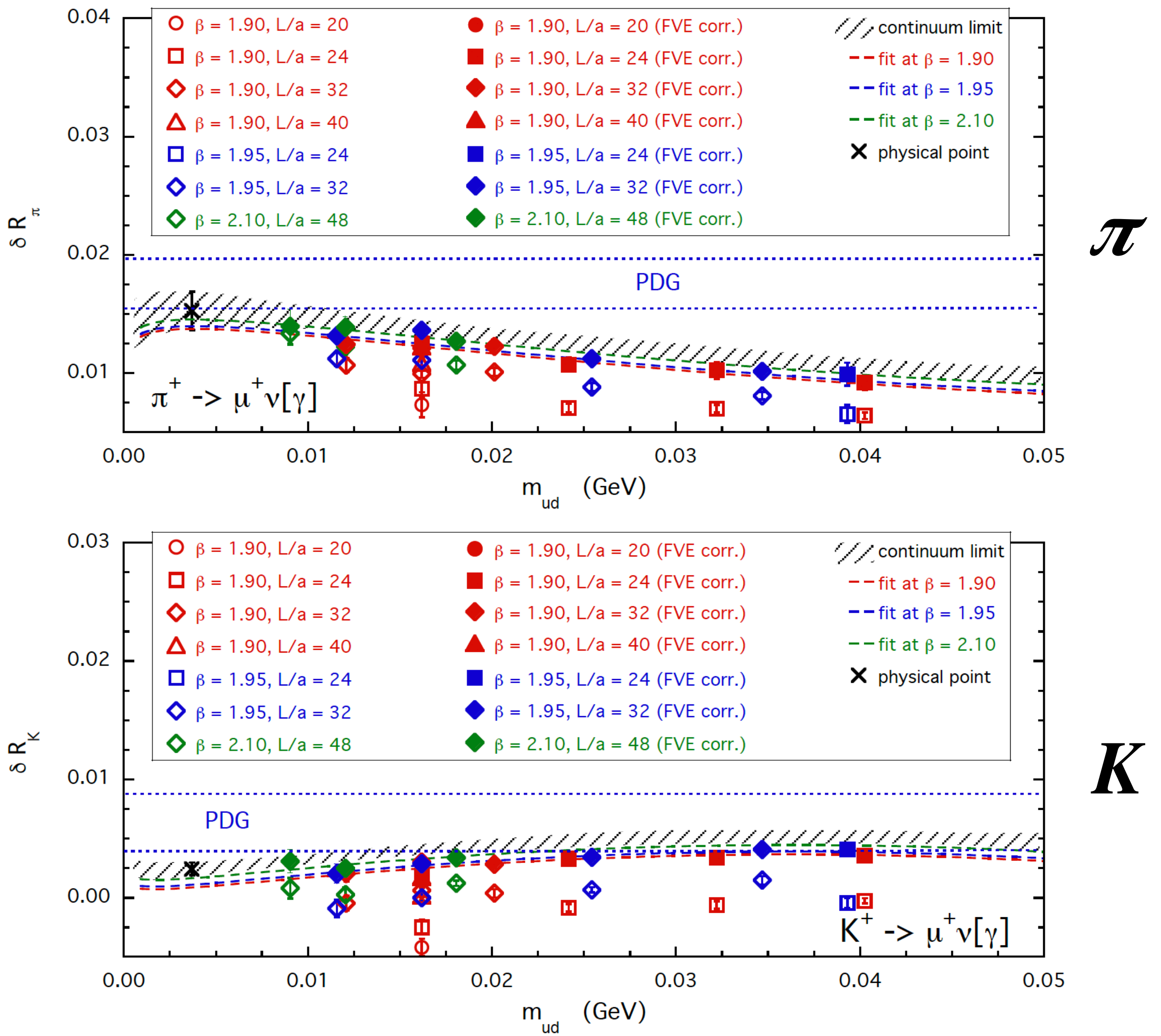
Extrapolation in 1/L²



Extrapolation to the physical point in the infinite volume limit

$$\delta R_P = R_P^{(0)} + R_P^{(\chi)}\,\log(m_{ud}) + R_P^{(1)} m_{ud} + R_P^{(2)} m_{ud}^2 + D_P\, a^2$$

$$+\ \frac{K_P}{M_P^2 L^2} + \frac{K_P^\ell}{(E_P^\ell)^2 L^2} + \delta\Gamma^{\rm pt}(\Delta E_\gamma^{max,P})\ ,$$


β = 1.90, L/a = 20
β = 1.90, L/a = 24
β = 1.90, L/a = 32
β = 1.90, L/a = 40
β = 1.95, L/a = 24
β = 1.95, L/a = 32
β = 2.10, L/a = 48
β = 1.90, L/a = 20 (FVE corr.)
β = 1.90, L/a = 24 (FVE corr.)
β = 1.90, L/a = 32 (FVE corr.)
β = 1.90, L/a = 40 (FVE corr.)
β = 1.95, L/a = 24 (FVE corr.)
β = 1.95, L/a = 32 (FVE corr.)
β = 2.10, L/a = 48 (FVE corr.)
continuum limit
fit at β = 1.90
fit at β = 1.95
fit at β = 2.10
physical point
PDG
π+ -> μ+ν[γ]
K+ -> μ+ν[γ]
δ Rπ
δ RK
mud (GeV)
π
K

Back to Physics: From the Kaon Decay Rate

at the physical pion mass:

$$\delta R_{\pi}^{\rm phys} = +0.0153\ (16)_{stat+fit}\ (4)_{input}\ (3)_{chiral}\ (6)_{FVE}\ (2)_{disc}\ (6)_{qQED}$$

$$= +0.0153\ (19)\ ,$$

$$\delta R_{K}^{\rm phys} = +0.0024\ (6)_{stat+fit}\ (3)_{input}\ (1)_{chiral}\ (3)_{FVE}\ (2)_{disc}\ (6)_{qQED}$$

$$= +0.0024\ (10)\ ,$$

electro-quenching

input quark masses

chiral log in the fitting formula

different extrapolations to infinite volume at $O(1/L^2)$

discretisation effects

Back to Physics

Taking the experimental values $\Gamma(\pi^- \to \mu^- \bar{\nu}_\mu[\gamma]) = 3.8408(7) \cdot 10^7 \text{ s}^{-1}$ and $\Gamma(K^- \to \mu^- \bar{\nu}_\mu[\gamma]) = 5.134(11) \cdot 10^7 \text{ s}^{-1}$ from the PDG [20] and using our results (101) - (102), we obtain

$$f_\pi^{(0)} |V_{ud}| = 127.28 \ (2)_{\rm exp} \ (12)_{\rm th} \, \text{MeV} = 127.28 \, (12) \, \text{MeV} \ , \qquad (103)$$

$$f_K^{(0)} |V_{us}| = 35.23 \ (4)_{\rm exp} \ (2)_{\rm th} \, \text{MeV} = 35.23 \, (5) \, \text{MeV} \ , \qquad (104)$$

where the first error is the experimental uncertainty and the second is that from our theoretical calculations. The result for the pion in Eq. (103) agrees within the errors with the updated value $f_\pi^{(0)} |V_{ud}| = 127.12(13)$ MeV [20], obtained by the PDG and based on the model-dependent ChPT estimate of the e.m. corrections from Ref. [26]. Our result for the kaon in Eq. (104) however, is larger than the corresponding PDG value $f_K^{(0)} |V_{us}| = 35.09(5)$ MeV [20], based on the ChPT calculation of Ref. [26], by about 2 standard deviations.

using $\quad f_K^{(0)} = 156.11(21) \ \text{MeV}$ S. Aoki et al. [Flavour Lattice Averaging Group], arXiv:1902.08191 [hep-lat]

$$|V_{us}| = 0.22567(26)_{\rm exp} \, (33)_{\rm th} = 0.22567 \, (42)$$

with a precision of 0.2%

Back to Physics: from the ratio of decay rates

$$\delta R^{\rm phys}_{K\pi} = \delta R^{\rm phys}_{K} - \delta R^{\rm phys}_{\pi} = -0.0126\,(14)\ . \tag{106}$$

Using the pion and kaon experimental decay rates we get

$$\frac{|V_{us}|}{|V_{ud}|}\frac{f_K^{(0)}}{f_\pi^{(0)}} = 0.27683\,(29)_{\rm exp}\,(20)_{\rm th} = 0.27683\,(35)\,. \tag{107}$$

Using the best $N_f = 2+1+1$ lattice determination of the ratio of the QCD kaon and pion decay constants, $f_K^{(0)}/f_\pi^{(0)} = 1.1966\ (13)$ [3, 42–44], we find

$$\frac{|V_{us}|}{|V_{ud}|} = 0.23134\,(24)_{\rm exp}\,(30)_{\rm th} = 0.23134\,(38)\,. \tag{108}$$

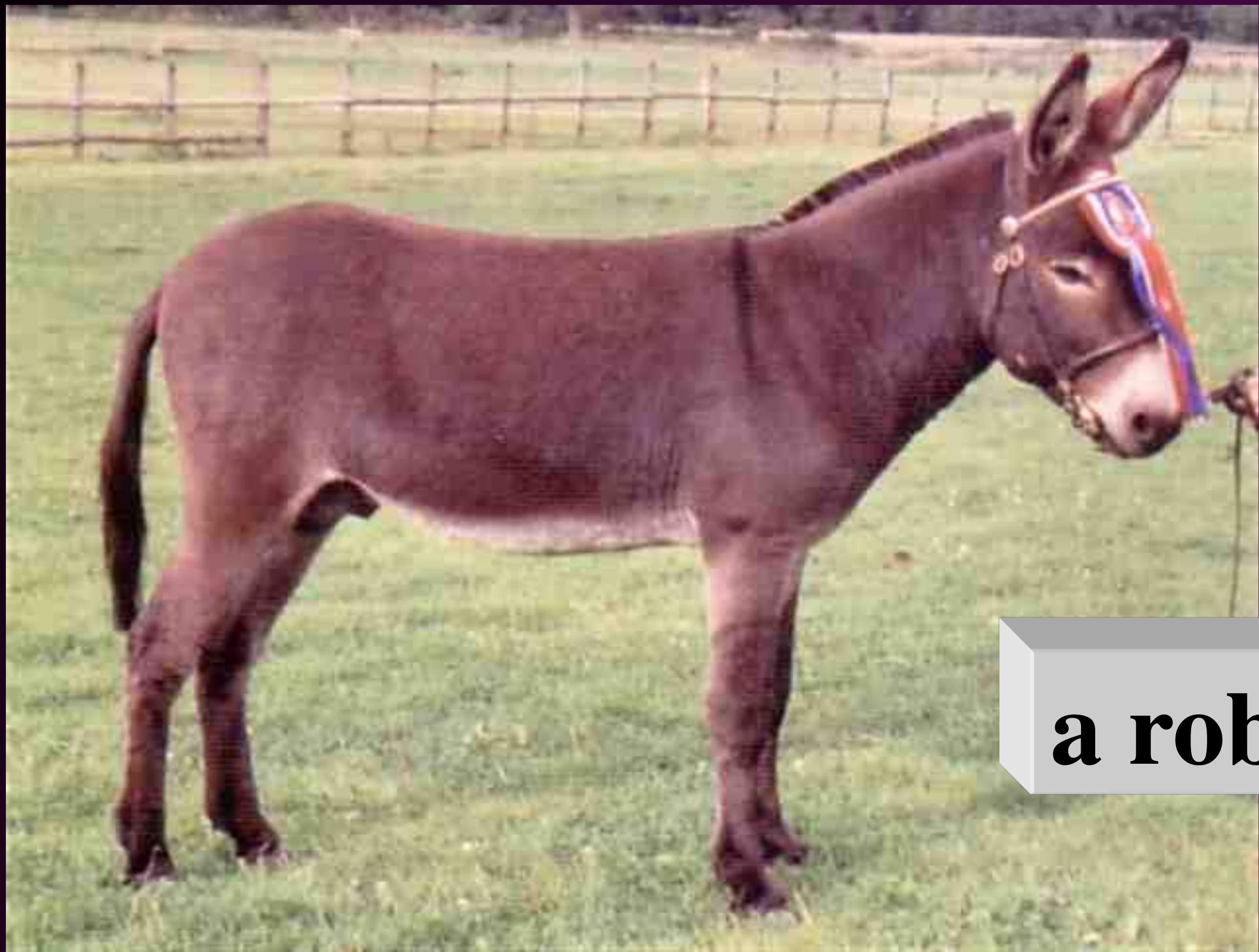
Taking the updated value $|V_{ud}| = 0.97420\,(21)$ from super-allowed nuclear beta decays [21], Eq. (108) yields the following value for the CKM element $|V_{us}|$:

$$|V_{us}| = 0.22538\,(24)_{\rm exp}\,(30)_{\rm th} = 0.22538\,(38)\,, \tag{109}$$

which agrees with our result (105) within the errors. Note that our result (109) agrees with the latest estimate $|V_{us}| = 0.2253(7)$, recently updated by the PDG [20], but it improves the error by a factor of approximately 1.8.

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 0.99988\,(44)\,.$$

the Standard Model



a robust animal

THANKS FOR YOUR ATTENTION