Non-perturbative Renormalization

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a selection, to be completed

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...

Introduction:

What are we here interested in?

QCD without CP-violating term, quark masses are real

$$\mathcal{L}_{\rm QCD} = -\frac{1}{2g_0^2}\,\text{tr}\,\{F_{\mu\nu}F_{\mu\nu}\} + \sum_f \overline{\psi}_f\{D + m_f\}\psi_f$$

$$\mathcal{L}_{\rm QCD}(g_0, m_f) \leftrightarrow \overbrace{\begin{bmatrix} m_{\rm proton} \\ m_\pi \\ m_{\rm K} \\ m_{\rm D} \\ m_{\rm B} \end{bmatrix}}^{\rm Experiment} \qquad (m_{\rm u} = m_{\rm d}, \quad \text{ignore top})$$

bare parameters $\rightarrow$ masses, observables

theory parametrized in terms of observables

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NP renormalization

What are we interested in?

Strong interactions at large energies

LHC (and other collider physics):

$$p\bar{p} \to H \to \dots$$

SM (or MSSM) predictions depend on

renormalized perturbation theory (PT) in $\alpha_s(\mu) \equiv \alpha_R(\mu)$

$\mu = \mathrm{O}(10\mathrm{GeV}) \dots \mathrm{O}(300\mathrm{GeV})$

What is $\alpha_R(\mu)$ in a given renormalization scheme?

What is $\Lambda_{\rm QCD}$:

$$\begin{aligned} m_{\rm proton} &= \# \times \Lambda_{\rm QCD} \\ \alpha_R(\mu) &\overset{\mu/\Lambda \gg 1}{\sim} \frac{1}{b_0 \ln(\mu/\Lambda)} \left\{ 1 - \frac{b_1}{b_0^2 \ln(\mu/\Lambda)} \ln(\ln(\mu/\Lambda)) + \mathrm{O}(\ln(\mu/\Lambda)^{-2}) \right\} \end{aligned}$$

What are we interested in?

Weak interactions

Weak decays (search for BSM physics) of quarks:

$$\begin{matrix}\text{low energy effective theory} \\ \text{2-quark op's, 4-quark op's}\end{matrix} \quad \leftarrow \quad \begin{cases}\mathrm{SM} \\ +\mathrm{BSM}\end{cases}$$

necessitates the **renormalization of composite fields**

What will we do?

- Renormalization in PT (repetition)
- RGE's, RGI
- NP renormalization (principle)
- Large scale ratios, step scaling functions (SSF)
- Finite volume schemes
- Gradient flow (recent development)

- very incomplete coverage of techniques
  concentrate on concepts

  recommend to study yourself
  - RI-sMOM
  - chirally rotated SF

Renormalization in PT

Repetition

Consider continuum PT, $D = 4 - 2\epsilon$ dimensions as a regularisation

gauge-invariant, physical observable

$$G$$

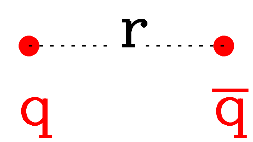
bare, regularised

$G_0(\epsilon, q, g_0, m_{0i})$

Example

force between static quarks

$F(r)$



$q \equiv 1/r$

G_0 is singular as $\epsilon \to 0$ *at fixed* q, g_0, m_{0i}

Renormalization in PT

MS scheme

Renormalizability:

all observables G become **finite** after the

Renormalization:

dimensionful coupling in D dimensions

$$g_\mathrm{R}^2 \equiv g^2 \quad = Z_g(\epsilon, g^2)\mu^{-2\epsilon} g_0^2$$

$$m_{\mathrm{R},i} \equiv m_i \quad = Z_m(\epsilon, g^2) m_{0i}$$

$$G_\mathrm{R}(\mu, q, g, m_i) = \lim_{\epsilon \to 0} G_0(\epsilon, q, \underbrace{Z_g^{-1/2} g \mu^\epsilon}_{g_0}, \underbrace{Z_m^{-1} m_i}_{m_{0i}})$$

The limit exists with

$$Z_x = 1 + g^2 z_{x,1}\epsilon^{-1} + g^4[z_{x,2}\epsilon^{-2} + z_{x,3}\epsilon^{-1}] + \ldots$$

"minimal subtraction" (of ϵ poles; only those)

Renormalization in PT

$\overline{\text{MS}}$ scheme

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mass–independent renormalization scheme

$$G_{\mathrm{R}}(\mu, q, g, m_i) = \lim_{\epsilon \to 0} G_0(\epsilon, q, \underbrace{Z_g^{-1/2} g \mu^{\epsilon}}_{g_0}, \underbrace{Z_m^{-1} m_i}_{m_{0i}})$$

The limit exists with

$$Z_x = 1 + g^2 z_{x,1}\epsilon^{-1} + g^4[z_{x,2}\epsilon^{-2} + z_{x,3}\epsilon^{-1}] + \ldots$$

"minimal subtraction" (of ϵ poles; only those) on the lattice: $G_0(a, q, g_0, m_{0i})$

$$\begin{aligned} g_{\rm lat}^2 \equiv g^2 &= Z_g(\ln(a\mu), g^2)\, g_0^2 \\ m_{{\rm lat},i} \equiv m_i &= Z_m(\ln(a\mu), g^2)\, \hat{m}_{{\rm q},i} \qquad (*) \\ G_{\rm R}^{\rm lat}(\mu, q, g_{\rm lat}, m_i) &= \lim_{a\to 0} G_0(a, q, \underbrace{Z_g^{-1/2} g}_{g_0}, \underbrace{m_{0i}}_{Z_m^{-1} m_{{\rm lat},i}}) \end{aligned}$$

when $r_m = 1$

The limit exists (continuum limit) with different Z_x !

$$Z_x = 1 + g^2 z_{x,1} \ln(a\mu) + g^4 [z_{x,2}(\ln(a\mu))^2 + z_{x,3} \ln(a\mu)] + \ldots$$

"**lattice minimal subtraction**" (of logs $\ln(a\mu)$; only those) $g = g_{\rm lat}, m = m_{\rm lat}$

Proven to all orders of PT for Wilson reg'n [T. Reisz].

Expected also non-perturbatively and for other regularisations (universality).

(*) $\hat{m}_{{\rm q},i}$ are bare subtracted masses,

$$\begin{aligned} \hat{m}_{{\rm q},i} &= m_{{\rm q},i} + (r_m(g_0) - 1)\frac{1}{N_{\rm f}}\, \mathrm{tr}\, M_{\rm q} \\ m_{{\rm q},i} &= m_{0i} - m_{\rm c}(g_0)\,, \quad M_{\rm q} = \mathrm{diag}(m_{{\rm q},1}, m_{{\rm q},2}, \ldots) \end{aligned}$$

with sufficient chiral symmetry: $r_m = 1, m_c = 0$

The limit is universal (does not depend on the regularisation) after changing the renormalization scheme: finite renormalization

$$
\begin{aligned}
g^2_{\rm lat} &= \chi_g(g_{\rm MS})\, g^2_{\rm MS}\,, \quad \chi_g(g) = 1 + \chi_g^{(1)} g^2 + \ldots \\
m_{{\rm lat},i} &= \chi_m(g_{\rm MS})\, m_{{\rm MS},i}\,, \quad \chi_m(g) = 1 + \chi_m^{(1)} g^2 + \ldots \\
G_{\rm R}(\mu, q, g_{\rm MS}, m_{{\rm MS},i}) &= G_{\rm R}^{\rm lat}(\mu, q, \underbrace{\chi_g(g_{\rm MS})\, g^2_{\rm MS}}_{g_{\rm lat}}, \underbrace{\chi_m(g_{\rm MS})\, m_{{\rm MS},i}}_{m_{{\rm lat},i}})
\end{aligned}
$$

Renormalized masses and coupling depend on μ:

$$\lim_{a\to 0} \mu\partial_\mu g_{\rm lat}\big|_{g_0,m_{{\rm q},i}} \equiv \beta_{\rm lat}(g_{\rm lat}) = -g_{\rm lat}^3\,(b_0 + b_1 g_{\rm lat}^2 + \ldots)$$

$$\lim_{a\to 0} \mu\partial_\mu m_{{\rm lat},i}\big|_{g_0,m_{{\rm q},i}} \equiv \tau_{\rm lat}(g_{\rm lat})\, m_{{\rm lat},i}$$

$$\tau_{\rm lat}(g_{\rm lat}) = -g_{\rm lat}^2\,(d_0 + d_1^{\rm lat} g_{\rm lat}^2 + \ldots)$$

$$b_0 = \frac{1}{(4\pi)^2}\left(11 - \frac{2}{3}N_{\rm f}\right), \qquad d_0 = \frac{8}{(4\pi)^2}$$

$$b_1 = \frac{1}{(4\pi)^4}\left(102 - \frac{38}{3}N_{\rm f}\right)$$

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or in the MS-scheme

$$\lim_{\epsilon\to 0} \mu\partial_\mu g_{\rm MS}|_{g_0,m_{0,i}} \equiv \beta_{\rm MS}(g_{\rm MS}) = -g_{\rm MS}^3\,(b_0 + b_1 g_{\rm MS}^2 + \ldots)$$

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$$\tau_{\rm MS}(g_{\rm MS}) = -g_{\rm MS}^2\,(d_0 + d_1^{\rm MS} g_{\rm MS}^2 + \ldots)$$

A physical quantity G_R does not depend on μ, since G_0 does not depend on μ:

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}G_0 = 0 \quad \rightarrow \quad \mu\frac{\mathrm{d}}{\mathrm{d}\mu}G_\mathrm{R}(\mu,q,g,m_i) = 0$$
$$\left(\mu\partial_\mu + \beta(g)\partial_g + \tau(g)\, m_i\partial_{m_i}\right) G_\mathrm{R} = 0$$

The general solution of the RGE can be expressed in terms of special solutions:

1. m_i, q-independent function $\Lambda(\mu, g)$: $m_i\partial_{m_i}\Lambda = 0$

$$\left(\mu\partial_\mu + \beta(g)\partial_g\right)\Lambda = 0$$
$$\Lambda = \mu\,\varphi_g(g)\,, \qquad \varphi_g \text{ dimensionless}$$
$$(1+\beta(g)\partial_g)\varphi_g = 0$$
$$\varphi_g = \exp\left\{-\int^g \mathrm{d}x\,\frac{1}{\beta(x)}\right\} \times \text{constant}$$

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$$= (b_0 g^2)^{-b_1/(2b_0^2)}\,\mathrm{e}^{-1/(2b_0 g^2)}\,\exp\left\{-\int_0^g \mathrm{d}x\left[\frac{1}{\beta(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0^2 x}\right]\right\}$$

A physical quantity $G_{\rm R}$ does not depend on an arbitrarily introduced μ:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\mu} G_{\rm R} &= 0 \\ \left(\mu\partial_\mu + \beta(g)\partial_g + \tau(g)\, m_i \partial_{m_i}\right) G_{\rm R} &= 0 \end{aligned}$$

The general solution can be expressed in terms of special solutions:

2. m_i dependent functions, independent of μ and q: $M_i(m_i, g)$:

$$\left(\tau(g)\, m_i \partial_{m_i} + \beta(g)\, \partial_g\right) M_i = 0$$

$$M_i = m_i\, \varphi_m(g)\,, \qquad \varphi_m \text{ dimensionless}$$

$$(\tau(g) + \beta(g)\partial_g)\varphi_m = 0$$

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$$= \left(2b_0 g^2\right)^{-d_0/(2b_0)} \exp\left\{-\int_0^g \mathrm{d}x \left[\frac{\tau(x)}{\beta(x)} - \frac{d_0}{b_0 x}\right]\right\}$$

Now take G_R independent of q; example: $G_\mathrm{R} = m_\mathrm{hadron}$

$$\begin{aligned} G_\mathrm{R} &= G_\mathrm{R}(\mu, g(\mu), m_i(\mu)) \\ \mu \text{ independent :} \quad &= G_\mathrm{R}(k\Lambda, \underbrace{g(k\Lambda)}_{\varphi_g^{-1}(1/k)}, M_i / \underbrace{\varphi_m(g(k\Lambda))}_{\varphi_m(\varphi_g^{-1}(1/k))}) \\ \to \quad G_\mathrm{R} &= G^\mathrm{RGI}(\Lambda, M_i) \qquad \text{any } k\text{, e.g. } k=1 \end{aligned}$$

with mass dimension 1: $[G_\mathrm{R}] = 1$, e.g. m_hadron

$$m_\mathrm{hadron} = \Lambda \bar{f}_h(M_i/\Lambda)$$

Λ, M_i: fundamental parameters of QCD ($N_\mathrm{f}+1$ parameters)

Renormalization Group Invariants (RGI)

Renormalization Group Invariants (RGI)

- non–perturbatively defined

  with the standard (undoubted) assumtions:
  NP "corrections" to RG functions vanish as $\mu^{-\eta}$, $\eta > 0$
  e.g. renormalons, instantons

- our job is to determine them

in the chiral limit $M_i = 0$

$$\begin{aligned} m_{\rm hadron} &= \Lambda \bar{f}_h(0) = \bar{f}_h(0)\;\mu\;{\rm e}^{-1/(2b_0 g(\mu)^2)} \times \ldots \\ & \partial_g^n m_{\rm hadron}\big|_{g=0} = 0 \\ \to \quad & m_{\rm hadron} = 0 \text{ to all orders of PT} \end{aligned}$$

$m_{\rm hadron}, \Lambda, M_i$ are non-perturbative quantities

Exercises

- Show that
$$M_i^s = M_i^{s'}$$
where s, s' are different schemes.

- Show that
$$\Lambda^s = k\,\Lambda^{s'}$$
Determine k in terms of $\chi_g^{(1)}\,,\ b_0$.

- What is needed to determine $\chi_g^{(1)}$?

Renormalization Group

Application: short distance behavior

$q = 1/r$ large: short (Euclidean) distances

$$\begin{aligned} G_\mathrm{R} &= G_\mathrm{R}(\mu, q, g(\mu), m_i(\mu)) \qquad \text{dimensionless (e.g. } r^2 F(r)) \\ &= P(q/\mu, g(\mu), m_i(\mu)/q) \\ &= P(1, g(q), m_i(q)/q)\,, \quad \varphi_g(g(q)) = \Lambda/q \\ & \qquad m_i(q) = M_i/\varphi_m(g) \end{aligned}$$

- yields the RG improved prediction for P
- becomes more and more accurate for $q \to \infty$

$$\begin{aligned} g^2(q) &= \frac{1}{b_0 t}\left\{1 - \frac{b_1}{b_0^2 t}\ln(t) + \mathrm{O}(t^{-2})\right\} \\ &\to 0 \text{ as } t \to \infty \qquad t = 2\ln(q/\Lambda) \\ m_i(q) &= M_i \left(\tfrac{2}{t}\right)^{d_0/2b_0} \{1 + \ldots\} \end{aligned}$$

- unphysical μ-dependence of the coupling turned into physical q dependence

$q = 1/r$ large: short (Euclidean) distances
we also see that

$$G_{\mathrm{R}} \quad = \quad P(1, g(q), m_i(q)/q) \overset{q \gg \Lambda, M_i}{\sim} P(1, g(q), 0)$$

mass effects disappear at short distances

Renormalization of composite fields

For weak interactions, chiral symmetry breaking order parameter, . . .

Local composite fields ("operators")

$$
\begin{aligned}
S^{rs}(x) &= \overline{\psi}_r(x)\psi_s(x)\,, \quad P^{rs}(x) = \overline{\psi}_r(x)\gamma_5\psi_s(x) \quad r \neq s \text{ flavor indices} \\
S(x) &= S^{rr}(x) \equiv \sum_{r=1}^{N_\mathrm{f}} S^{rr}(x)\,, \quad P(x) = P^{rr}(x) \\
A_\mu^{rs}(x) &= \overline{\psi}_r(x)\gamma_\mu\gamma_5\psi_s(x)\ldots \\
O_{\mathrm{LL}}^{rs}(x) &= \overline{\psi}_r(x)\gamma_\mu(1-\gamma_5)\psi_s(x)\overline{\psi}_r(x)\gamma_\mu(1-\gamma_5)\psi_s(x)
\end{aligned}
$$

In contrast to non-local composite fields

- Wilson loop
- smeared fields

$S_t^{rs}(x)$ $\qquad$ t a proper smearing parameter

$\rightarrow$ see the final lecture

Renormalization of composite fields

mixing with operators of same dimension

$$\langle \phi_{\mathrm{R}1}(x_1)\phi_{\mathrm{R}2}(x_2)\phi_{\mathrm{R}3}(x_3)\phi_{\mathrm{R}4}(x_4)\ldots\rangle_{\text{path integral average}}$$

is finite for $x_i \neq x_j$ for $i \neq j$ with

dimensional regularisation, MS

$$\phi_{\mathrm{R},i}^{(D)} = \sum_j Z_{ij}(\epsilon, g^2)\, \Phi_j^{(D)}\,, \qquad [\Phi_j^{(D)}] = [\Phi_i^{(D)}] = D$$

e.g. $[S] = [P^{rs}] = 3$

lattice MS

$$\phi_{\mathrm{R},i}^{(D)} = \sum_j Z_{ij}(\ln(a\mu), g^2)\, \Phi_{\mathrm{sub},j}^{(D)}\,, \quad [\Phi_{\mathrm{sub},j}^{(D)}] = [\Phi_i^{(D)}] = D$$

$$\Phi_{\mathrm{sub},j}^{(D)} = \Phi_j^{(D)} + \sum_{n\geq 1} \mathbf{a}^{-\mathbf{n}} \sum_k d_{jk}(g_0)\Phi_k^{(D-n)}$$

Subtraction coefficients d_{jk} can be chosen purely as functions of g_0, not $\ln(a\mu)$ [M. Testa, hep-th/9803147, Sect. 2]

Exercise: Go through the argument in hep-th/9803147. Does it hold beyond PT?

Renormalization of composite fields

$$\phi_{\mathrm{R},i}^{(D)} = \sum_j Z_{ij}(\ln(a\mu), g^2)\, \Phi_{\mathrm{sub},j}^{(D)}\,, \quad [\Phi_{\mathrm{sub},j}^{(D)}] = [\Phi_i^{(D)}] = D$$

$$\Phi_{\mathrm{sub},j}^{(D)} = \Phi_j^{(D)} + \sum_{n\geq 1} \mathbf{a}^{-\mathbf{n}} \sum_k d_{jk}(g_0)\Phi_k^{(D-n)}$$

An example

$$S_\mathrm{R}(x) = Z_\mathrm{S}^{\mathrm{sing}}(\ln(a\mu), g^2)\, \left[\overline{\psi}(x)\psi(x) + \mathbf{a}^{-\mathbf{3}} d_1(g_0)\right]$$

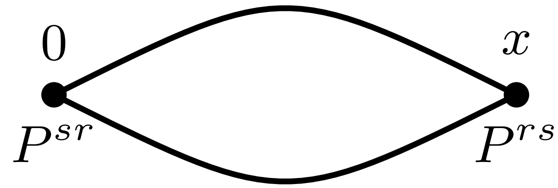
- “mixing with the unit-operator”
- in theories without exact chiral symmetry
- in the chiral limit
  otherwise: $\frac{m^n}{a^{3-n}}$ terms

Renormalization of composite fields

RGI fields and short distance behavior

Just work with a simple example:

$$
\begin{aligned}
G_0(a,x,g_0) &= \langle P^{rs}(x) P^{sr}(0) \rangle \\
G_{\rm R}^{\rm cont}(\mu,x,g) &= \lim_{a\to 0} G_{\rm R}^{\rm lat}(\mu,x,g,a\mu) \\
G_{\rm R}^{\rm lat}(\mu,x,g,a\mu) &= \langle P_{\rm R}^{rs}(x) P_{\rm R}^{sr}(0) \rangle \\
&= Z_{\rm P}^2(a\mu,g_0)\, G(a,x,g_0)
\end{aligned}
$$



RGE:

$$
\mu \frac{\rm d}{{\rm d}\mu} G_0(a,x,g_0) = 0 = \mu \frac{\rm d}{{\rm d}\mu} Z_{\rm P}^{-2} G_{\rm R}
$$

$$
\rightarrow \qquad Z_{\rm P}^2 \mu \frac{\rm d}{{\rm d}\mu} [Z_{\rm P}^{-2} G_{\rm R}] = 0
$$

$$
(\mu\partial_\mu + \beta(g)\partial_g + \tau(g)\, m_i \partial_{m_i} - 2\gamma)\, G_{\rm R} = 0
$$

$$
\gamma = Z_{\rm P}^{-1}\, \mu\partial_\mu Z_{\rm P}(\mu a, g_0)
$$

Now turn to a renormalization group invariant form:

$$P_{\rm RGI}^{rs} = \varphi_{\rm P}(\mu, g) P_{\rm R}^{rs}$$

with $$(\mu\partial_\mu + \beta\partial_g)\varphi_{\rm P} = -\gamma\,\varphi_{\rm P}$$

then

$$G_{\rm RGI} = \langle P_{\rm RGI}^{rs}(x) P_{\rm RGI}^{sr}(0)\rangle = \varphi_{\rm P}^2\, G_{\rm R}$$

$$\varphi_{\rm P} = \exp\left\{-\int^g {\rm d}x\, \frac{\gamma(x)}{\beta(x)}\right\} \times \text{constant} \ldots$$

Then we get the RGE for a renormalization group invariant (without an anomalous dimension term).

$$(\mu\partial_\mu + \beta(g)\partial_g + \tau(g)\, m_i \partial_{m_i})\, G_{\rm RGI} = 0$$

- We have the prediction for the short distance behavior as before.
- $G_{\rm RGI}(x, \Lambda, M_i)$: scheme-independent functions, uniquely given by QCD.

Now turn to a renormalization group invariant form:

$$P^{rs}_{\mathrm{RGI}} = \varphi_{\mathrm{P}}(\mu, g) P^{rs}_{\mathrm{R}}$$

with $$(\mu\partial_\mu + \beta\partial_g)\varphi_{\mathrm{P}} = -\gamma\,\varphi_{\mathrm{P}}$$

then

$$G_{\mathrm{RGI}} = \langle P^{rs}_{\mathrm{RGI}}(x) P^{sr}_{\mathrm{RGI}}(0)\rangle = \varphi_{\mathrm{P}}^2\, G_{\mathrm{R}}$$

$$\varphi_{\mathrm{P}} = \exp\left\{-\int^g \mathrm{d}x\,\frac{\gamma(x)}{\beta(x)}\right\} \times \text{ constant } \ldots$$
$$= \left(2b_0 g^2\right)^{-\gamma_0/(2b_0)} \exp\left\{-\int_0^g \mathrm{d}x \left[\frac{\gamma(x)}{\beta(x)} - \frac{\gamma_0}{b_0 x}\right]\right\}$$

Then we get the RGE for a renormalization group invariant (without an anomalous dimension term).

$$(\mu\partial_\mu + \beta(g)\partial_g + \tau(g)\, m_i \partial_{m_i})\, G_{\mathrm{RGI}} = 0$$

- We have the prediction for the short distance behavior as before.
- $G_{\mathrm{RGI}}(x, \Lambda, M_i)$: scheme-independent functions, uniquely given by QCD.
- **It is the job of lattice QCD to determine them.**

The general principle (lot's of evidence)

- Mixing with all local operators of same and lower dimensions, allowed by the symmetries
  in renormalizable theories (normal propagators, no couplings with negative mass dimension)

The general principle (lot's of evidence)

- Mixing with all local operators of same and lower dimensions, allowed by the symmetries

$$S = a^4 \sum_{\text{space-time}} \sum_{n=1}^{4} \sum_i g_{in} \Phi_i^{(n)}(x) + a^3 \sum_{\text{boundary}} \sum_{n=1}^{3} \sum_i c_{in} \Phi_i^{(n)}(x)$$

$[g_{in}] = 4 - n\,, \quad [c_{in}] = 3 - n$ bare couplings and masses

- adjust (= tune = renormalize) all coefficienst g_{in}, c_{in} such that the continuum limit exists
- no couplings with negative mass dimensions!
- Including theories with boundaries (Schrödinger functional , Gradient Flow) all-order proof for GF and for SF in ϕ^4.
- $\mathrm{O}(a)$ effects: go higher in powers of a and include $[g_{in}] = 5 - n\,, \quad [c_{in}] = 4 - n$ Symanzik *effective* theory

$$S = a^4 \sum_{\text{space-time}} \sum_{n=1}^{5} \sum_i g_{in} \Phi_i^{(n)}(x) + a^3 \sum_{\text{boundary}} \sum_{n=1}^{4} \sum_i c_{in} \Phi_i^{(n)}(x)$$

$g_{i5} \sim a \qquad c_{i4} \sim a$

Wilson fermions

We had:

$$\begin{aligned} m_{\mathrm{lat},i} &= Z_m(\ln(a\mu), g^2)\, \hat{m}_{\mathrm{q},i} \\ \hat{m}_{\mathrm{q},i} &= m_{\mathrm{q},i} + (r_m(g_0) - 1)\frac{1}{N_\mathrm{f}}\,\mathrm{tr}\, M_\mathrm{q} \\ m_{\mathrm{q},i} &= m_{0i} - m_\mathrm{c}(g_0)\,,\;\; M_\mathrm{q} = \mathrm{diag}(m_{\mathrm{q},1}, m_{\mathrm{q},2}, \ldots) \end{aligned}$$

Why is that?

Write the mass-term as

$$\begin{aligned} \mathcal{L}_\mathrm{mass} &= \sum_i \overline{\psi}_i m_{0i} \psi_i \\ &= \sum_{a=3,8,\ldots} \mu_0^a \underbrace{\overline{\psi} T^a \psi}_{S^a\text{, nonsinglet}} + \underbrace{\overline{\psi}\psi}_{S_0\text{ singlet}} \frac{1}{N_\mathrm{f}}\,\mathrm{tr}\, M_0 \end{aligned}$$

Therefore there is (in general) $Z_m = (Z_\mathrm{S}^\mathrm{NS})^{-1}$ and $(r_m - 1) Z_m = (Z_\mathrm{S}^\mathrm{sing})^{-1}$

There is a **non**–anomalous chiral Ward identity: PCAC–relation

$$\langle \, [\partial_\mu A_\mu^{rs}(x) - (m_r + m_s) P^{rs}(x)] \;\; [\text{fields not at } x] \, \rangle = 0$$

- Can be obtained formally, performing a chiral rotation in the PI
- Can be obtained with lattice exact chiral symmetry (overlap)
- Is therefore (universality) a property of QCD in the continuum limit after renormalization

Quark mass renormalization on the lattice

In general, for $N_\mathrm{f} > 2$, any regularisation

Renormalized relation

$$\langle\, [Z_\mathrm{A}\partial_\mu A_\mu^{rs}(x) - (m_r + m_s)_\mathrm{R} Z_\mathrm{P} P^{rs}(x)] \;\; [\text{fields not at } x] \,\rangle = 0$$

$$(m_r + m_s)_\mathrm{R} = \frac{Z_\mathrm{A}}{Z_\mathrm{P}}(m_r + m_s)$$

- defines $(m_r + m_s)_\mathrm{R} \longrightarrow$ with $N_\mathrm{f} > 2$ enough combinations to define/determine $m_r\,,\;\; r = 1 \ldots N_\mathrm{f}$ with $N_\mathrm{f} = 2$ use also PCVC
- $Z_\mathrm{A}, Z_\mathrm{P}$ standard problem which we will discuss
- RGI masses from μ-dependent masses as discussed. Unambiguous.
- NB: Z_A is actually more simple

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- This does not say that anything special happens at $M_\mathrm{u} = 0$. There is no symmetry enhancement as explained by Mike Creutz.