Non-perturbative Renormalization

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Beijing, Lattice Field Theory school, June 2019

Nonperturbative Renormalization

First consider just the renormalization of the coupling, set

$$m_i = 0\,.$$

Properties of a renormalised coupling

a finite: $g = f(g_0, \mu a)$, such that $\lim_{a\to 0}\, G_0(qa, g_0)|_g$ exists

b gauge invariant (physical)

most natural

$$\begin{aligned} & G_0 = G_0(qa, g_0) \\ \rightarrow \qquad & G_\mathrm{R} = G_\mathrm{R}(q/m_\mathrm{proton}, qa)\,, \quad am_\mathrm{proton} = f(g_0) \\ & G_\mathrm{R}^\mathrm{cont} = G_\mathrm{R}(q/m_\mathrm{proton}, 0) \end{aligned}$$

"hadronic scheme"

- but we want a **coupling**, i.e. the relation to the Λ - parameter, the relation to perturbative QCD

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c

$$g^2_{\rm NP} \overset{g_{\rm lat}\to 0}{\sim} g^2_{\rm lat}\chi_g^{\rm NP,lat}(g_{\rm lat})$$

$$\chi_g^{x,y}(g) = \underset{\substack{\uparrow \\ \text{convention, good idea}}}{1} + \chi_{g,1}^{x,y} g^2 + \ldots$$

d It depends on a single scale μ → RGE!
For $\mu \to \infty$ it is purely short distance

Nonperturbative Renormalization

Generic definition of a renormalized coupling

Take $G_0(\mu a, g_0)$ dimensionless (in the massless theory) satisfying a,b,d.

G_0 has a regular PT

$$G_0(qa, g_0) \;=\; G_0^{(0)}(qa) + G_0^{(1)}(qa) g_0^2 + G_0^{(2)}(qa) g_0^4 + \ldots$$

lat scheme: $g_0^2 = g_{\rm lat}^2 + 2b_0 \ln(a\mu) g_{\rm lat}^4 + \ldots$

$$\rightarrow \quad G_0 = G_{\rm R} \;=\; G_{\rm R}^{(0)}(qa) + G_{\rm R}^{(1)}(qa) g_{\rm lat}^2 + G_{\rm R}^{(2)}(q/\mu, qa) g_{\rm lat}^4 + \ldots$$

$i = 0, 1$: $G_{\rm R}^{(i)}(qa) = G_0^{(i)}(qa) = C^{(i)} + {\rm O}(a^2 q^2) \qquad (*)$

$$G_{\rm R}^{(2)}(q/\mu, qa) = G_0^{(2)}(qa) + 2b_0 \ln(a\mu) G_0^{(1)}(qa)$$

$$= H(qa) + 2b_0 \ln(\mu/q) G_0^{(1)}(qa) \qquad (*)$$

$$H(qa) = C^{(2)} + {\rm O}(a^2 q^2) \qquad (*)$$

(*): the continuum limit exists

Nonperturbative Renormalization

Generic definition of a renormalized coupling

Set $\mu = q$:

$$\begin{aligned} G_0(\mu a, g_0^2) &= G_{\rm R}^{(0)}(\mu a) + G_{\rm R}^{(1)}(\mu a) g_{\rm lat}^2(\mu) + G_{\rm R}^{(2)}(1, \mu a) g_{\rm lat}^4(\mu) + \ldots \\ &= G_0^{(0)}(\mu a) + G_0^{(1)}(\mu a) g_{\rm lat}^2(\mu) + \underbrace{G_{\rm R}^{(2)}(1, \mu a)}_{C^{(2)} + {\rm O}(a^2 q^2)} g_{\rm lat}^4(\mu) + \ldots \end{aligned}$$

then

$$\bar{g}_G^2(\mu) \equiv \frac{G_0(\mu a, g_0^2) - G_0^{(0)}(\mu a)}{G_0^{(1)}(\mu a)}$$

satisfies a,b,c,d

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satisfies a,b,c,d

“physical” coupling

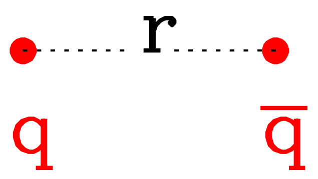
note

$$\bar{g}_G^2(\mu) = \underset{\substack{\uparrow \\ \text{no } a^2 \text{ effects here}}}{1} \times g_{\rm lat}^2 + {\rm O}(g_{\rm lat}^4)$$

Nonperturbative Renormalization

Example 1

$Q\bar{Q}$ potential, force:



$$G_0(\mu a, g_0^2) \;=\; r^2 F_{\rm impr}(r)\,, \qquad \mu = 1/r$$

$$G_0^{(0)} = 0\,, \qquad G_0^{(1)} = \frac{C_F}{4\pi} \qquad\qquad C_F = \frac{N^2-1}{6N}$$

def. of $F_{\rm impr}$ later

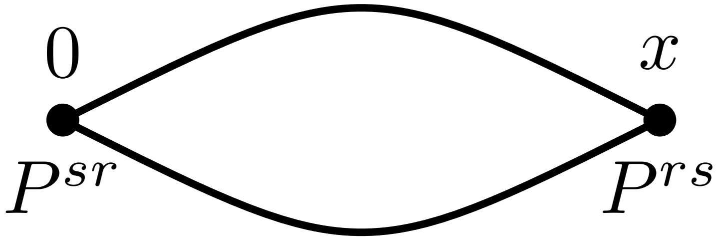
$$\rightarrow \qquad \bar{g}^2_{\rm qq}(\mu) = \frac{4\pi}{C_F} r^2 F_{\rm impr}(r)$$

[there is a little caveat with this ... not 100% short distance ... but ok]

Nonperturbative Renormalization

Example 2

Two-point function of multiplicatively renormalisable field



$$G_0(\mu a, g_0^2) \quad = \quad \frac{\langle P^{rs}(x) P^{sr}(0)\rangle_{x=(0,0,0,1/\mu)}}{\langle P^{rs}(x) P^{sr}(0)\rangle_{x=(0,0,0,2/\mu)}}$$

- Factors Z_{P} cancel
- Theoretically fine, but not really recommended (by me) in practise

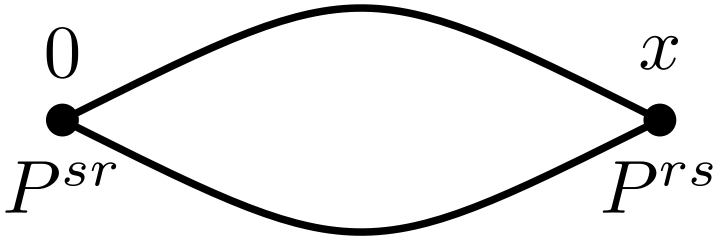
$$\langle P^{rs}(x) P^{sr}(0)\rangle \overset{x\to 0}{\sim} x^{-6}$$

steep function, large $(a/x)^n$ effects

[Martinelli, Rossi, Sachrajda, Sharpe, talevi, Testa, 1997; . . . , Cichy, Jansen, Korcyl, 2012]

Nonperturbative Renormalization

Renormalization of composite fields



- case by case
- eg. (in principle)

$$Z_\mathrm{P}^2(a\mu, g_0)\,\langle P^{rs}(x)P^{sr}(0)\rangle_{x=(0,0,0,1/\mu)} = \langle P^{rs}(x)P^{sr}(0)\rangle_{x=(0,0,0,1/\mu),g_0=0}$$

this defines $Z_\mathrm{P}(a\mu, g_0)$

- many correlation functions of P^{rs} can be used, as long as they are sufficiently short distance dominated
- but be careful with integrals, e.g.

$$\int \mathrm{d}^4x \langle P^{rs}(x)P^{sr}(0)\rangle$$

does not exist, since $\langle P^{rs}(x)P^{sr}(0)\rangle \overset{x\to 0}{\sim} x^{-6}$

Nonperturbative Renormalization

RI-MOM: the principle idea [G. Martinelli, C. Pittori, C. T. Sachrajda, M. Testa & A. Vladikas 1995]

drop gauge invariance requirement (b): fix a gauge (e.g. Landau gauge)
numerical evidence that this can be done non-perturbatively

then

$$\begin{aligned}
S(p) &= a^8 \sum_{x_1,x_2} \exp(-ip(x_1-x_2)) \; \langle \psi_r(x_1)\bar{\psi}_r(x_2)\rangle \\
G_P(p_1,p_2) &= a^8 \sum_{x_1,x_2} \exp(-ip_1x_1+ip_2x_2) \; \langle \; \psi_r(x_1) \; P^{rs}(0) \; \bar{\psi}_s(x_2) \; \rangle \\
\Lambda_P(p_1,p_2) &= S^{-1}(p_1) \; G_P(p_1,p_2) \; S^{-1}(p_2) \\
\Gamma_P(p_1,p_2) &= \tfrac{1}{12}\mathrm{Tr} \; [\; \gamma_5 \; \Lambda_P(p,p) \;] \;\; , \qquad \Gamma_V(p_1,p_2) = \ldots
\end{aligned}$$

Define $\Gamma_V(p)$ similarly from the conserved vector current, then

$$Z_\mathrm{P}\Gamma_P(p_1,p_2)/\Gamma_V(p_1,p_2) = [\Gamma_P(p_1,p_2)/\Gamma_V(p_1,p_2)]_{g_0=0}$$

defines $Z_\mathrm{P}(\mu)$ [or use a similar condition for the quark propagator to define the quark field renormalization constant and divide it out in Λ_P]

symmetric point $p^2 = p_1^2 = p_2^2 = (p_1-p_2)^2 = \mu^2$
$\rightarrow$ short distance dominated “RI-sMOM”

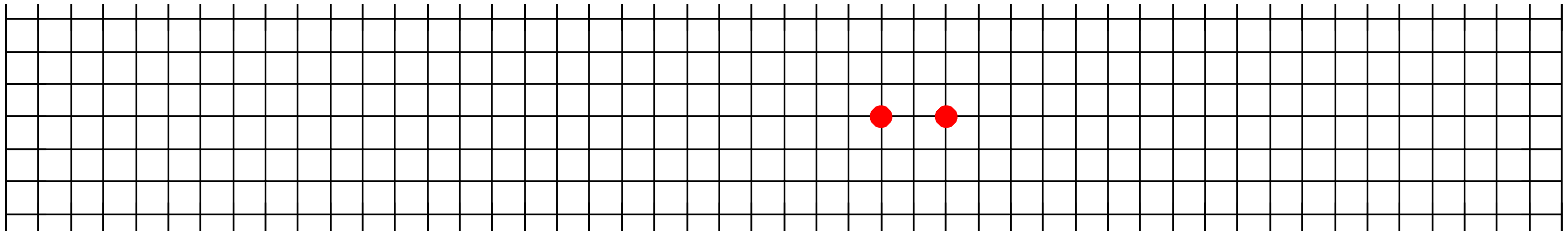
[C. Sturm, Y. Aoki, N. H. Christ, T. Izubuchi, C. T. C. Sachrajda & A. Soni, ARXIV:0901.2599]

Scale Problem

(α_{qq} as an example)

We need to reach large μ where perturbation theory is reliable to be able to use the perturbative relation (perturbative β-function) in

$$\frac{\Lambda}{\mu} = \varphi_g(g(\mu))$$



$$\underset{\substack{\uparrow \\ \text{box size}}}{L} \gg \underset{\substack{\uparrow \\ \text{QCD scale, } m_\pi}}{\frac{1}{0.2\mathrm{GeV}}} \gg \frac{1}{\mu} \sim \frac{1}{10\mathrm{GeV}} \gg \underset{\substack{\uparrow \\ \text{spacing}}}{a}$$

$$\Downarrow$$

$$L/a \gg 50$$

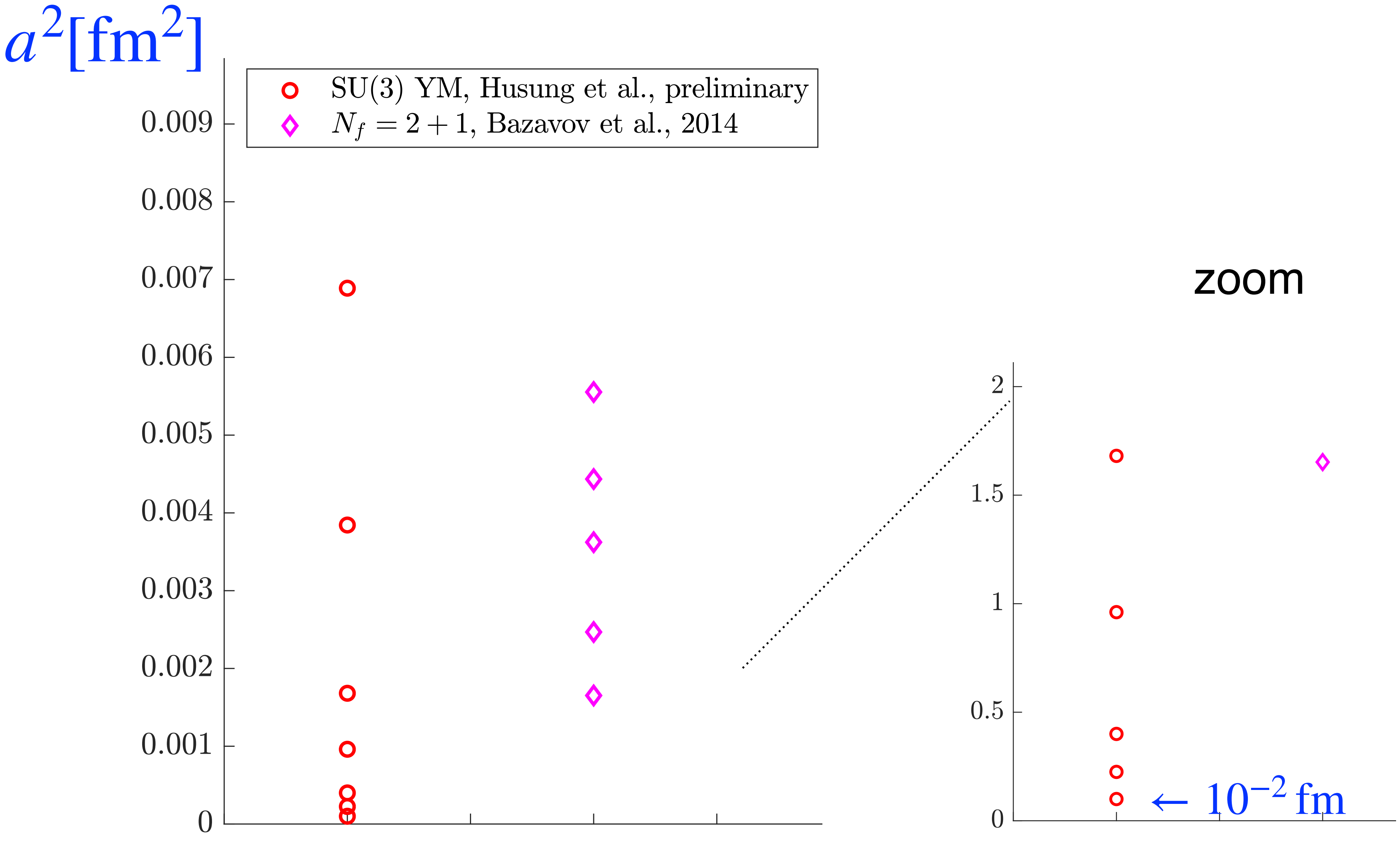
Let us see this in more detail

From Lattice 2019, Yang Mills potential …

Husung, Krah, Nada, RS

Scales, lattices

- L/a=32, … 192, L=2fm (big enough in YM), open BC (no topology freezing)



Strategy to get to small r (see also arXiv:1711.01860)

- basic scale from t_0 :

$$\alpha_{\rm qq}(\mu, a^2\mu^2)\,, \quad \mu = 1/r = (x\sqrt{8t_0})^{-1}$$

on ensembles with a > 0.02 fm

- Then step scaling functions

$$\Sigma(u, a/r) = \bar{g}^2_{\rm qq}(sr)\Big|_{\bar{g}^2_{\rm qq}(r)=u}$$

with $s = 3/4$ including $a = \{1.0, 1.4, 2.0\} \times 10^{-2}\,\mathrm{fm}$

Strategy to get to small r (see also arXiv:1711.01860)

- basic scale from t_0 :

$$\alpha_{\rm qq}(\mu, a^2\mu^2)\,, \quad \mu = 1/r = (x\sqrt{8t_0})^{-1}$$

$$0.25 \leq x \leq 0.4$$

on ensembles with a > 0.02 fm

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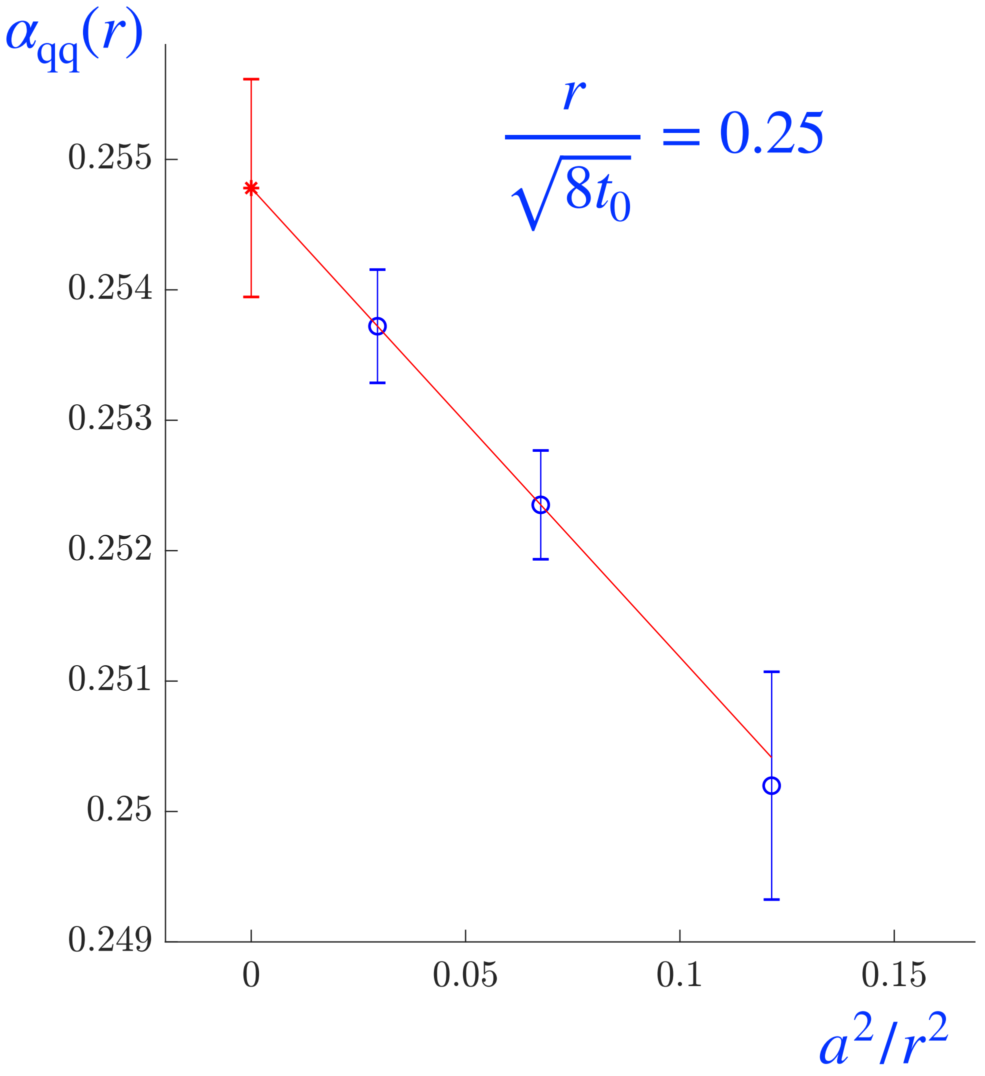
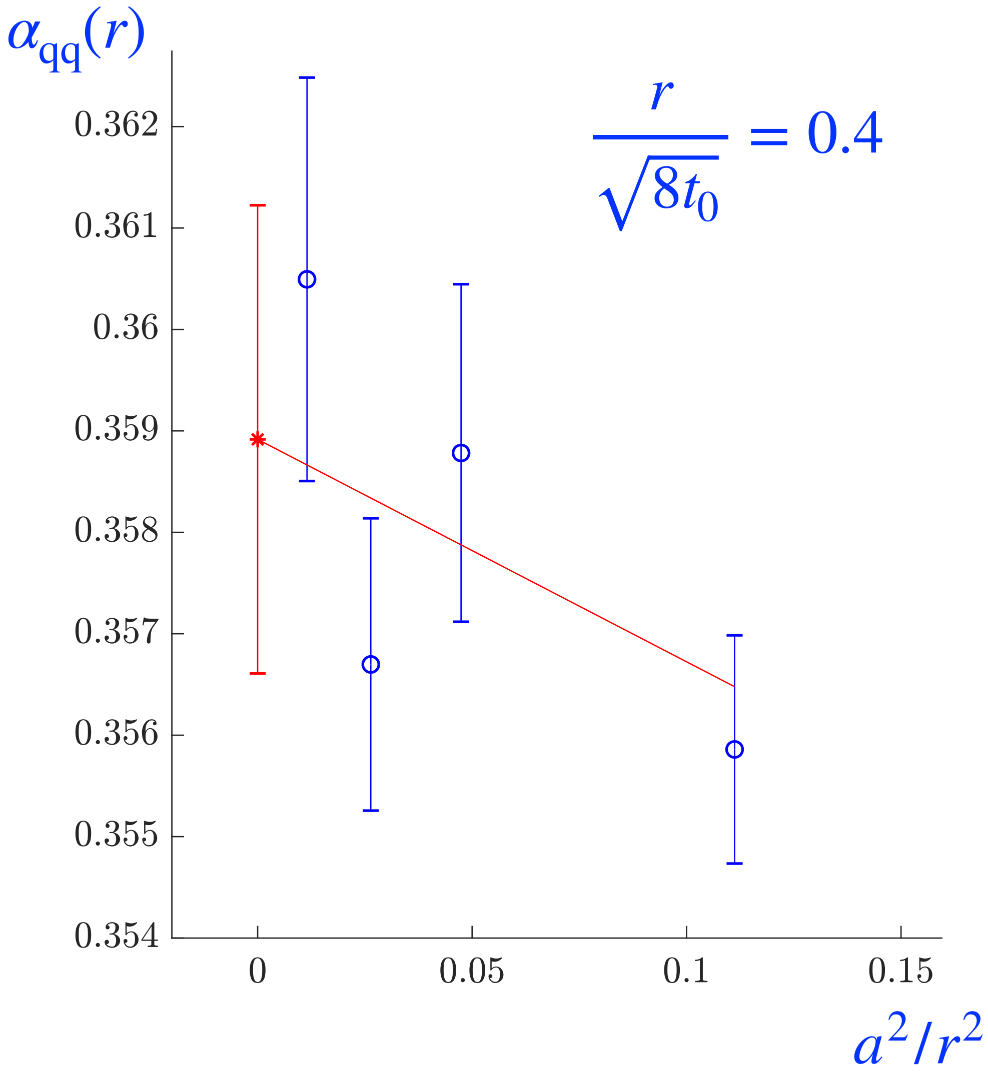
Continuum limits

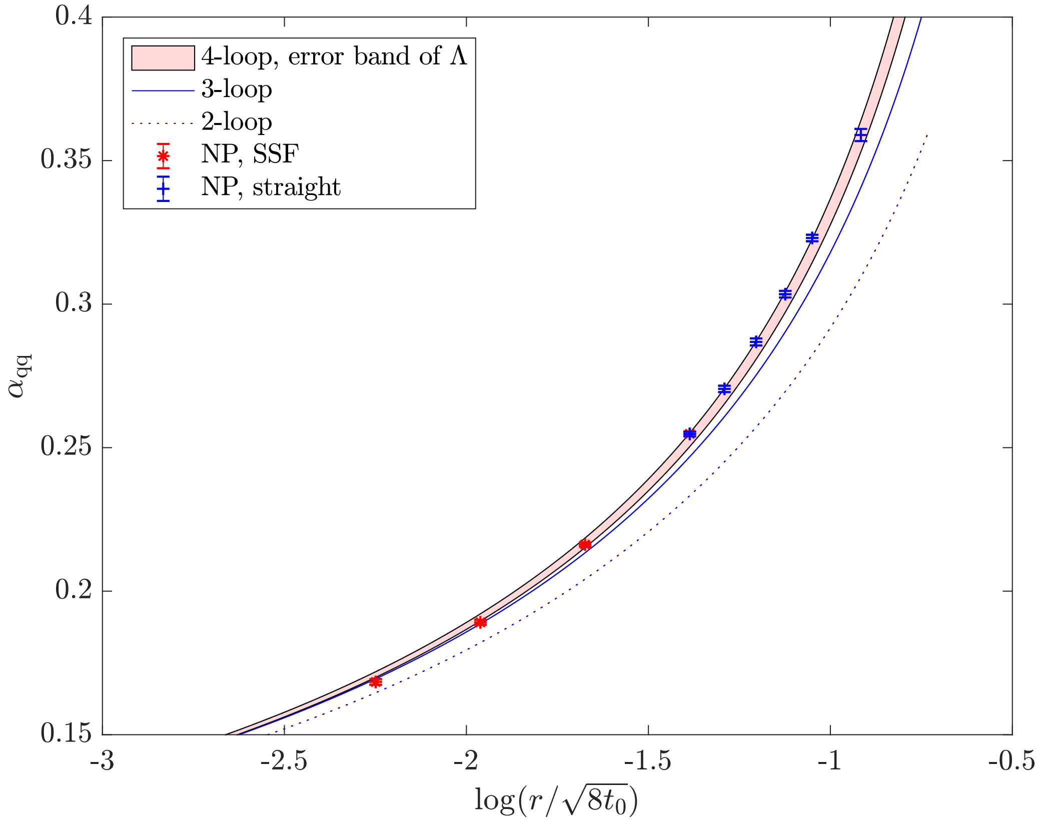
- Large r region (r > 0.1fm)





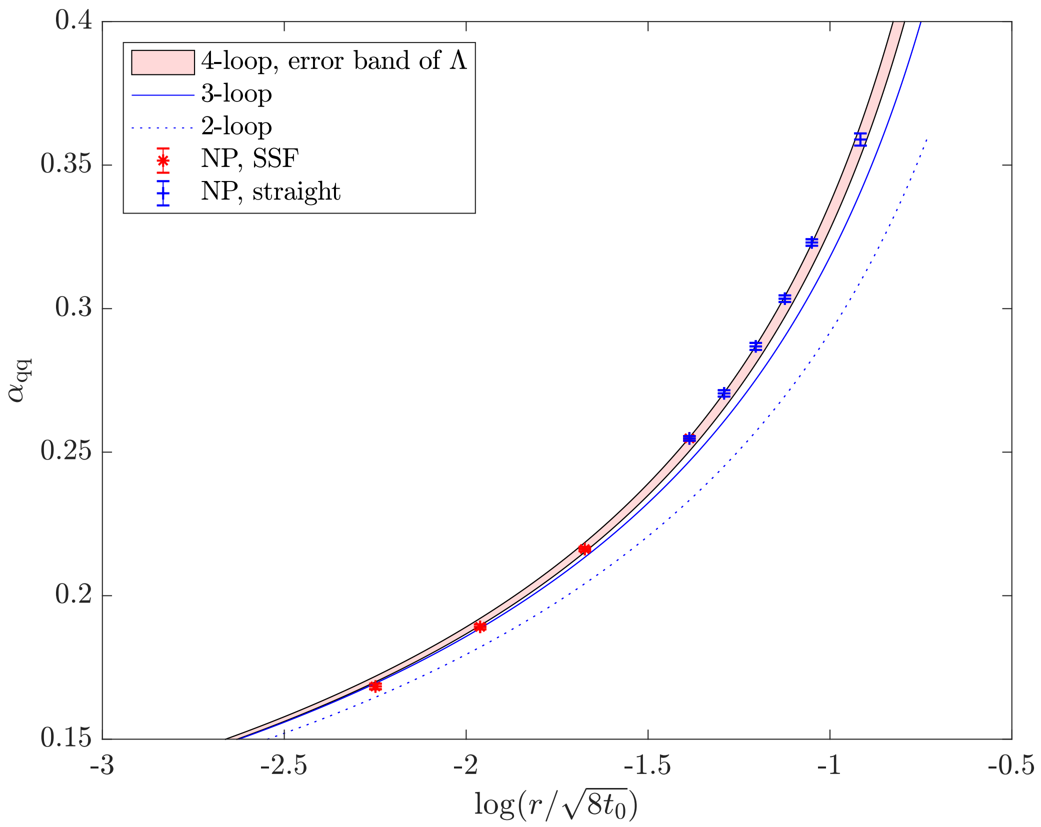
- Gradient flow: log-corrections to a^2 not yet known.

Force



- perturbative prediction with known Λ

Force



- perturbative prediction with known Λ
- Qualitative contact to PT is made

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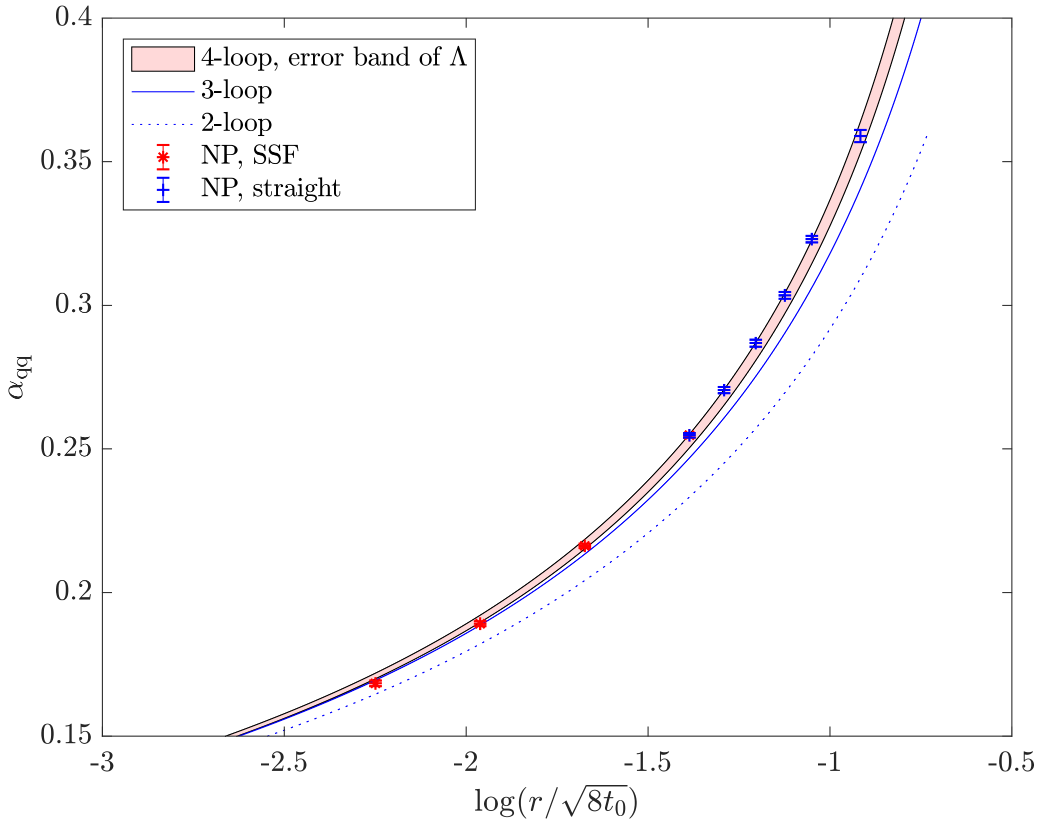

- perturbative prediction with known Λ
- Qualitative contact to PT is made
- But is this safe to determine the Λ-parameter?

Lambda parameter from $\alpha_{\rm qq}$

$$\Lambda/\mu = \varphi_g(g) = \quad = \left(b_0 g^2\right)^{-b_1/(2b_0^2)}\, \mathrm{e}^{-1/(2b_0 g^2)} \exp\left\{-\int_0^g \mathrm{d}x \left[\frac{1}{\beta(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0^2 x}\right]\right\}$$

approximations:

$$\left.\frac{\Lambda}{\mu}\right|^{\rm eff}_{n-\text{loop}} = \left(b_0 g^2\right)^{-b_1/(2b_0^2)}\, \mathrm{e}^{-1/(2b_0 g^2)} \exp\left\{-\int_0^g \mathrm{d}x \left[\frac{1}{\beta_{n-\text{loop}}(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0^2 x}\right]\right\}$$

$$\left.\frac{\Lambda}{\mu}\right|^{\rm eff}_{2-\text{loop}} = \frac{\Lambda}{\mu} + \mathrm{O}(\alpha_{\rm qq})$$

$$\left.\frac{\Lambda}{\mu}\right|^{\rm eff}_{n-\text{loop}} = \frac{\Lambda}{\mu} + \mathrm{O}(\alpha_{\rm qq}^{n-1})$$

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A specialty of qq-coupling:
at high orders there are infrared
divergencies, need to be resummed,
produce log(α) terms —>

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computed from

Peter 97; Schröder 99
Anzai, Kiyo, Sumino, 10
Smirnov, Smirnov, Steinhauser, 10
Brambilla, Pineda, Soto, Vairo, 99
Kniehl, Penin, 99
Brambilla, Garcia i Tormo, Soto, Vairo, 07, 09

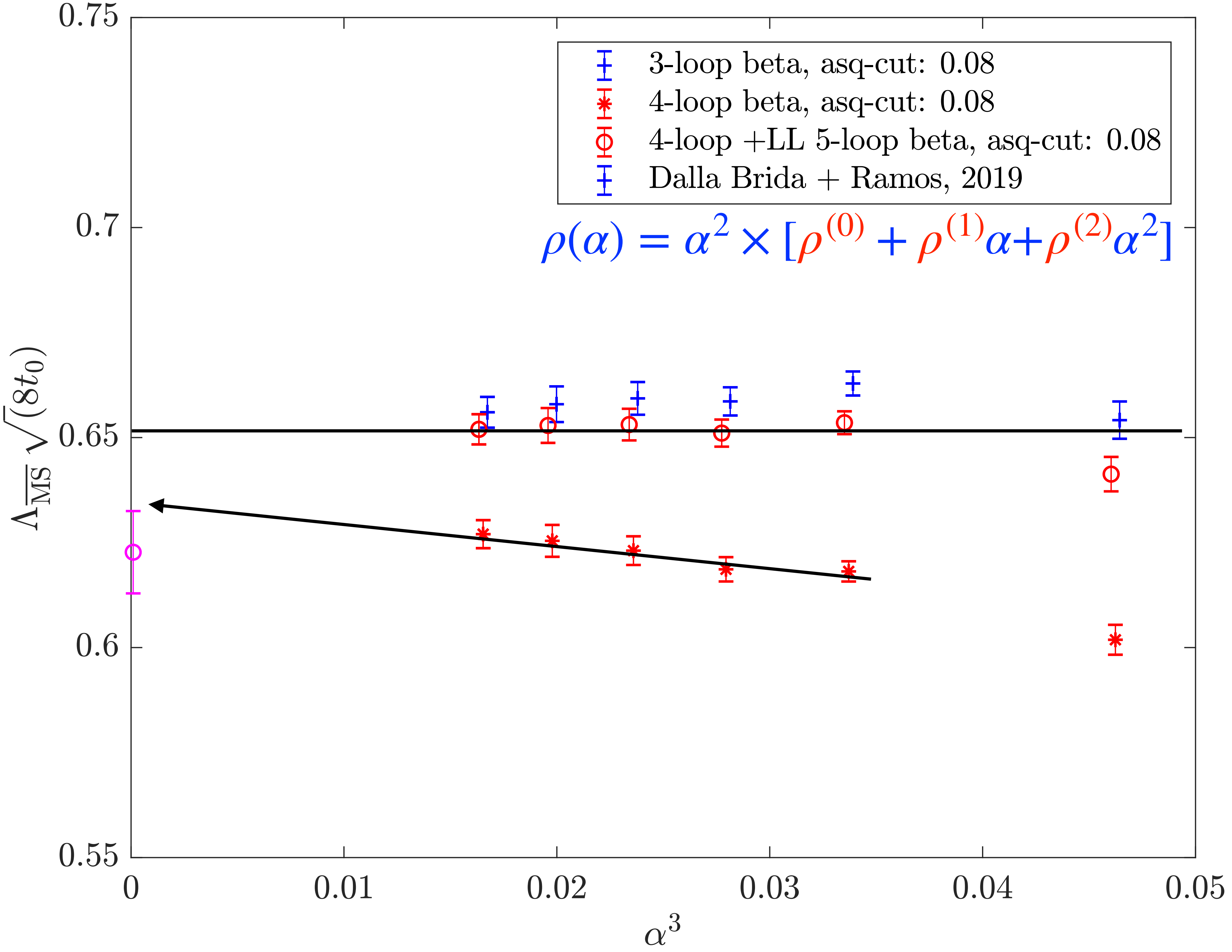
A specialty of qq-coupling:
at high orders there are infrared divergencies, need to be resummed, produce log(α) terms —>

$$\beta_{3-\mathrm{loop}}(g) = -\, g^3\, [\, b_0 + b_1 g^2 + b_2 g^4]$$

4-loop: $+b_3 g^6 + b_{3\mathrm{L}} g^6 \log(\alpha)$

4-loop LL: $+b_{4\mathrm{L}} g^8 \log(\alpha) + b_{4\mathrm{LL}} g^8 [\log(\alpha)]^2$

Results (from 2-stage continuum limit, standard derivative)



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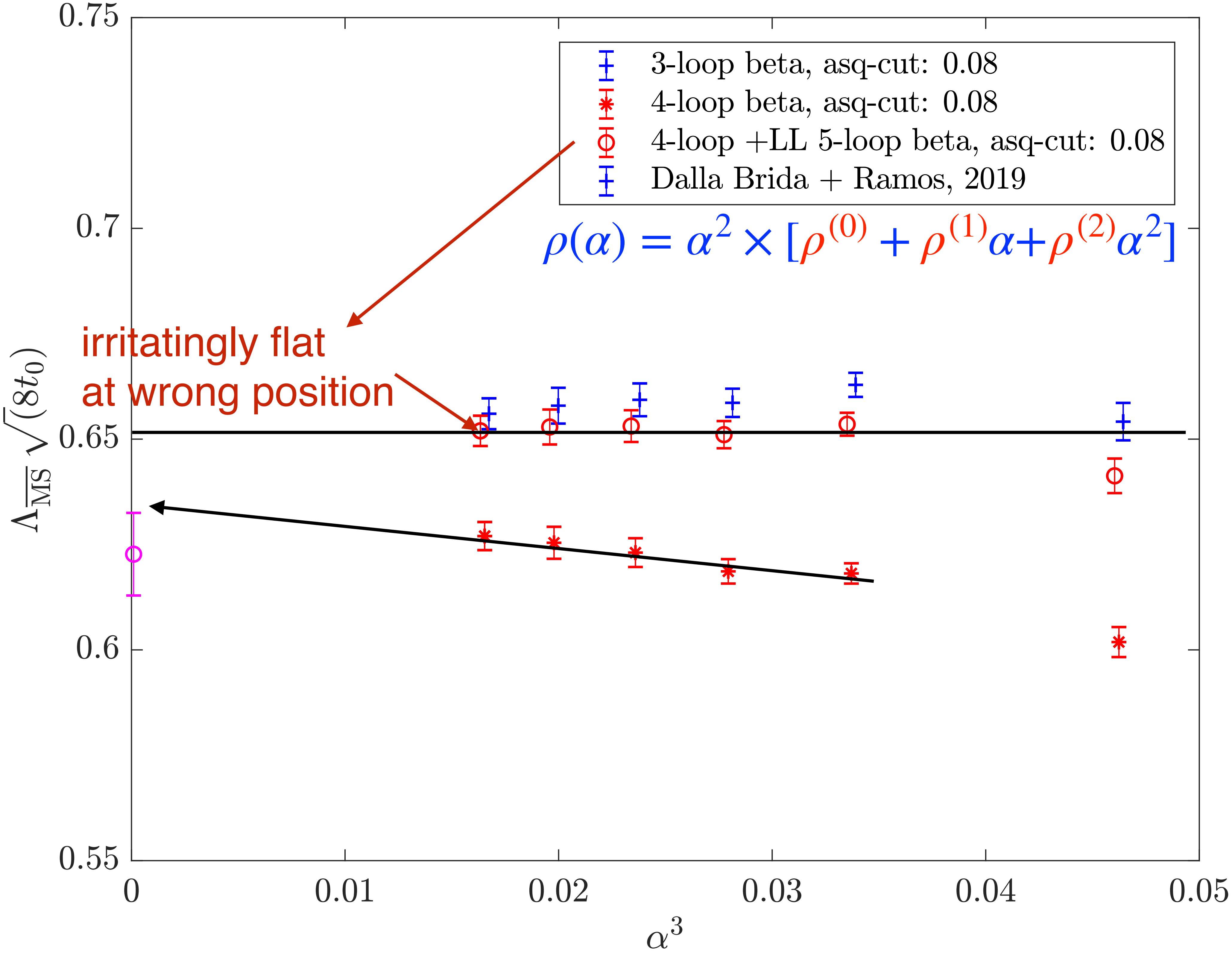

$\rho(\alpha) = \alpha^2 \times [\rho^{(0)} + \rho^{(1)}\alpha + \rho^{(2)}\alpha^2]$

irritatingly flat
at wrong position

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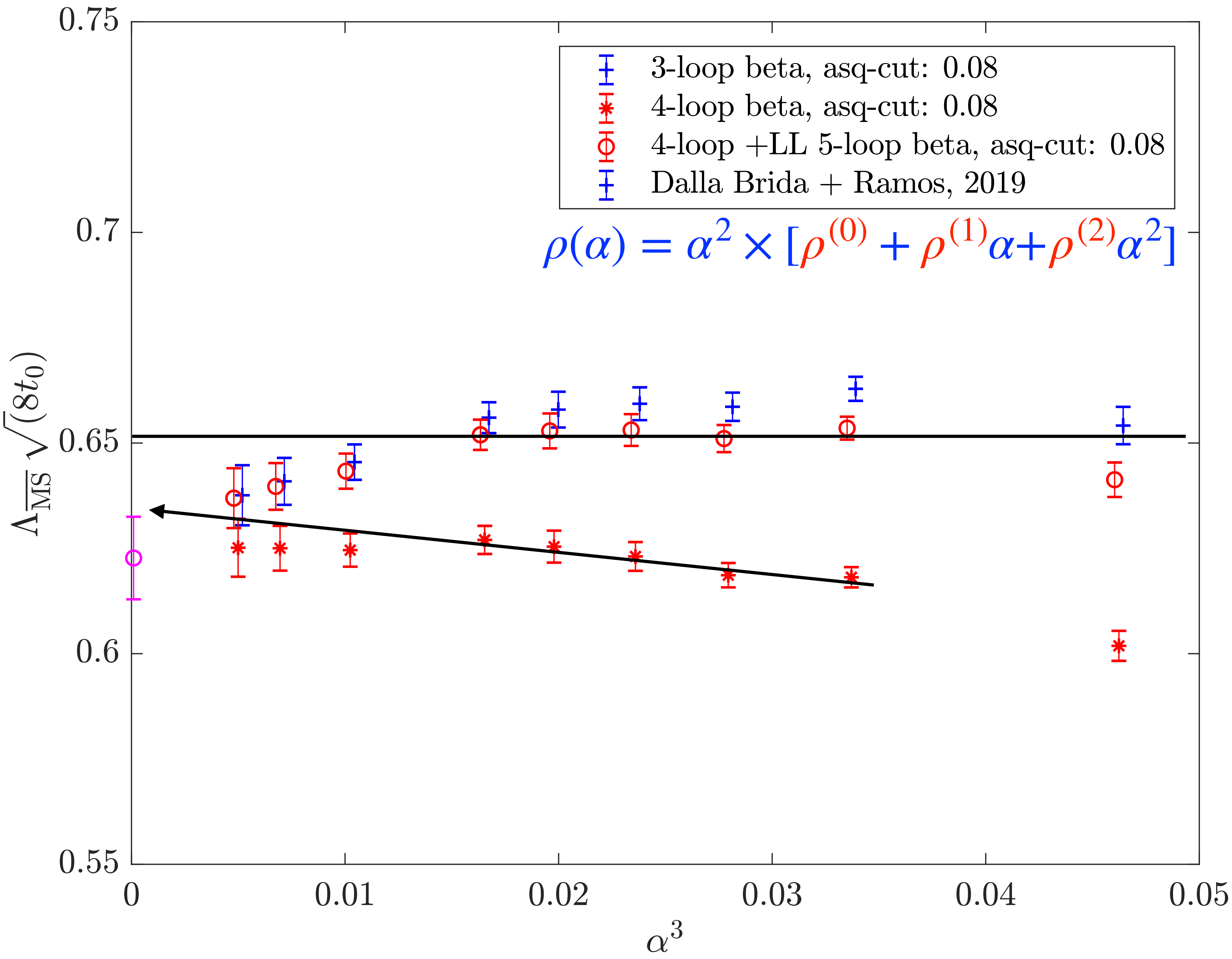

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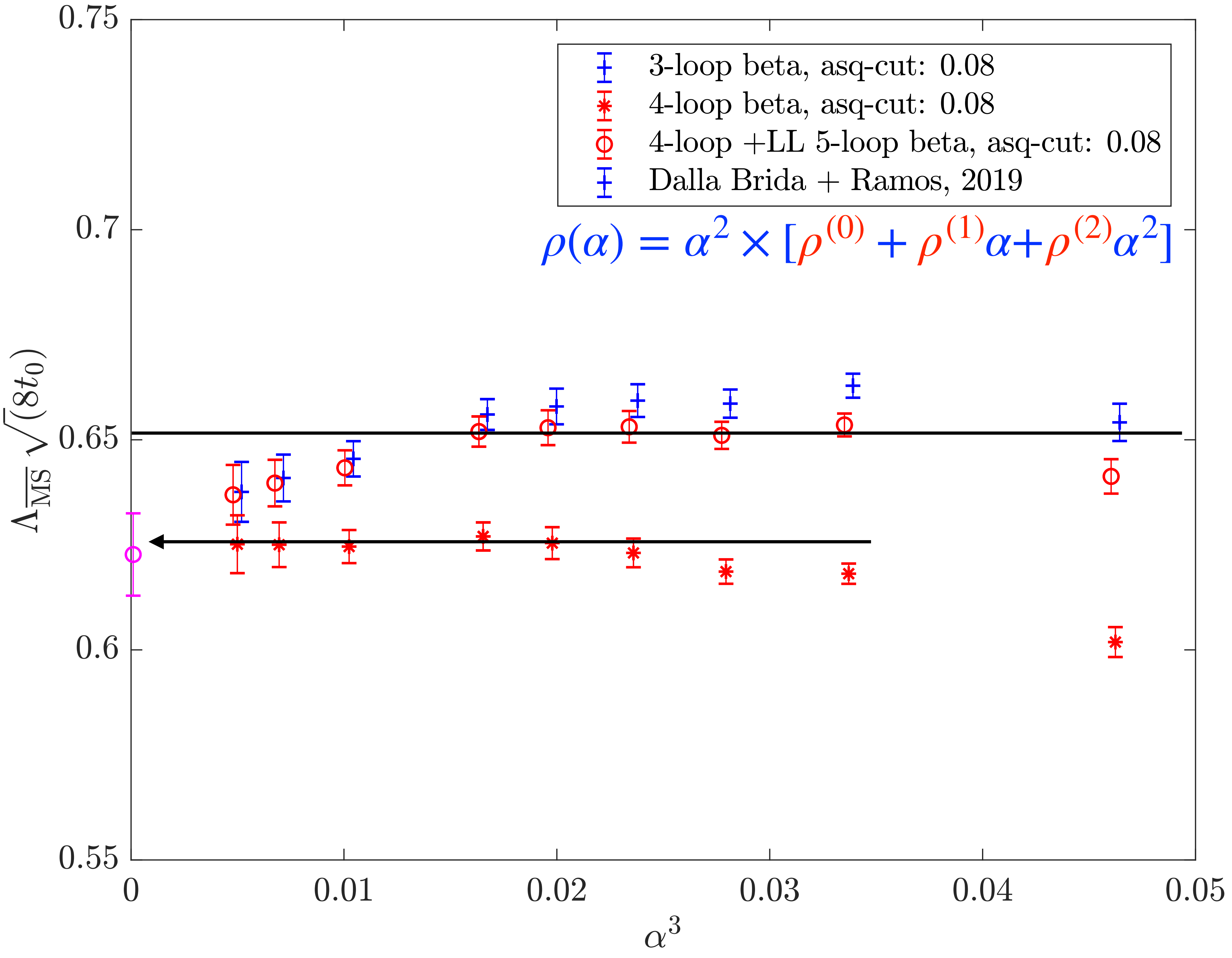

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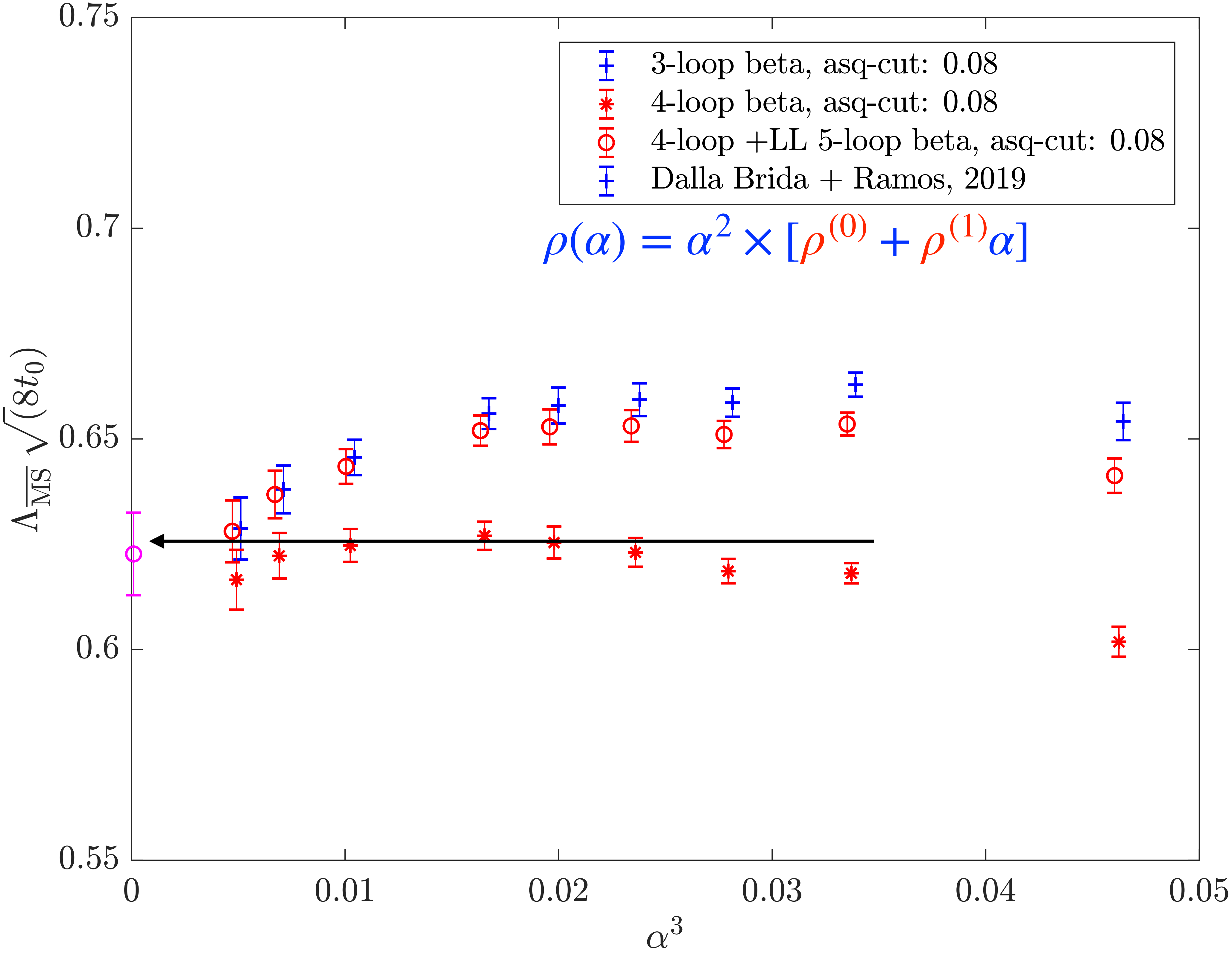

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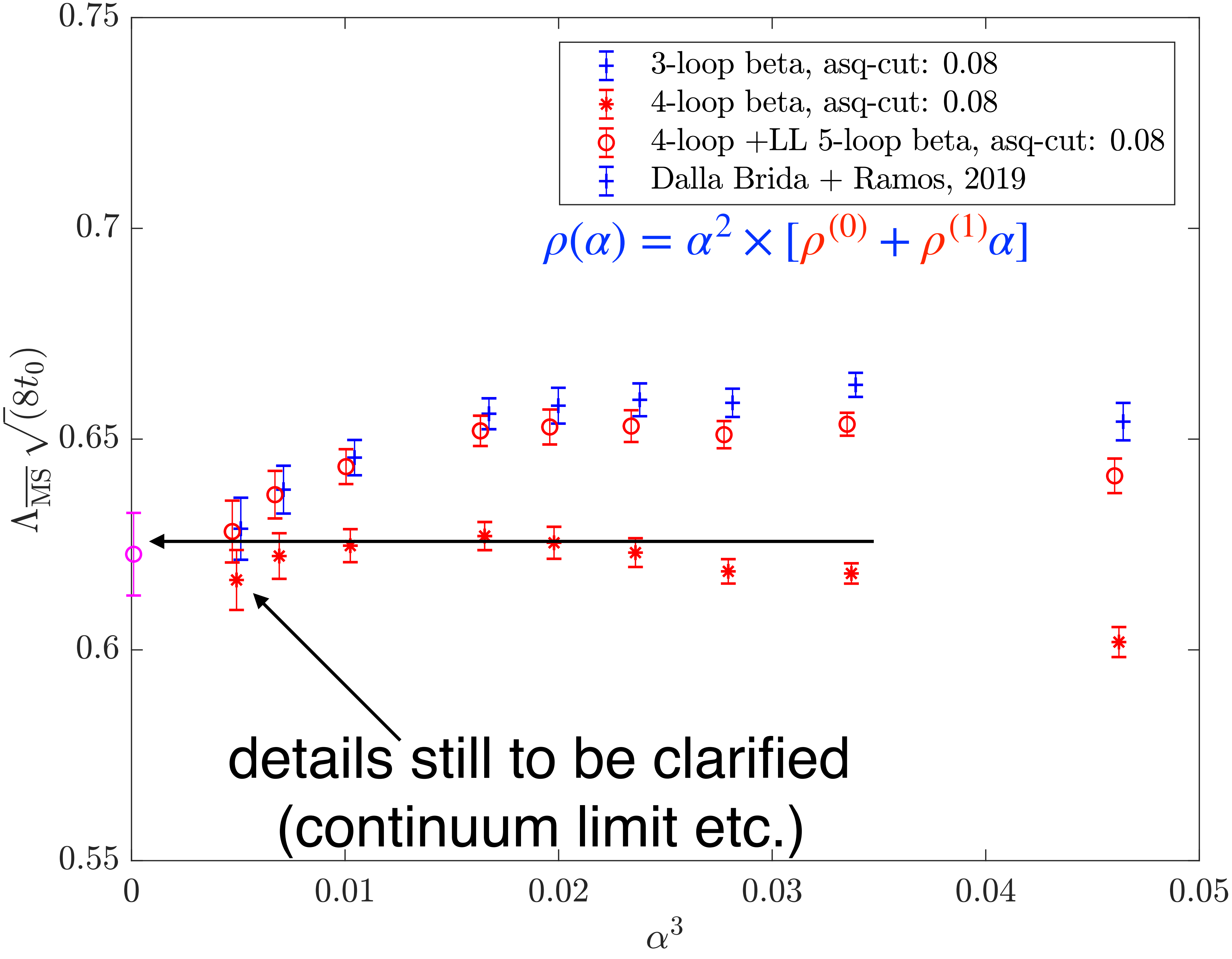
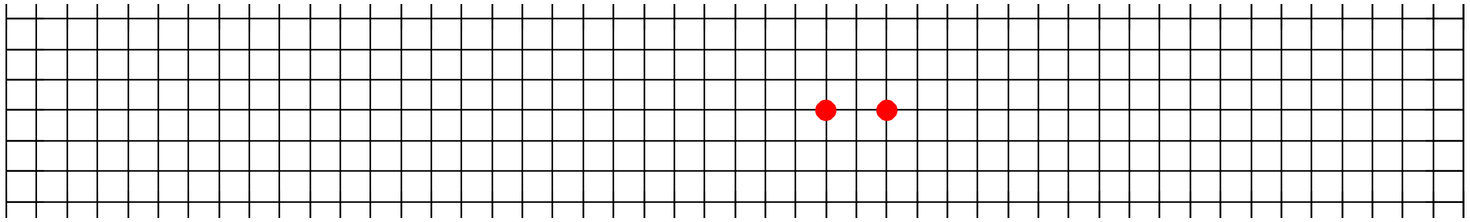


$\rho(\alpha) = \alpha^2 \times [\rho^{(0)} + \rho^{(1)}\alpha]$

details still to be clarified
(continuum limit etc.)

Scale Problem

(α_{qq} as an example)



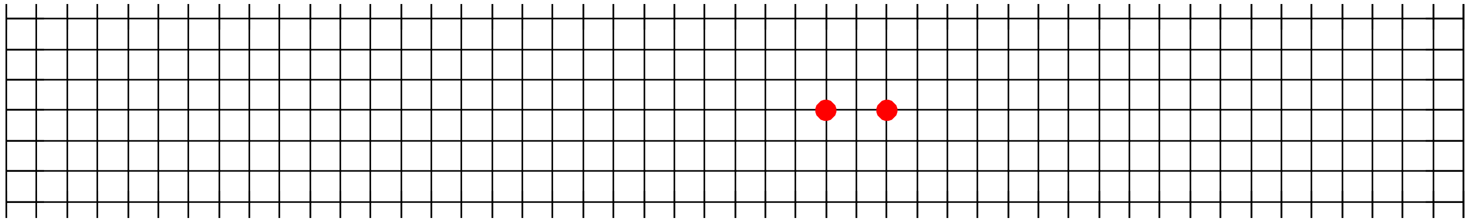
$$\underset{\substack{\uparrow\\ \text{box size}}}{L} \gg \underset{\substack{\uparrow\\ \text{QCD scale, } m_\pi}}{\frac{1}{0.2\mathrm{GeV}}} \gg \frac{1}{\mu} \sim \frac{1}{10\mathrm{GeV}} \gg \underset{\substack{\uparrow\\ \text{spacing}}}{a}$$

$$\Downarrow$$

$$L/a \gg 50$$

Scale Problem

($\alpha_{\rm qq}$ as an example)



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$$\Downarrow$$

$$L/a \gg 50$$

Solution: $\boxed{L = 1/\mu}$ $\longrightarrow$ left with $L/a \gg 1$ [Wilson, ... , Lüscher, Weisz, Wolff]

Finite size effect as a physical observable; finite size scaling!

Strategy

finite volume coupling $\alpha_{\mathrm{SF}}(\mu), \mu = 1/L$
defined at zero quark mass

$$\begin{tikzcd}
L_{\mathrm{max}} = \mathrm{const.}/m_{\mathrm{prot}} = \mathrm{O}(\tfrac{1}{2}\mathrm{fm}) : \arrow[r] & \alpha_{\mathrm{SF}}(\mu = 1/L_{\mathrm{max}}) \arrow[d] \\
 & \alpha_{\mathrm{SF}}(\mu = 2/L_{\mathrm{max}}) \arrow[d] \\
 & \bullet \\
 & \bullet \\
 & \bullet \arrow[d] \\
 & \alpha_{\mathrm{SF}}(\mu = 2^n/L_{\mathrm{max}} = 1/L_{\mathrm{min}}) \arrow[d, "\text{PT:}"'] \\
 & \Lambda_{\mathrm{SF}} L_{\mathrm{max}} = \#
\end{tikzcd}$$

always $a/L \ll 1$

Result is a value for $\boxed{\Lambda_{\mathrm{SF}}/m_{\mathrm{prot}} = \#}$

The step scaling function

We leave the discussion of a finite volume coupling for later.
Discuss first the

Step scaling function

- It is a discrete β function:

$$\sigma(s,\bar{g}^2(L)) = \bar{g}^2(sL) \qquad \text{mostly } s=2$$

The step scaling function

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Discuss first the

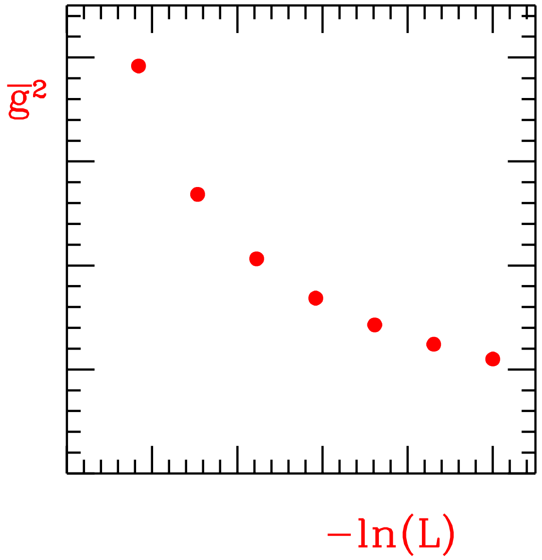
Step scaling function

- It is a discrete β function:

$$\sigma(s,\bar{g}^2(L)) = \bar{g}^2(sL) \qquad \text{mostly } s=2$$

- determines the non-perturbative running:

$$\begin{aligned} u_0 &= \bar{g}^2(L_{\max}) \\ &\downarrow \\ \sigma(2,u_{k+1}) &= u_k \\ &\downarrow \\ u_k &= \bar{g}^2(2^{-k}L_{\max}) \end{aligned}$$



The step scaling function

$\sigma(s,u) = \bar{g}^2(sL)$ with $u = \bar{g}^2(L)$

On the lattice:
additional dependence on the resolution a/L

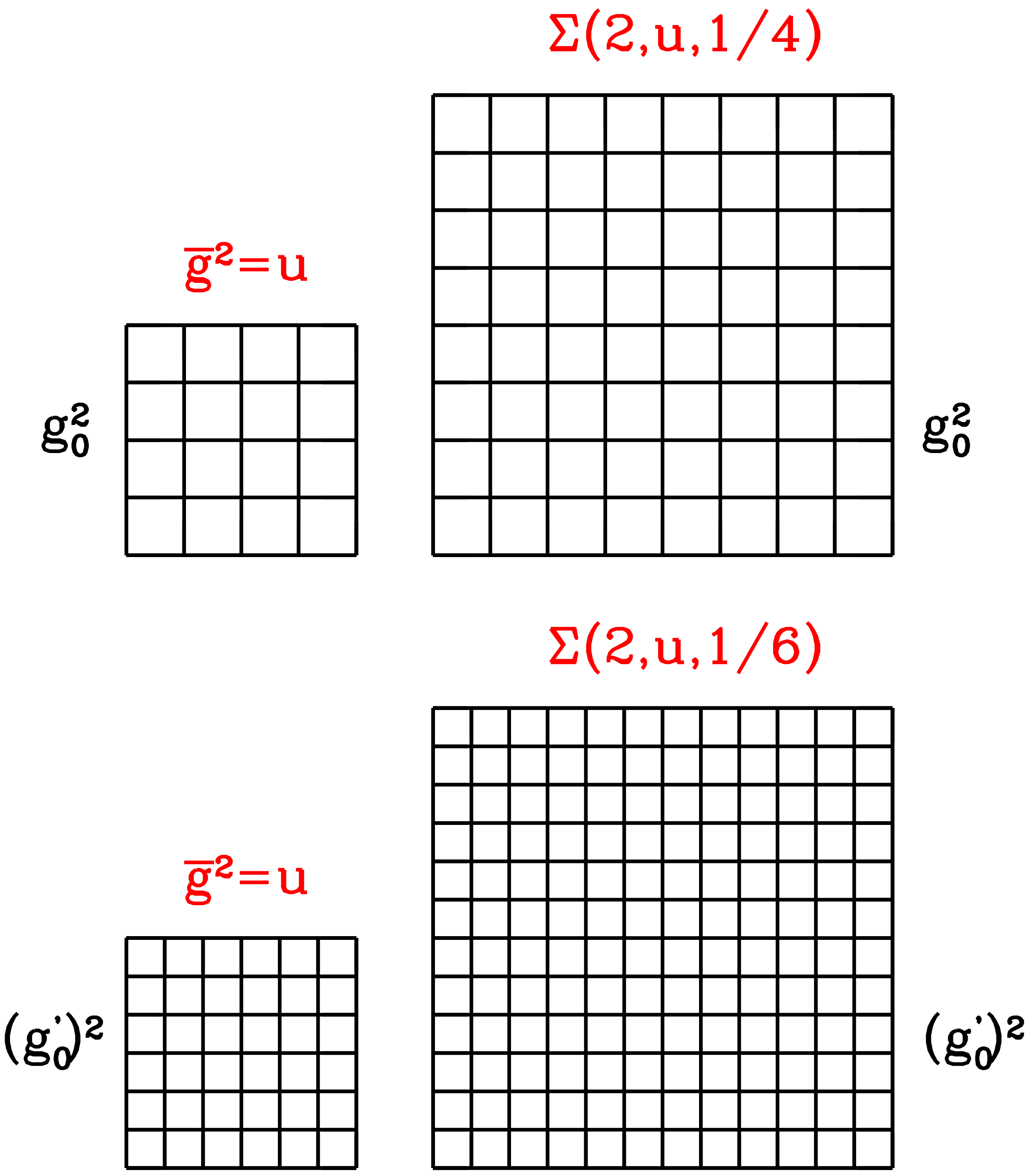
g_0 fixed, L/a fixed:

$$\begin{aligned} \bar{g}^2(L) = u, \qquad & \bar{g}^2(sL) = u'\,, \\ \Sigma(s,u,a/L) \quad = \quad & u' \end{aligned}$$

continuum limit:

$$\Sigma(s,u,a/L) = \sigma(s,u) + \mathrm{O}(a/L)$$

in the following always $s = 2$



everywhere: $m = 0$ (PCAC mass defined in $(L/a)^4$ lattice)

The step scaling function

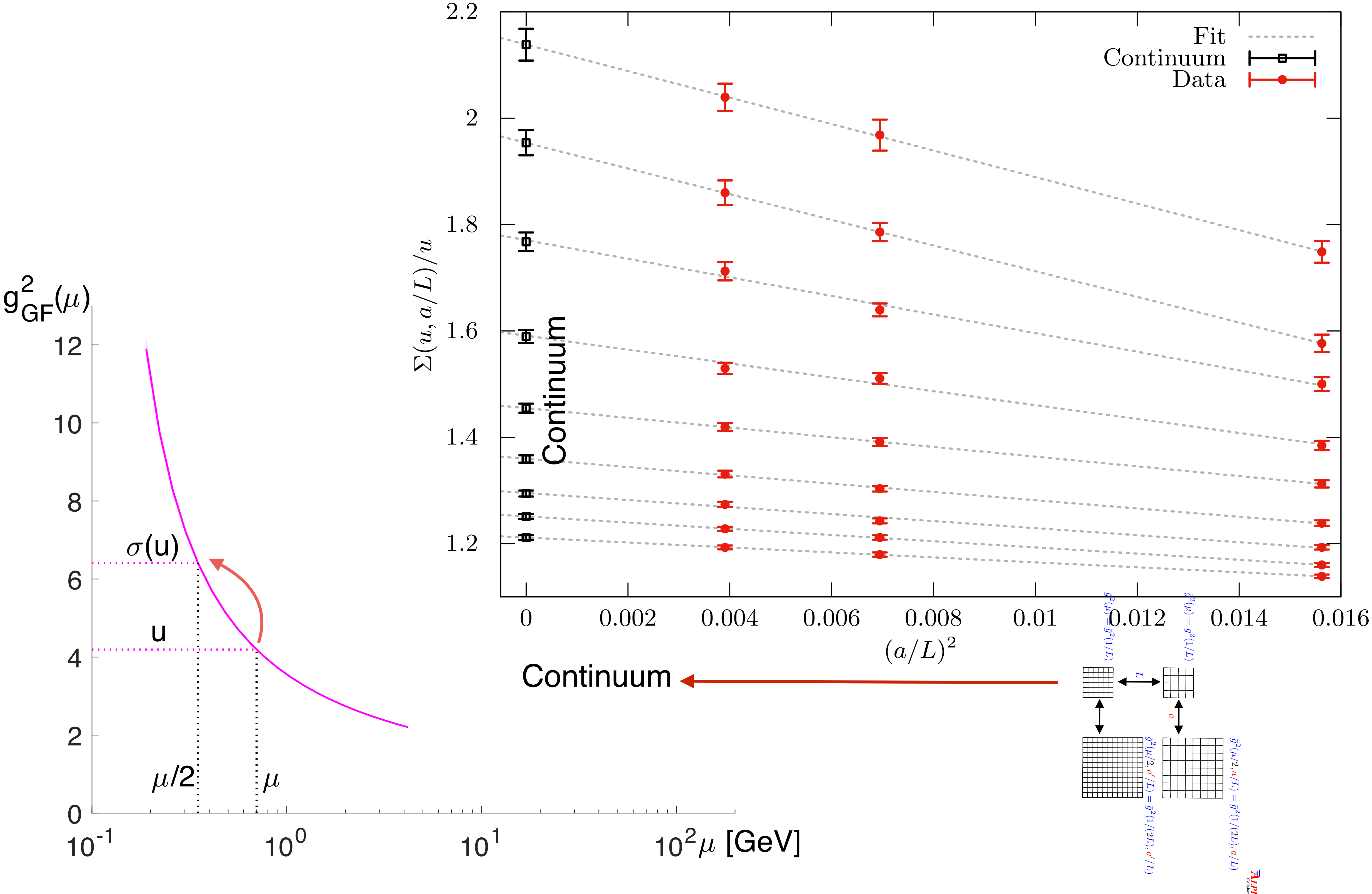
(Table from $N_\mathrm{f} = 2$, ALPHA Collaboration)

L/a	β	κ	$\bar{g}^2$	$d\bar{g}^2$	m	dm
$u = 1.1814$						
4	8.2373	0.1327957	1.1814	0.0005	0.00100	0.00011
5	8.3900	0.1325800	1.1807	0.0012	-0.00018	0.00009
6	8.5000	0.1325094	1.1814	0.0015	-0.00036	0.00003
8	8.7223	0.1322907	1.1818	0.0029	-0.00115	0.00004
8	8.2373	0.1327957	1.3154	0.0055	0.00020	0.00005
10	8.3900	0.1325800	1.3287	0.0059	0.00097	0.00007
12	8.5000	0.1325094	1.3253	0.0067	-0.00102	0.00002
16	8.7223	0.1322907	1.3347	0.0061	-0.00194	0.00002
L/a			$\Sigma(1.1814, a/L)$	$\delta\Sigma$		
4			1.3154	0.0055		
5			1.3296	0.0061		
6			1.3253	0.0070		
8			1.3342	0.0071		

- tune κ, g_0 to have desired $m \approx 0$, fixed $\bar{g}^2(L)$
- propagate errors from $\bar{g}^2(L)$, shift mean values if necessary
$\longrightarrow \Sigma, \delta\Sigma$

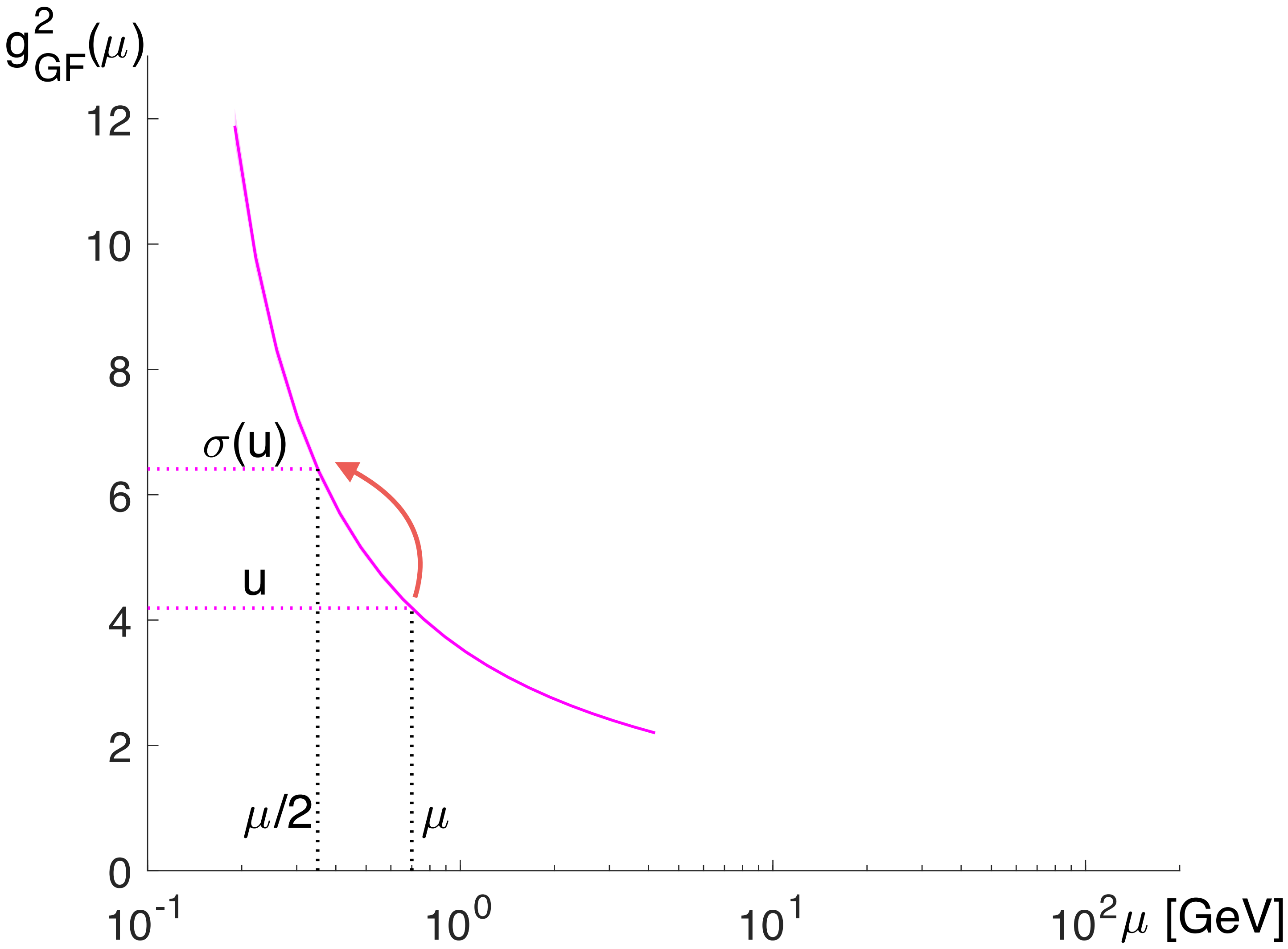
Example continuum extrapolation of step scaling functions

Continuum extrapolations of σ(u)=Σ(u,0), N$_f$=3



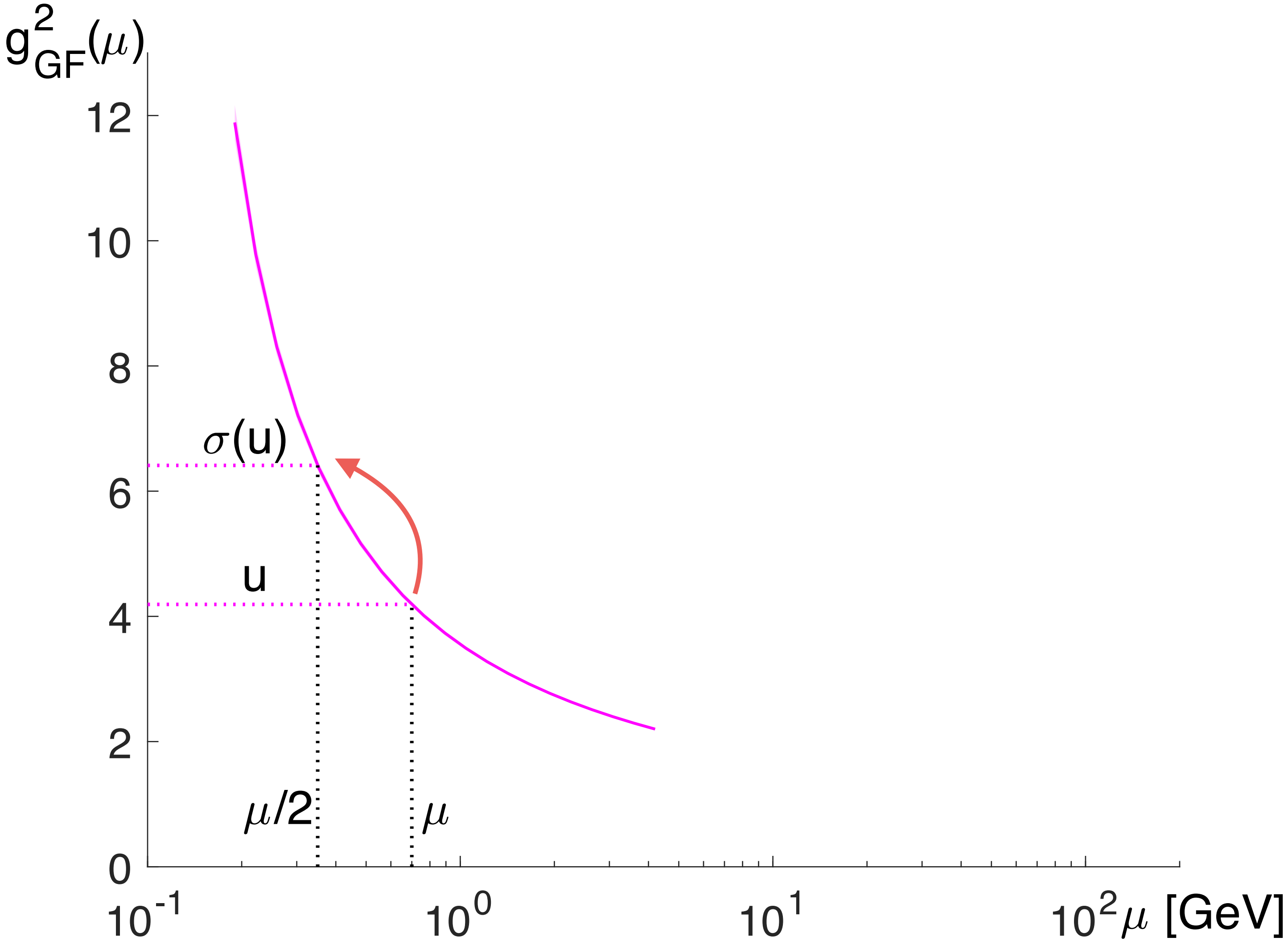
The β-function from the step scaling function

$$\int_{g(\mu)}^{\sigma(g(\mu))} \frac{-1}{\beta(x)} \mathrm{d}x = \log(2)$$



The β-function from the step scaling function

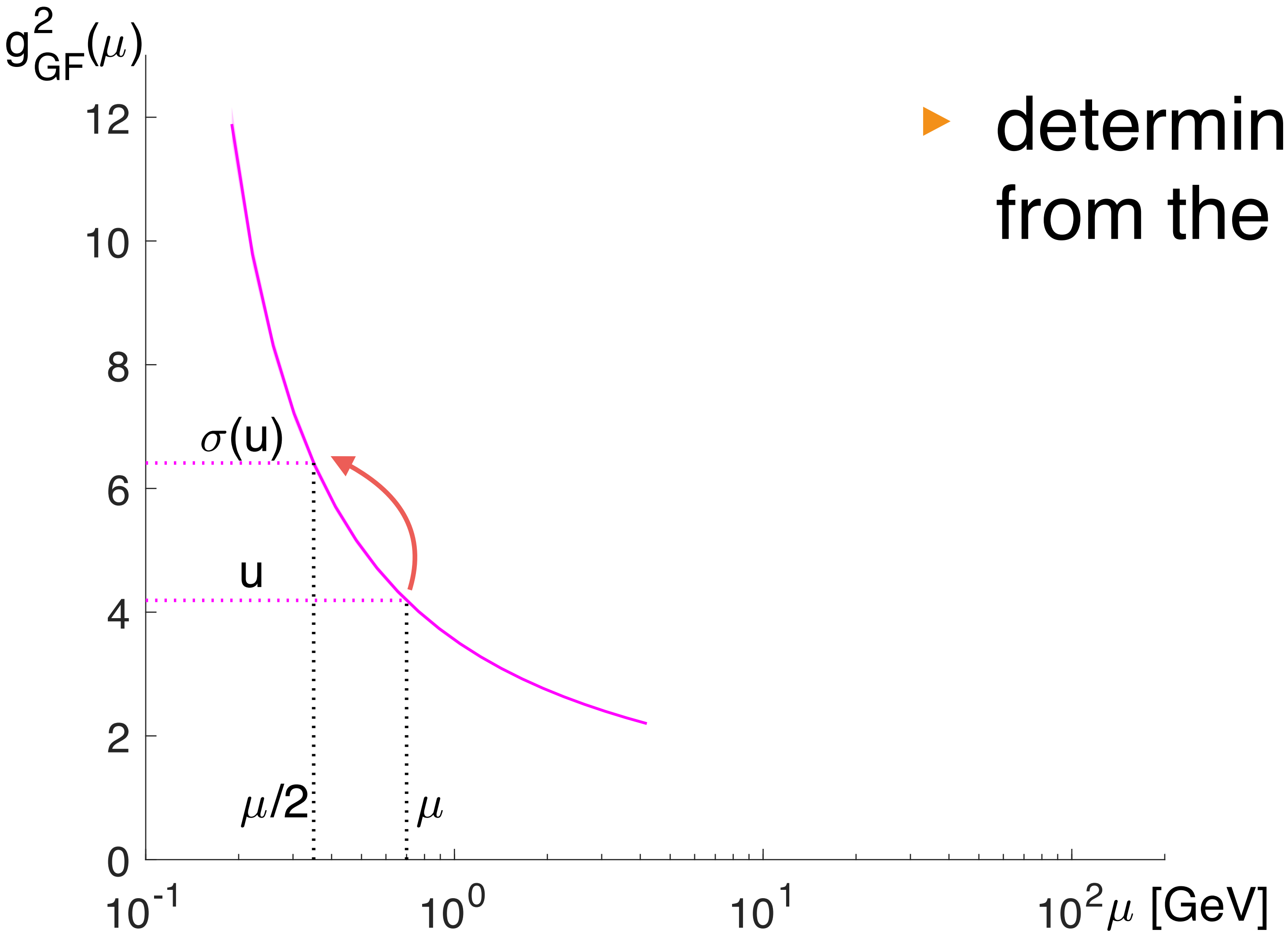
$$\int_{\sqrt{u}}^{\sqrt{\sigma(u)}} \frac{-1}{\beta(x)} \mathrm{d}x = \log(2)$$



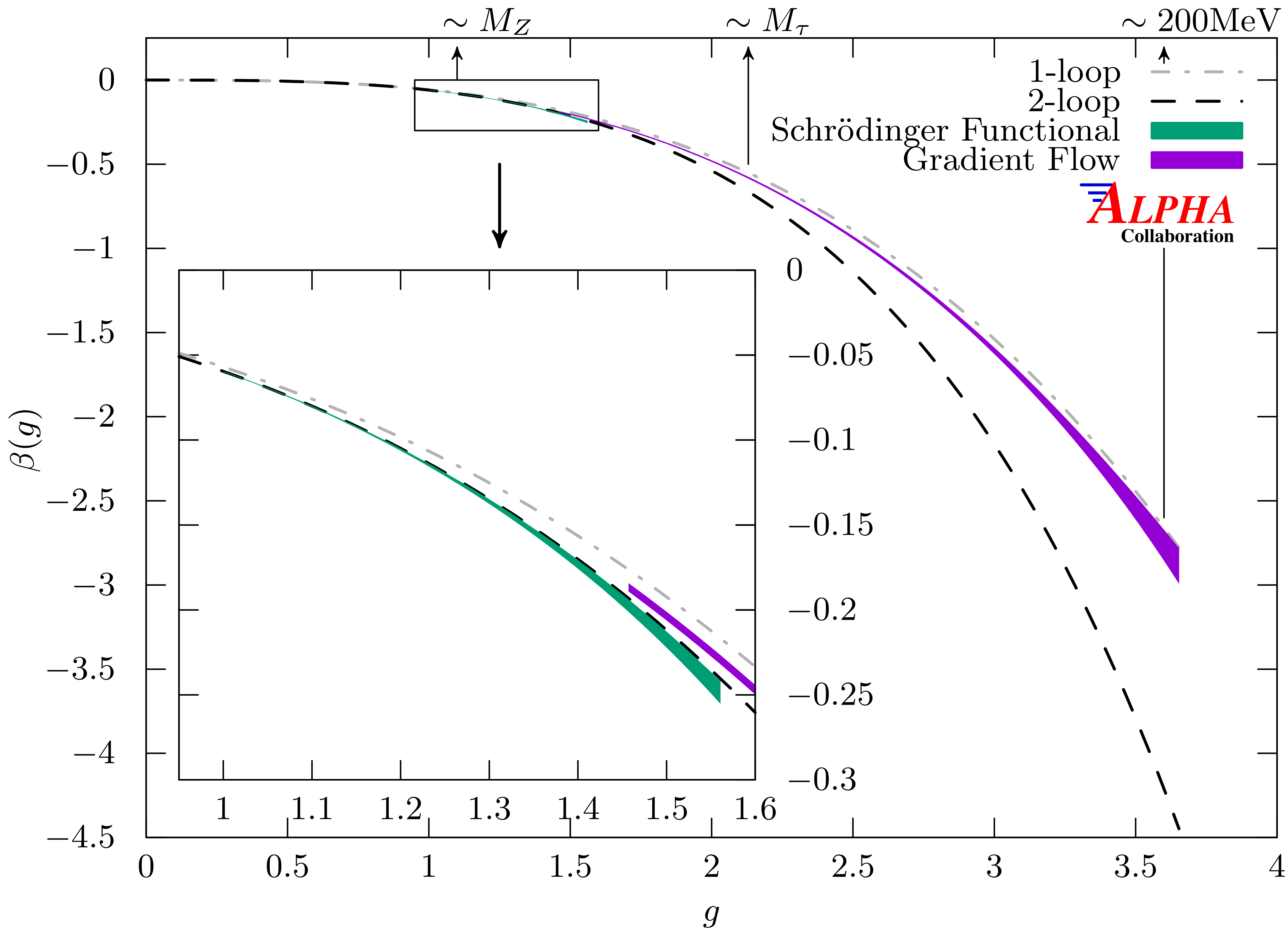
The β-function from the step scaling function

$$\int_{\sqrt{u}}^{\sqrt{\sigma(u)}} \frac{-1}{\beta(x)} \mathrm{d}x = \log(2)$$

- smooth fit function for β(x)
- determine parameters in fit fct from the data points σ(u)



Non-perturbative β-functions for N_f=3 QCD



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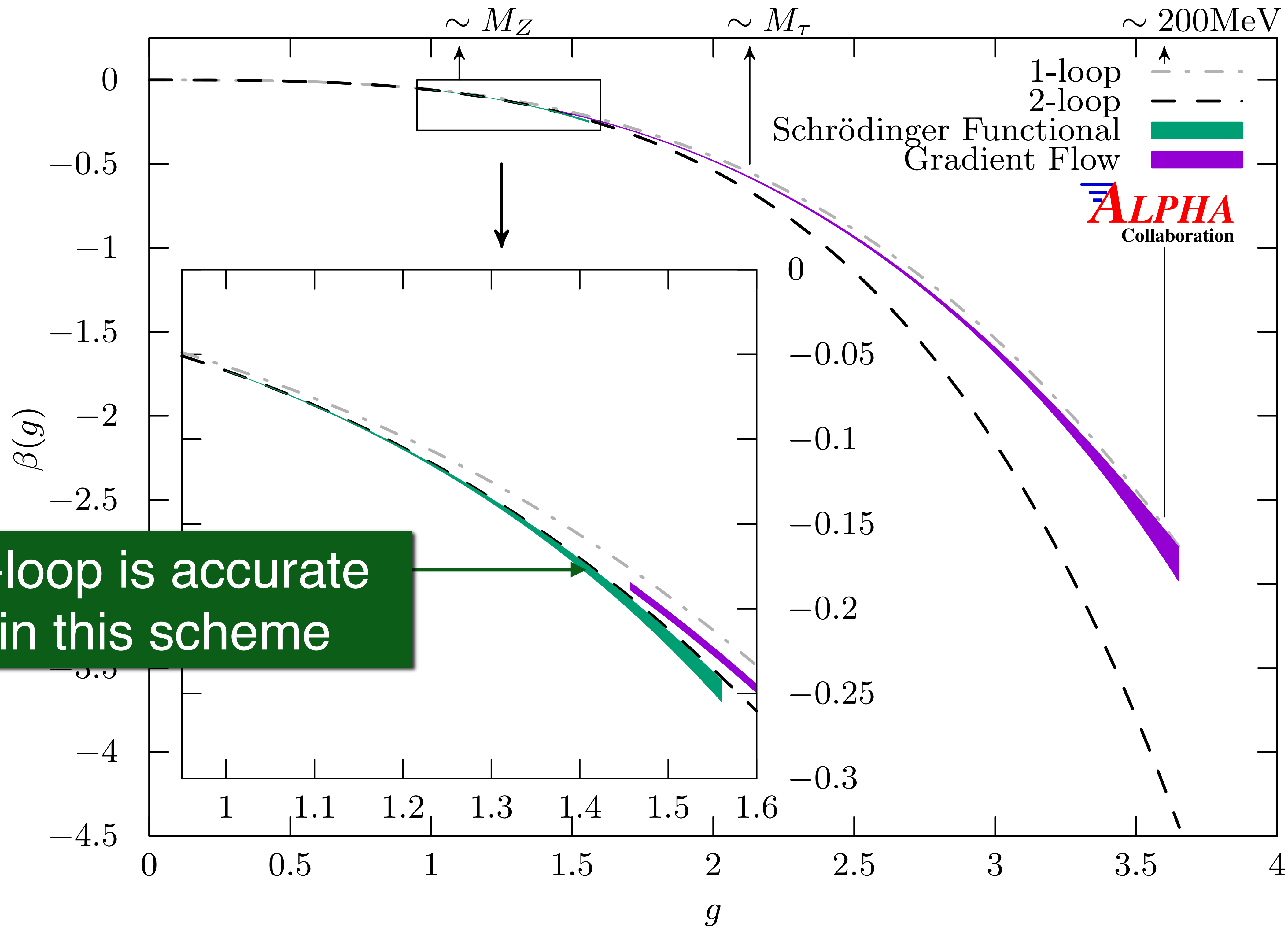
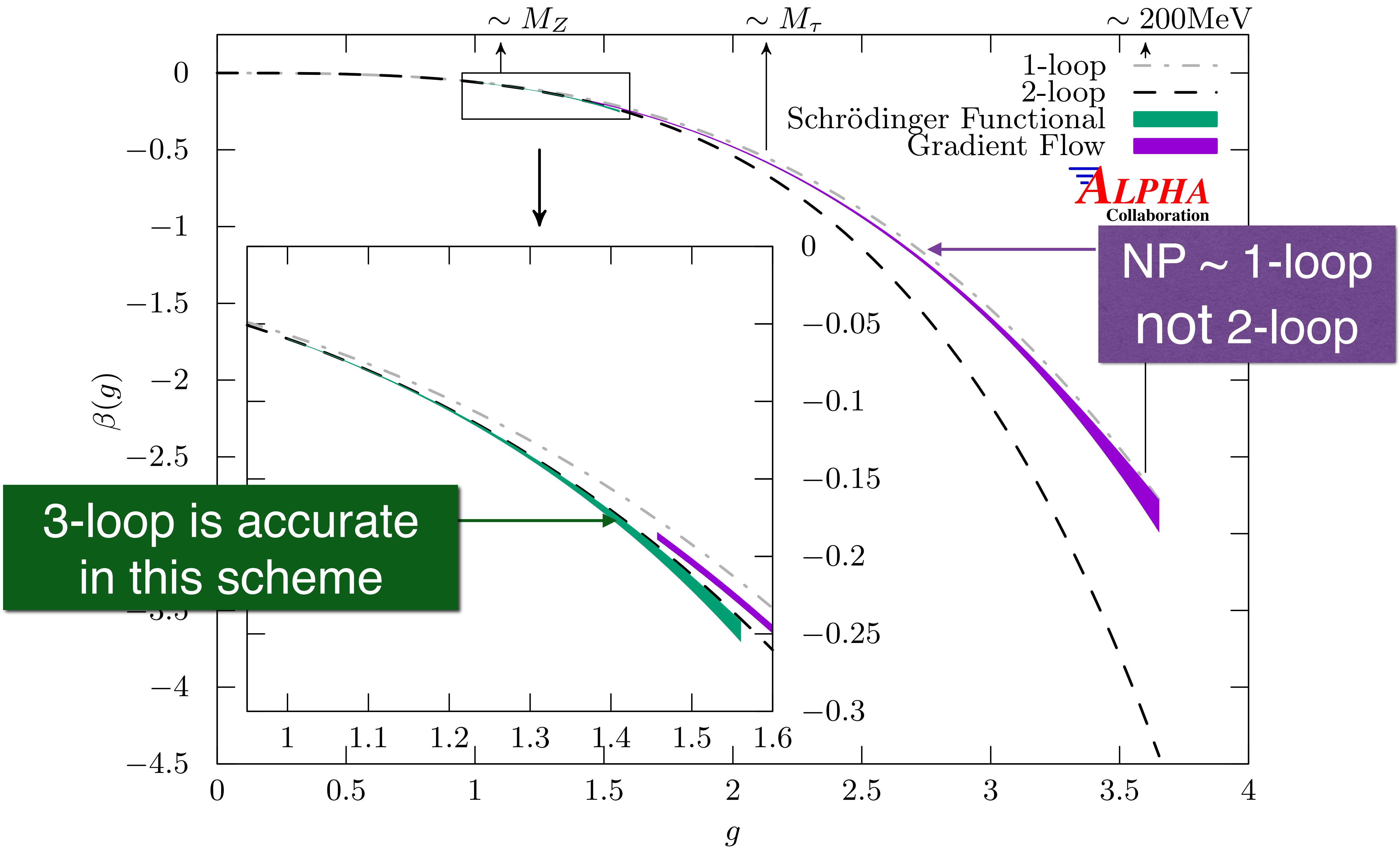


3-loop is accurate in this scheme

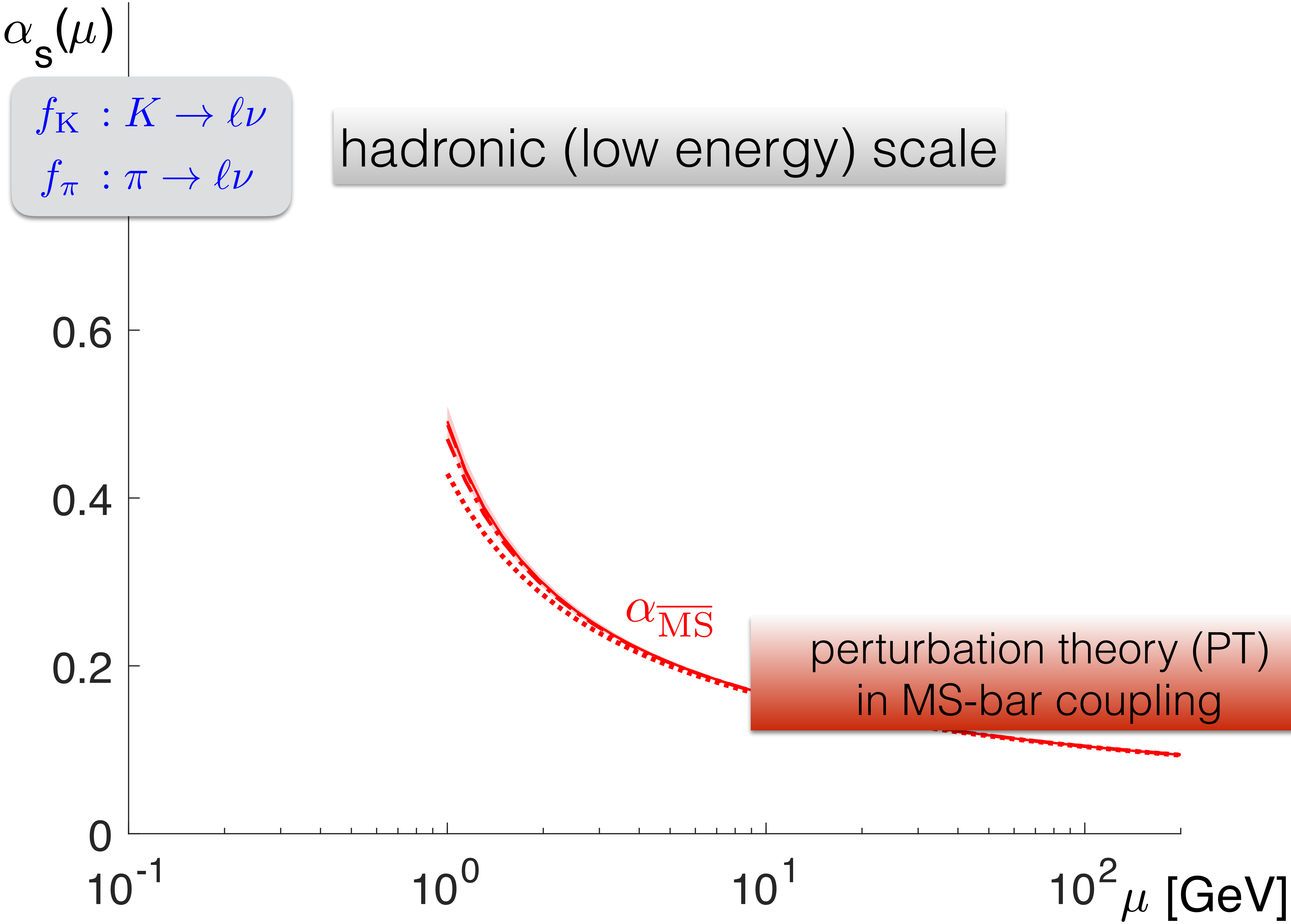
Non-perturbative β-functions for N_f=3 QCD



NP ~ 1-loop not 2-loop

3-loop is accurate in this scheme

Overall strategy



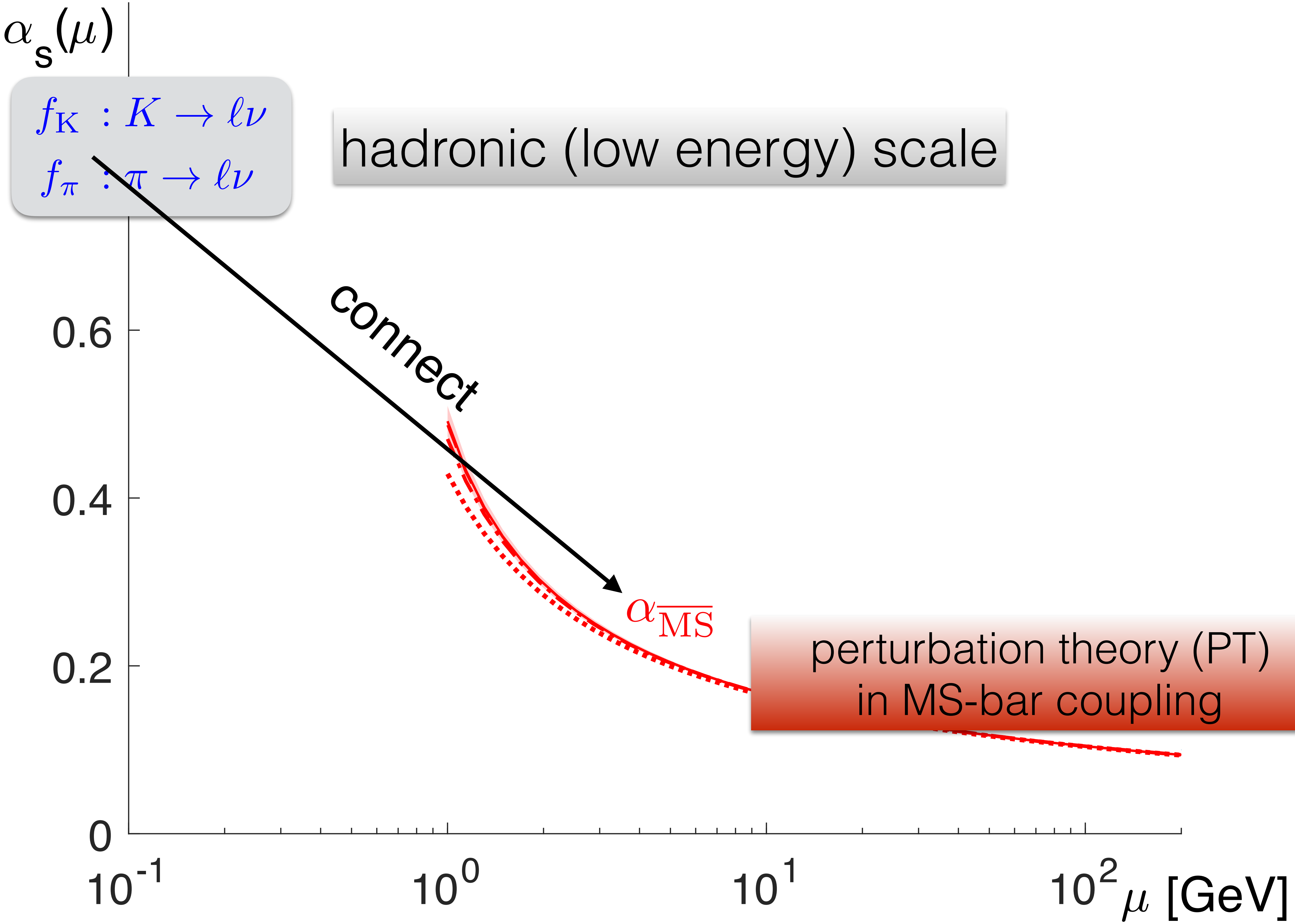
$f_{\mathrm{K}}\ : K \to \ell\nu$

$f_{\pi}\ : \pi \to \ell\nu$

hadronic (low energy) scale

perturbation theory (PT)
in MS-bar coupling

Overall strategy



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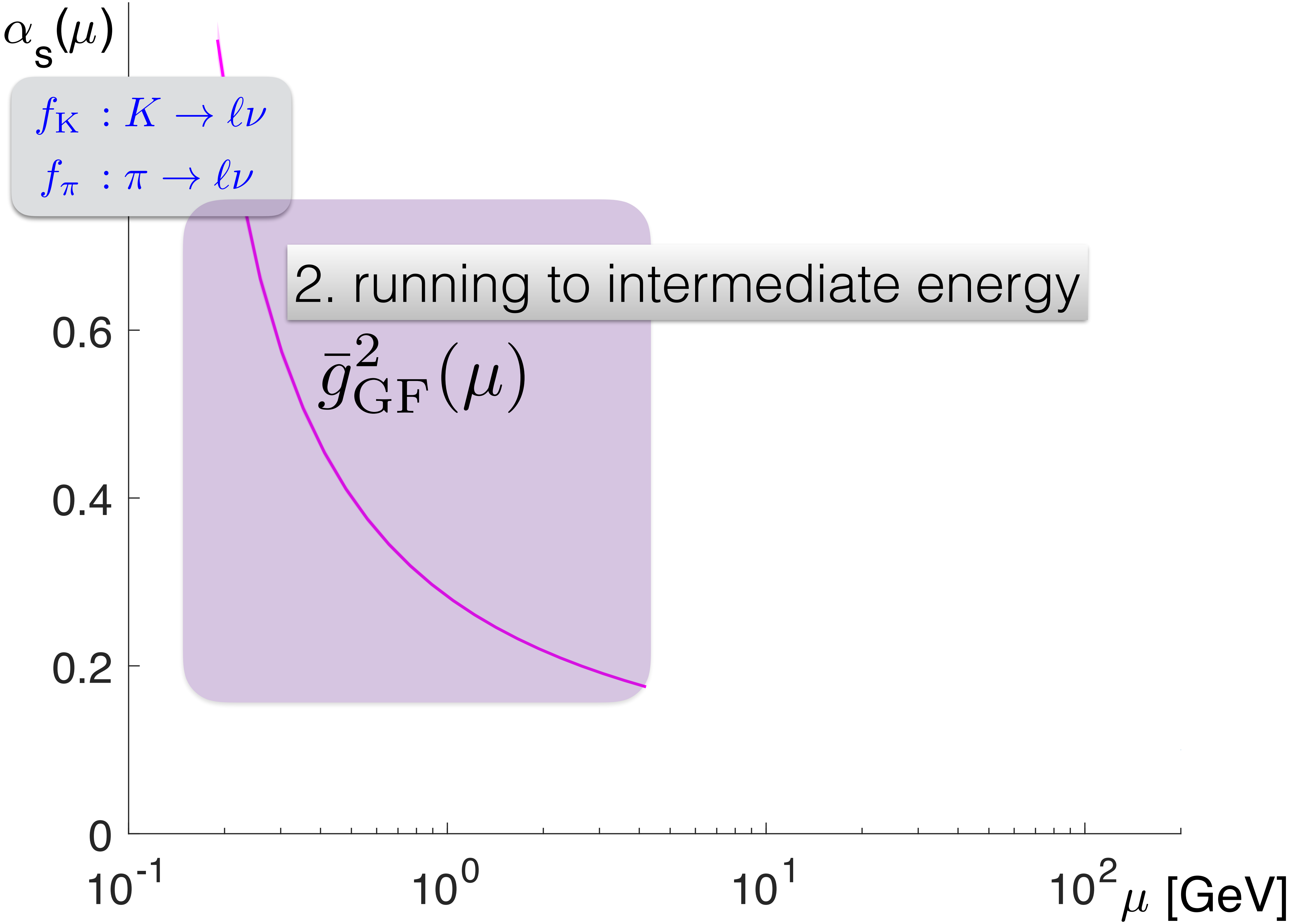


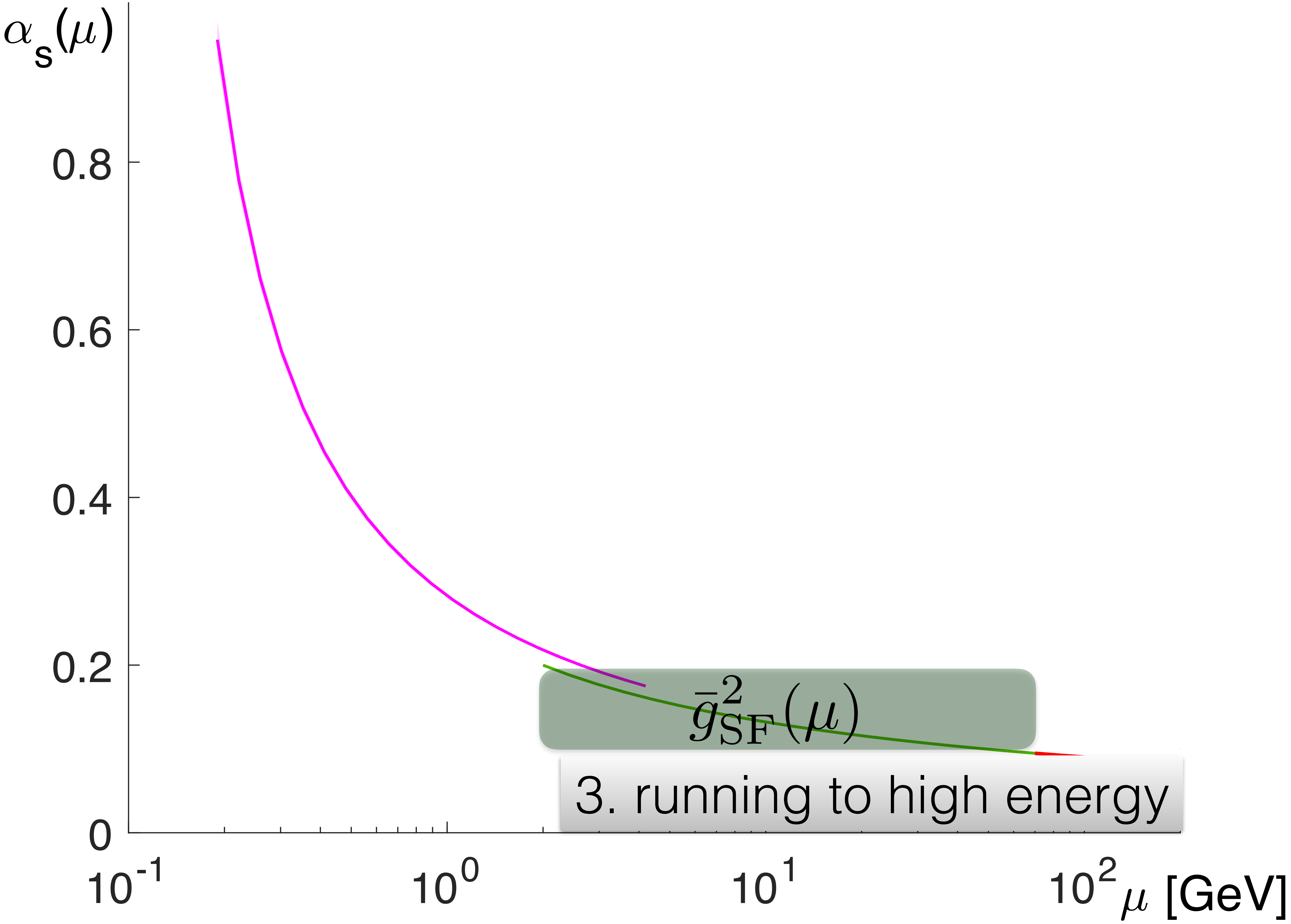
f_K : $K \to \ell\nu$

f_π : $\pi \to \ell\nu$

2. running to intermediate energy

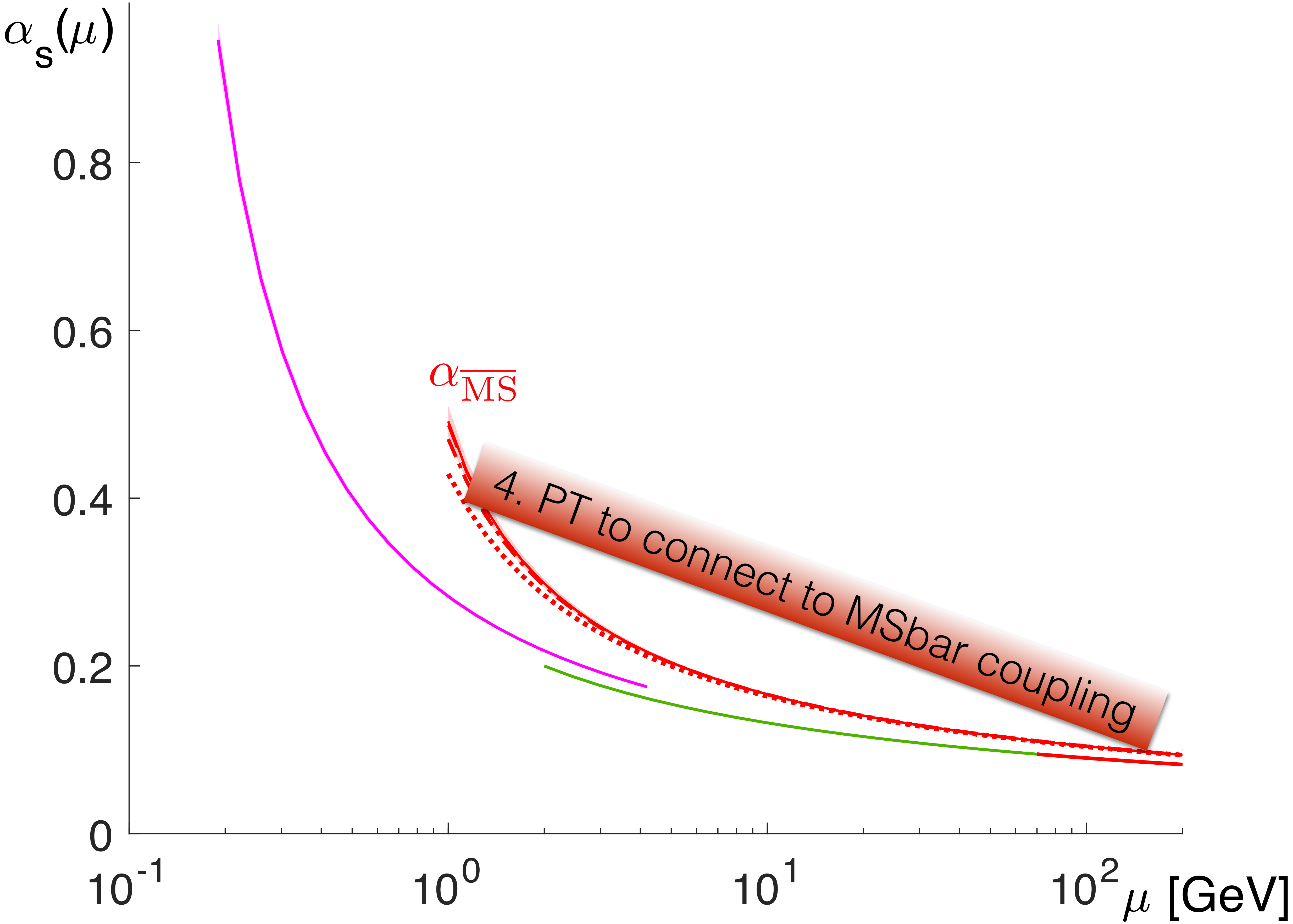
$\bar{g}^2_{\mathrm{GF}}(\mu)$

Overall strategy



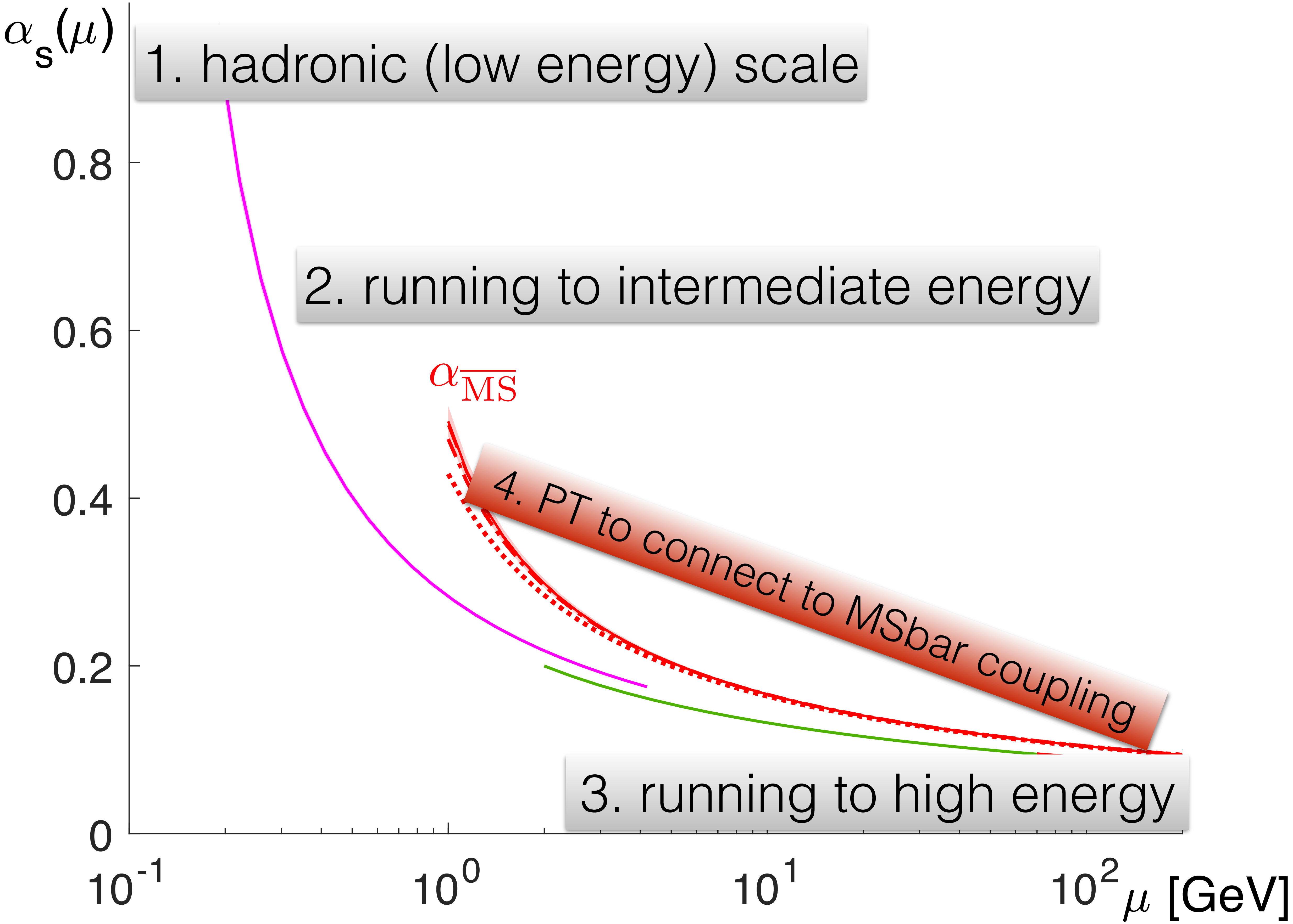
3. running to high energy

Overall strategy



4. PT to connect to MSbar coupling

Overall strategy



1. hadronic (low energy) scale

2. running to intermediate energy

4. PT to connect to MSbar coupling

3. running to high energy

1. Determination of hadronic scale: CLS Ensembles

- CLS Ensembles
  - Large volume, large scale simulations, with theoretically well founded improved Wilson action
  - coordinated between

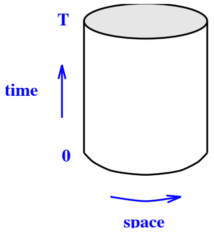
CERN
MADRID
MAINZ
MILANO + ROMA
REGENSBURG
DESY, Standort ZEUTHEN

coordinated by S. Schaefer, Data management H. Simma

1. Determination of hadronic scale: **CLS** Ensembles

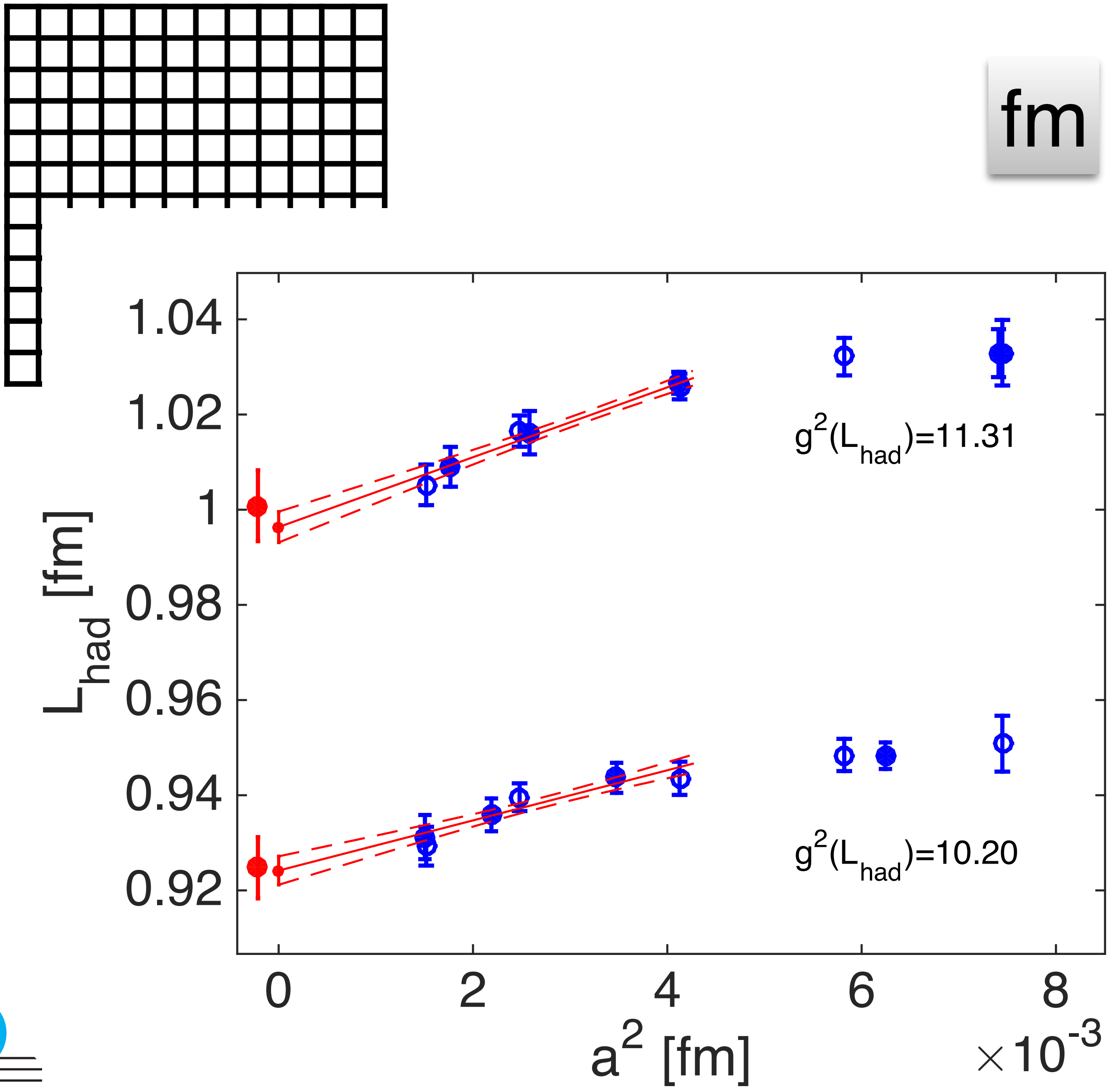
Bruno et al, 1411.3982
Bruno, Korzec, Schaefer, 1608.089000

- finite L

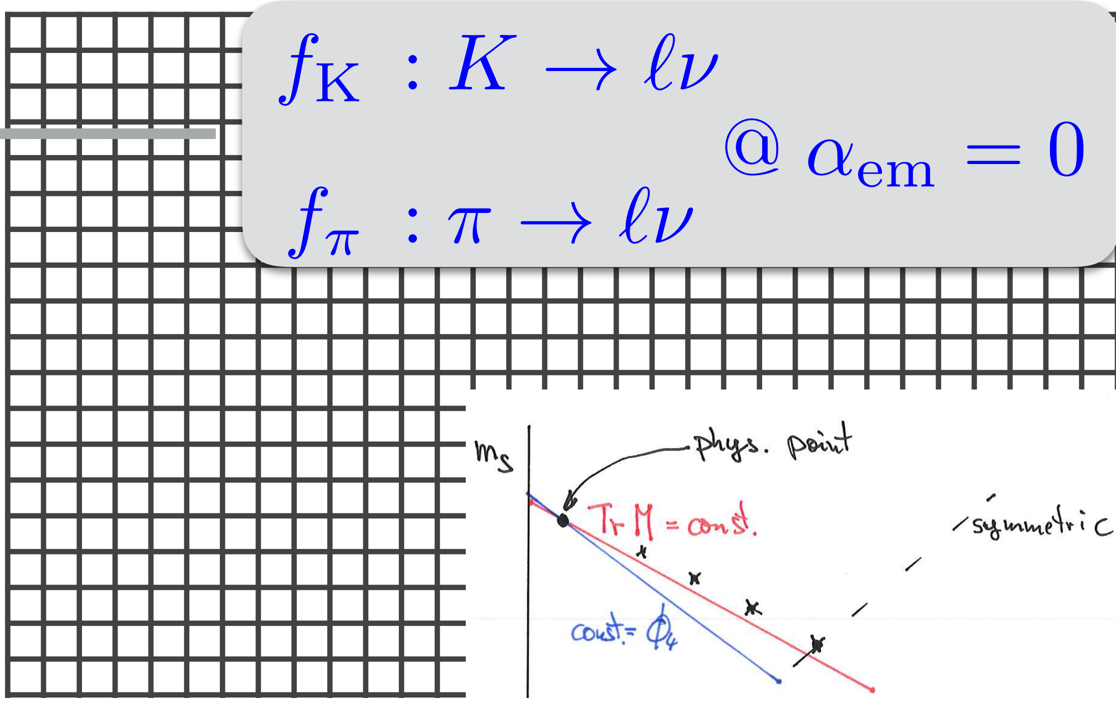


large L

simulated at common g_0 ⇔ common lattice spacing a

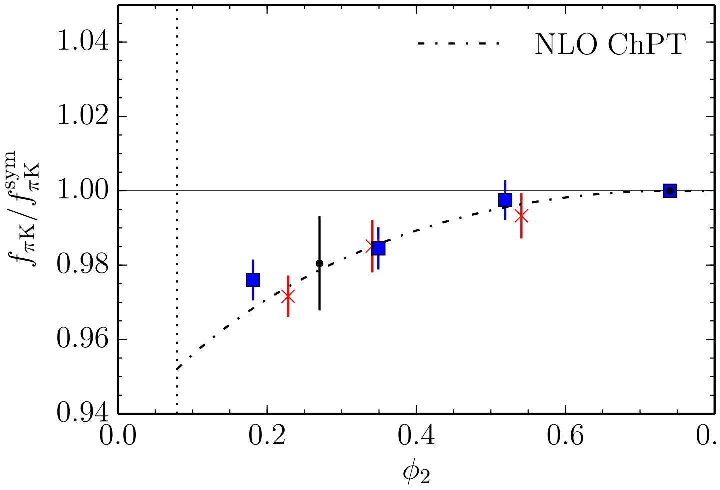


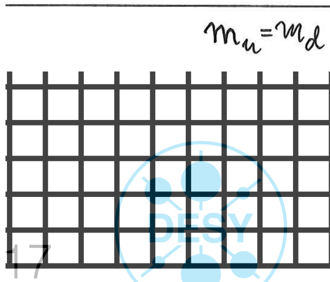
fm



$f_K : K \to \ell\nu$

$f_\pi : \pi \to \ell\nu$ @ $\alpha_{em} = 0$





Adding in c, b, t - quarks by perturbation theory (see later)

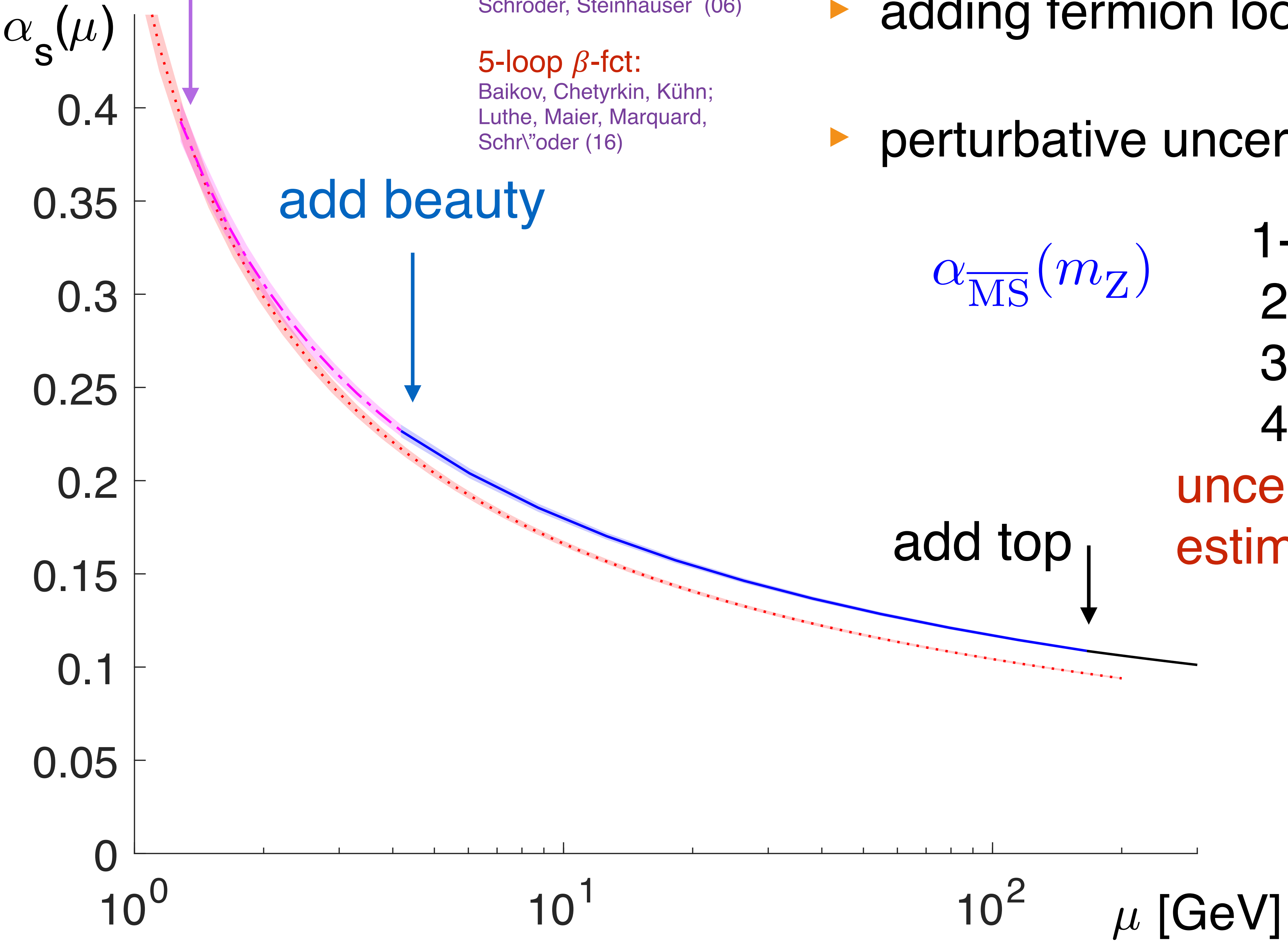
- 4-loop PT available
- adding fermion loops, “only”
- perturbative uncertainties are tiny

Weinberg (80),
Bernreuther&Wetzel (82),
…

Chetyrkin, Kühn & Sturm;
Schröder, Steinhauser (06)

5-loop β-fct:
Baikov, Chetyrkin, Kühn;
Luthe, Maier, Marquard,
Schr\"oder (16)

add charm



add beauty

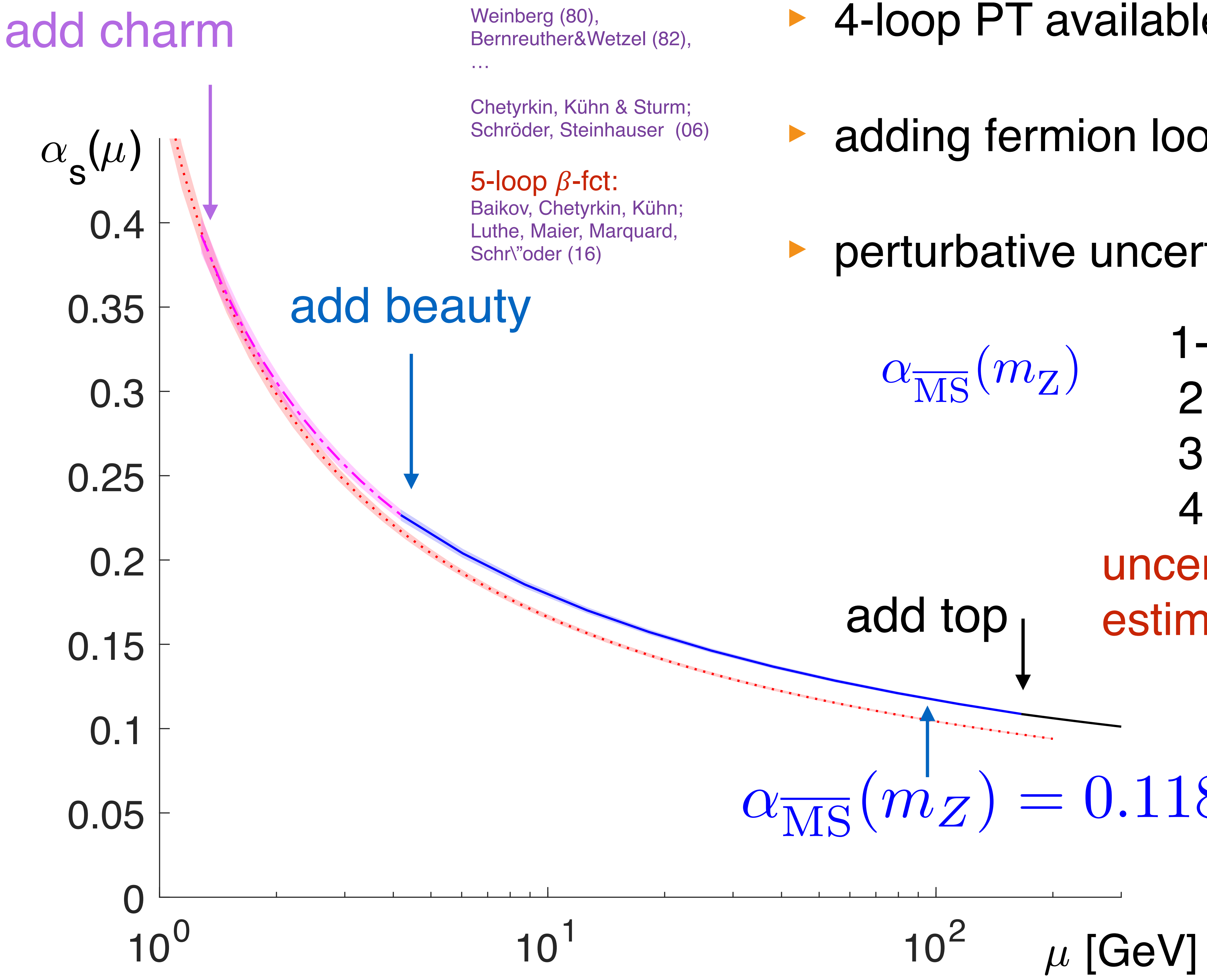
add top

$\alpha_{\overline{\mathrm{MS}}}(m_Z)$

1-loop:	0.11701
2	0.00128
3	0.00019
4	0.00006

uncertainty
estimate= 0.00025

Adding in c, b, t - quarks by perturbation theory (see later)



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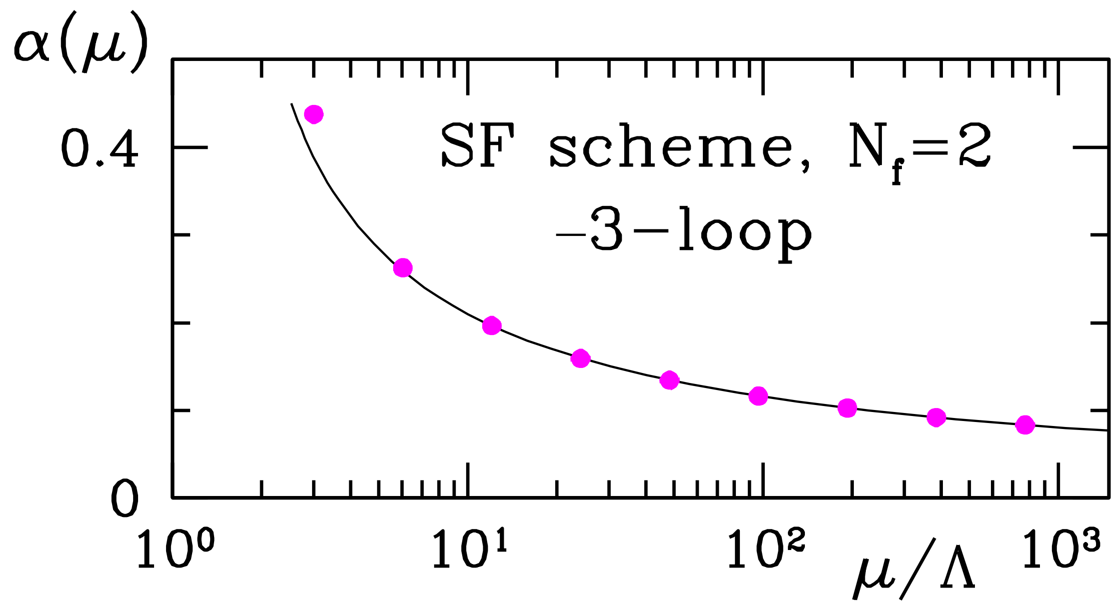
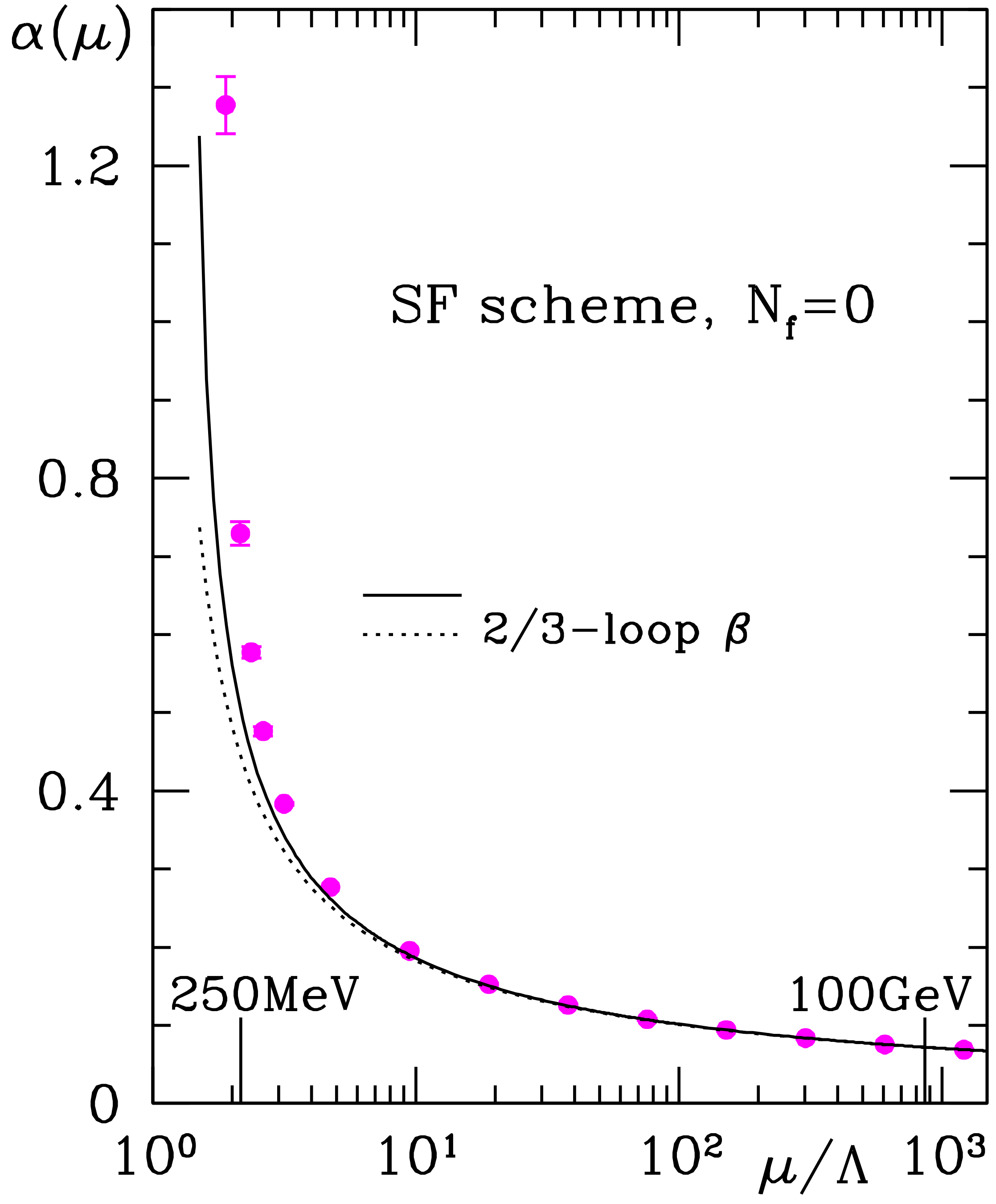
uncertainty
estimate= 0.00025

$$\alpha_{\overline{\mathrm{MS}}}(m_Z) = 0.1185(8)(3)$$

Non-perturbative running of α_{SF}



[ALPHA Collaboration, 2005]



[ALPHA Collaboration, 2001]

Finite volume schemes

Zero modes

Boundary conditions matter in finite volume. Which ones?

A most relevant criterion is zero modes

- Zero modes of gauge fields
  → perturbative expansion (+ MC)
- Zero modes of Dirac operator
  → HMC stability

Finite volume schemes

Gauge field zero modes

Path integral w.o. fermions

$$\begin{aligned}\langle O(U)\rangle &= \frac{1}{\mathcal{Z}}\int \mathrm{D}[U]\mathrm{e}^{-\beta\bar{S}(U)}O(U)\\ \bar{S}(U) &= \sum_p \text{tr}\,(1-U(p))\,, \quad \beta=\frac{6}{g_0^2}\end{aligned}$$

PT, sketchy

$$\beta\to\infty \qquad U\approx U_{\rm min}\equiv V \quad \text{dominates (classical solution)}$$

$$U(x,\mu)=V(x,\mu)\mathrm{e}^{\bar{q}_\mu^b(x)T^b}\,, \quad \bar{q}_\mu^b(x)\ll 1\,, \quad \int\mathrm{D}[U]\to\int\mathrm{D}[\bar{q}]$$

$$\begin{aligned}\bar{S}(U) &= \bar{S}(V)+\sum_{n,m}q_m K_{mn}q_n+\mathrm{O}(q^3)\,, \quad q_n=\bar{q}_\mu^b(x)\,,\ n=(\tfrac{x}{a},\mu,b)\\ O(U) &= O(V)+\ldots\end{aligned}$$

Gauss intergrals → Wick contractions ... IFF K has no zero modes ($Kv=\lambda v\,,\ \lambda>0$)

Finite volume schemes

Gauge field zero modes

Generically there are zero modes

- gauge modes → gauge fixing
- finite volume modes (gauge invariant)

  "Ground state metamorhosis"[Gonzales Arroyo, Jurkiewicz, Korthals-Altes] with periodic BC's

Finite volume schemes

Ground state metamorphosis

Toy example: SU(2), $L^4, L = a$ lattice, PBC, $d = 2$, single point

- $\bar{S} = 2 -$ tr $(U_2 U_1 U_2^\dagger U_1^\dagger)$
- tr U_i is gauge invariant, U_i can't be gauged away
- minima: $U_1 = U_2 = V$ …pick $U_1 = U_2 = 1$.
- fluctuations $U_i = \mathrm{e}^{i\sigma^b q_i^b}$

$$\bar{S} \quad = \quad 2 - \mathrm{tr}\, \mathrm{e}^{i\sigma^b q_2^b} \mathrm{e}^{i\sigma^b q_1^b} \mathrm{e}^{-i\sigma^b q_2^b} \mathrm{e}^{-i\sigma^b q_1^b} = \mathrm{O}(q^4) \qquad \to K = 0$$

- $q = \mathrm{O}(\beta^{-1/4}) = \mathrm{O}(g_0^{1/2})$
  PT in powers of g_0, not g_0^2 NOT regular
- In general: mixture of gaussian and non-gaussian modes
  integrate over non-gaussian ones exactly ...
  complicated, non-universal β-function
  it can be worse, divergent behavior, $1/\log(g)$ terms , see [Nogradi et al., 2012]
- think of these U_i as Polyakov loops → relevant for 4-d gauge theory.
  "Ground state metamorhosis"[Gonzales Arroyo, Jurkiewicz, Korthals-Altes]

Finite volume schemes

Zero modes of the Dirac operator

$$V(x,\mu)=1\,,\qquad \text{PBC: } \psi(x+L\hat\mu)=\psi(x)$$

massless Dirac operator has a zero mode (constant mode, $p=0$)

easily fixed by

$$\psi(x+L\hat\mu)=\mathrm{e}^{i\alpha}\psi(x)$$

e.g. $\alpha=\pi/2$ in SU(2), $\alpha=\pi/3$ in SU(3)

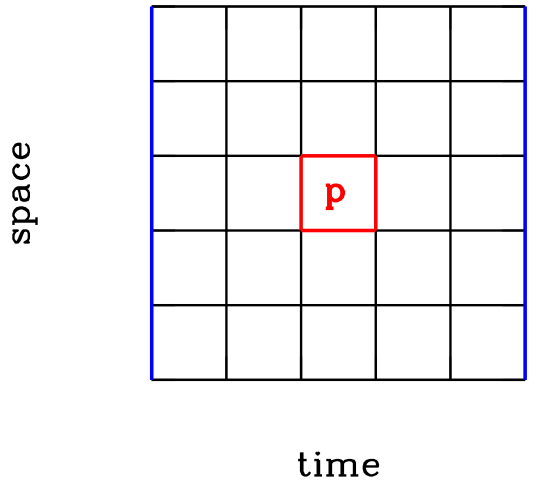
Exercise: why these values of α?

Finite volume schemes

Schrödinger functional

Boundary conditions

- Space: PBC
- Time: Dirichlet, breaks translation invariance!



Yang Mills theory [Lüscher, Narayanan, Weisz & Wolff]:

$$\mathcal{Z}(V,V') = \int \mathrm{D}(U)_{\text{inside}}\, \mathrm{e}^{-S_{\mathrm{SF}}(U)}$$

$$S_{\mathrm{SF}}(U) = \sum_{p \text{ inside}} \beta\, \mathrm{tr}\,(1-U(p))\,, \quad U(x,k) = \begin{cases} V(\mathbf{x},k) & x_0 = 0 \\ V'(\mathbf{x},k) & x_0 = T \end{cases}$$

Standard introduction of Hilbert space, transfer matrix:

$$\mathcal{Z}(V,V') = \langle V'|\underbrace{\mathrm{e}^{-\hat{H}T}}_{\mathbb{T}^{T/a}}\underbrace{\mathbb{P}_0}_{\uparrow}|V\rangle\,, \quad \hat{U}(\mathbf{x},k)|U\rangle = U(\mathbf{x},k)|U\rangle$$

projector onto gauge invariant states

$\mathcal{Z}(V,V') =$ Euclidean time propagation kernel by time T = Schrödinger functional

Finite volume schemes

Schrödinger functional : quarks

Wilson Dirac operator (also others are possible)

$$D_{\mathrm{W}} = \tfrac{1}{2}\{\gamma_\mu(\nabla_\mu + \nabla^*_\mu) - a\nabla^*_\mu\nabla_\mu\}$$

$$\begin{aligned} \nabla_\mu\psi(x) &= \frac{1}{a}\left[U(x,\mu)\psi(x+a\hat{\mu}) - \psi(x)\right] \\ \nabla^*_\mu\psi(x) &= \frac{1}{a}\left[\psi(x) - U(x-a\hat{\mu},\mu)^{-1}\psi(x-a\hat{\mu})\right] \end{aligned}$$

Schrödinger functional action

$$S_{\mathrm{F}} = a^4\sum_x \overline{\psi}(x)[m_0 + D_{\mathrm{W}}]\psi(x)\,,$$

with $\psi(x) = 0\,,\ \overline{\psi}(x) = 0$ for $x_0 \leq 0\,,$ and $x_0 \geq T$

In the continuum theory this corresponds to BC's [Sint, 1994]

$$P_+\psi(x)|_{x_0=0} = 0 \qquad \overline{\psi}(x)P_-\Big|_{x_0=0} = 0 \qquad P_\pm = \tfrac{1}{2}(1\pm\gamma_0)$$

$$P_-\psi(x)|_{x_0=T} = 0 \qquad \overline{\psi}(x)P_+\Big|_{x_0=T} = 0$$

These BC's are stable: emerge in the cont. limit without fine-tuning. Universality! [Lüscher, 2006]
The Universality class is characterised by Parity invariance, discrete rot. invariance (not chiral symm).

Finite volume schemes

Schrödinger functional : boundary quark fields

Correlation functions can be formed with the usual fields in the interior (bulk) **and the boundary quark fields**

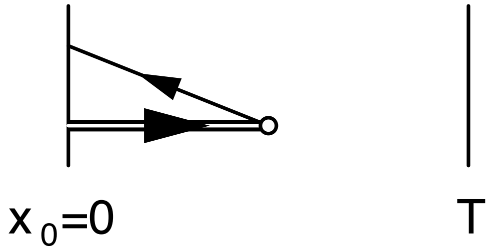
$$\zeta(\mathbf{x}) = P_- U(x,0)\psi(x+a\hat{0})\big|_{x_0=0} \qquad \bar{\zeta}(\mathbf{x}) = \bar{\psi}(x+a\hat{0})P_+ U(x,0)^{-1}\big|_{x_0=0}$$

$$\zeta'(\mathbf{x}) = P_+ U(x-a\hat{0},0)^{-1}\psi(x-a\hat{0})\big|_{x_0=T} \qquad \bar{\zeta}'(\mathbf{x}) = \bar{\psi}(x-a\hat{0})P_- U(x-a\hat{0},0)\big|_{x_0=T}$$

A very interesting feature of these is that one can form correlation functions where the **quark** fields are projected to $\mathbf{p}=0$. (Note that the gauge fields at the boundaries are fixed).

$$f_{\mathrm{P}}^{rs}(x_0) = a^6 \sum_{\mathbf{v},\mathbf{y}} \langle \bar{\zeta}_s(\mathbf{v})\gamma_5\zeta_r(\mathbf{y})P^{rs}(x)\rangle$$

$$P^{rs}(x) = \bar{\psi}_r(x)\gamma_5\psi_s(x)$$



Finite volume schemes

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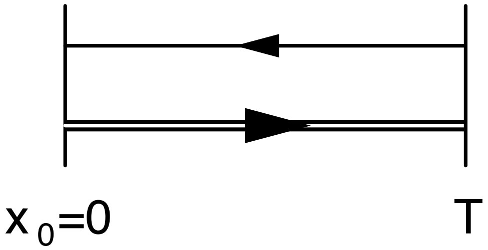
These boundary quark fields renormalize multiplicatively.

$$\zeta_{\mathrm{R}}(\mathbf{x}) = Z_\zeta \zeta(\mathbf{x}) \,, \; \ldots \; , \bar{\zeta}'_{\mathrm{R}} = Z_\zeta \bar{\zeta}'(\mathbf{x})$$

Define also boundary-to-boundary correlation functions

$$f_1^{rs} = \frac{a^{12}}{L^6} \sum_{\mathbf{v},\mathbf{y},\mathbf{u},\mathbf{x}} \langle \bar{\zeta}_s(\mathbf{v})\gamma_5\zeta_r(\mathbf{y})\bar{\zeta}'_r(\mathbf{u})\gamma_5\zeta'_s(\mathbf{x})\rangle$$



Then

$$(f_1^{rs})_{\mathrm{R}} = Z_\zeta^4\,(f_1^{rs}) \;, \quad (f_{\mathrm{P}}^{rs}(x_0))_{\mathrm{R}} = Z_\zeta^2\; Z_{\mathrm{P}}\;\, (f_{\mathrm{P}}^{rs}(x_0))$$

Finite volume schemes

Schrödinger functional : properties

- Regular PT (no gauge field zero modes)
- Gap for Dirac operators
- Momentum zero boundary quark fields
  (spatially one takes pbc up to a phase, cf "flavor twisted bc")
- Schrödinger functional coupling defined with non-trivial V, V'
  β-function known to 3-loops [LNWW; LW; Bode, Weisz, Wolff]

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- We can define Z-factors (schemes) for composite fields, e.g.

$$Z_{\rm P} = \frac{1}{c(a/L)} \frac{\sqrt{f_1^{rs}}}{f_{\rm P}^{rs}(T/2)}\,, \quad c(a/L) = \frac{\sqrt{f_1^{rs}}}{f_{\rm P}^{rs}(T/2)}\Big|_{g_0=0}$$

Finite volume schemes

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- There is also a new SF coupling . . .

Exercise

Consider the free Schrödinger functional , i.e. $U(x,\mu)=1$ with pbc in space for the fermions.

- Show that $f_{\rm P}(x_0) = \mathrm{constant}$ for mass-less quarks.
  hints:
  - write down the Wick-contraction in terms of the Schrödinger functional propagator
  - note that it is apropriate to go to momentum space concerning the space components, but to remain in coordinate space concerning the time coordinates
  - what is the the equation for the spatial $\mathbf{p}=0$ contribution to the propagator? note how it splits into $P_\pm$ pieces
  - solve the equation by “inspection”, iteration
  - obtain the result for arbitrary quark mass
- Could this result be guessed by dimensional reasoning?

Gradient Flow and SF-coupling

- Gradient flow [Lüscher, 2010; Lüscher & Weisz, 2011]

  new observables
  - UV finite (proven to all orders of PT)
  - excellent numerical precision
  - renormalized coupling in finite volume with pbc [BMW, 2012]
- Flow in finite volume, SF [P. Fritzsch & Ramos, `arXiv:1301.4388`]
  - lowest order PT to define a new coupling
  - numerical investigation shows excellent precision
- Flow with gauge fields AND quark fields [Lüscher,`arXiv:1302.5246`]
- General idea

$$x = (x_0, \mathbf{x})\,, \quad t = \text{ flow time}$$

$$A_\mu(x) = \text{ quantum gauge fields } : \quad \mathcal{Z} = \int \mathrm{D}[A_\mu(x)] \ldots$$

$$B_\mu(x,t) = \text{ smoothed gauge fields }, \quad B_\mu(x,0) = A_\mu(x)$$

$$\frac{\mathrm{d}B_\mu(x,t)}{\mathrm{d}t} = D_\nu G_{\nu\mu}(x,t) + \text{gauge fixing}$$

$$\sim -\frac{\delta S_{YM}[B]}{\delta B_\mu}$$

correlation functions of B-fields at arbitrary points are finite

Gradient Flow

Yang–Mills theory

- in PT: $A_\mu(x) = g_0 \bar{A}_\mu(x)$

$$\begin{aligned} B_\mu(x,t) &= B_{\mu,1}(x,t) g_0 + B_{\mu,2}(x,t) g_0^2 + \ldots \\ & \quad G_{\nu\mu} = \left[\partial_\nu B_{\mu,1} - \partial_\mu B_{\nu,1}\right] g_0 + \mathrm{O}(g_0^2)\,, \quad D_\nu = \partial_\nu + \mathrm{O}(g_0) \\ \to \quad \dot{B}_{\mu,1}(x,t) &= \partial_\nu \partial_\nu B_{\mu,1}(x,t) \end{aligned}$$

- heat equation

$$\begin{aligned} B_{\mu,1}(x,t) &= \int \mathrm{d}^D p \, \mathrm{e}^{ipx} \, b_\mu(p,t) \\ & \dot{b}_\mu = -p^2 b_\mu \quad \to \quad b_\mu(p,t) = b_\mu(p,0) \mathrm{e}^{-p^2 t} \\ B_{\mu,1}(x,t) &= \int \mathrm{d}^D y \, K_t(x-y) \, \bar{A}_\mu(y)\,, \quad K_t(z) = (4\pi t)^{-D/2} \mathrm{e}^{-z^2/(4t)} \end{aligned}$$

- smoothing over a radius of $\sqrt{8t}$
- gaussian damping of large momenta

Gradient Flow

Yang–Mills theory

- in PT: $A_\mu(x) = g_0 \bar{A}_\mu(x)$

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- smoothing over a radius of $\sqrt{8t}$
- gaussian damping of large momenta
- all correlation functions of B_μ are finite $(t > 0)$ [Lüscher & Weisz, 2011]

in particular $\langle E(t) \rangle$, $\quad E(t) = -\frac{1}{2}\,\mathrm{tr}\, G_{\mu\nu} G_{\mu\nu}$

Gradient Flow

Yang–Mills theory

- order by order iteration:

$$B_\mu(x,t) = \sum_k B_{\mu,n}(x,t) g_0^k$$

$$\dot{B}_{\mu,k}(x,t) - \partial_\nu \partial_\nu B_{\mu,k}(x,t) = R_{\mu,k}$$

$$R_{\mu,1} = 0, \qquad B_{\mu,1}(x,t) = \int \mathrm{d}^D y\; K_t(x-y)\; \bar{A}_\mu(y)$$

$$\begin{aligned} R_{\mu,2} &= 2[B_{\nu,1}, \partial_\nu B_{\mu,1}] - [B_{\nu,1}, \partial_\mu B_{\nu,1}], \\ R_{\mu,3} &= 2[B_{\nu,2}, \partial_\nu B_{\mu,1}] + 2[B_{\nu,1}, \partial_\nu B_{\mu,2}] \\ &\quad - [B_{\nu,2}, \partial_\mu B_{\nu,1}] - [B_{\nu,1}, \partial_\mu B_{\nu,2}] + [B_{\nu,1}, [B_{\nu,1}, B_{\mu,1}]], \end{aligned}$$

$$\ldots$$

$$B_{\mu,k}(t,x) = \int_0^t \mathrm{d}s \int \mathrm{d}^D y\, K_{t-s}(x-y) R_{\mu,k}(s,y) \quad k > 1$$

Gradient Flow

Yang–Mills theory

- order by order iteration:

$$
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&B_\mu(x,t) = \sum_k B_{\mu,n}(x,t) g_0^k \\
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&R_{\mu,1} = 0, \qquad B_{\mu,1}(x,t) = \int \mathrm{d}^D y\, K_t(x-y)\, \bar{A}_\mu(y) \\
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&R_{\mu,3} = 2[B_{\nu,2}, \partial_\nu B_{\mu,1}] + 2[B_{\nu,1}, \partial_\nu B_{\mu,2}] \\
&\qquad - [B_{\nu,2}, \partial_\mu B_{\nu,1}] - [B_{\nu,1}, \partial_\mu B_{\nu,2}] + [B_{\nu,1}, [B_{\nu,1}, B_{\mu,1}]], \\
&\ldots \\
&B_{\mu,k}(t,x) = \int_0^t \mathrm{d}s \int \mathrm{d}^D y\, K_{t-s}(x-y) R_{\mu,k}(s,y) \quad k > 1
\end{aligned}
$$

- For $\langle E \rangle$, $E = -\frac{1}{2}\,\mathrm{tr}\, G_{\mu\nu} G_{\mu\nu}$

$$
\begin{aligned}
\langle E \rangle &= E_0\, g_0^2 + E_0\, g_0^4 + \ldots \\
E_0 &= \langle\, \mathrm{tr}\, [\partial_\mu B_{\nu,1} \partial_\mu B_{\nu,1} - \partial_\mu B_{\nu,1} \partial_\nu B_{\mu,1}] \rangle \\
&\sim \int_p \mathrm{e}^{-p^2 2t} [p^2 \delta_{\mu\nu} - p_\mu p_\nu]\, D_{\mu\nu}(p) \quad \text{finite (also with cutoff reg'n)!}
\end{aligned}
$$

Gradient Flow and SF-coupling

The coupling in PT

use the flow in SF: $T \times L^3$ world with Dirichlet BC in time, $T = L$
define

$$\begin{aligned} \langle E(t) \rangle &\equiv -\tfrac{1}{2}\langle \mathrm{tr}\, G_{\mu\nu} G_{\mu\nu}(x,t) \rangle_{x_0=T/2} = \tfrac{\mathcal{N}}{t^2}\, \bar{g}^2_{\mathrm{MS}}(\mu)\, (1 + c_1 \bar{g}^2_{\mathrm{MS}} + \ldots) \\ \bar{g}^2_{\mathrm{GF}}(L) &\equiv \mathcal{N}^{-1}\, t^2 \langle E(t) \rangle \Big|_{t=c^2 L^2/8} \end{aligned}$$

This is a family of schemes characterized by c (dimensionless)

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This is a family of schemes characterized by c (dimensionless)

$$\begin{aligned} \mathcal{N}(c) &= \frac{c^4(N^2-1)}{128}\sum_{\mathbf{n},n_0} e^{-c^2\pi^2(\mathbf{n}^2+\frac{1}{4}n_0^2)} \\ &\quad\times \frac{2\mathbf{n}^2 s^2_{n_0}(T/2)+(\mathbf{n}^2+\frac{3}{4}n_0^2)c^2_{n_0}(T/2)}{\mathbf{n}^2+\frac{1}{4}n_0^2} \end{aligned}$$

- the lattice version is known (and needed)

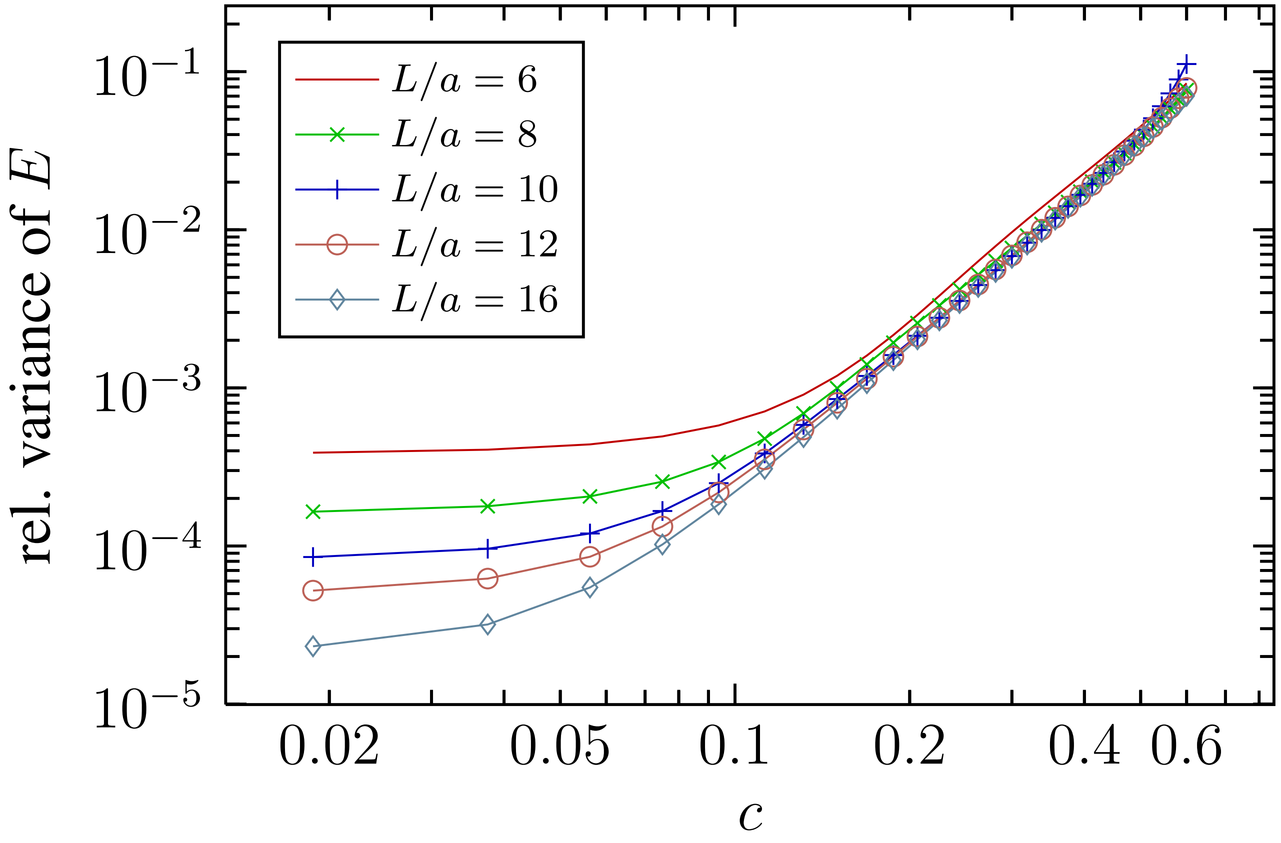
Gradient Flow and SF-coupling

statistical precision: variance

$$\text{relative variance } = \frac{\langle E^2\rangle - \langle E\rangle^2}{\langle E\rangle^2}$$

should be finite as $a \to 0$, $L/a \to \infty$

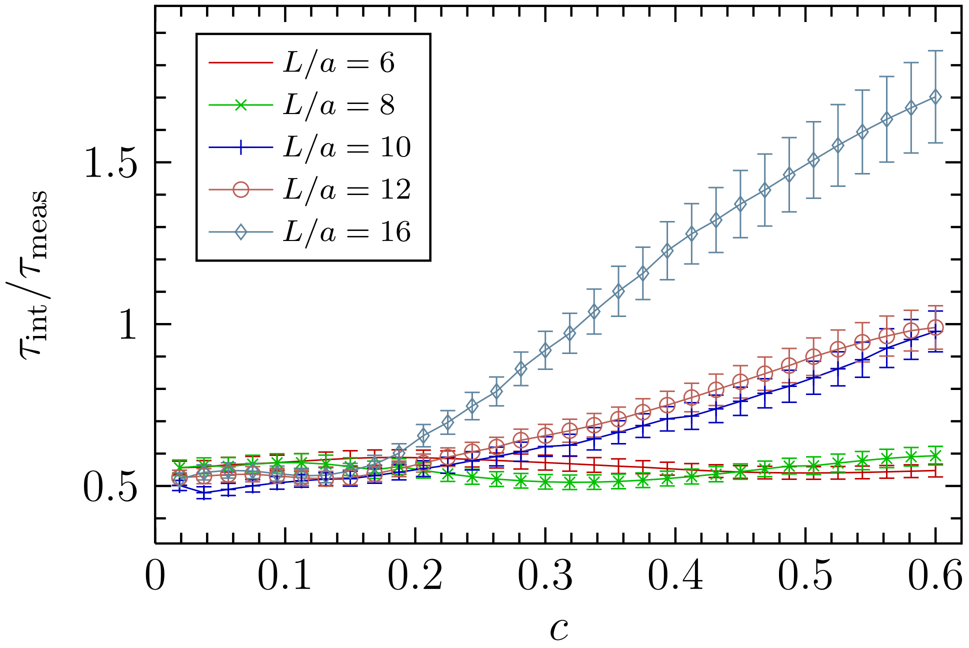
Numerically, Fritzsch & Ramos:

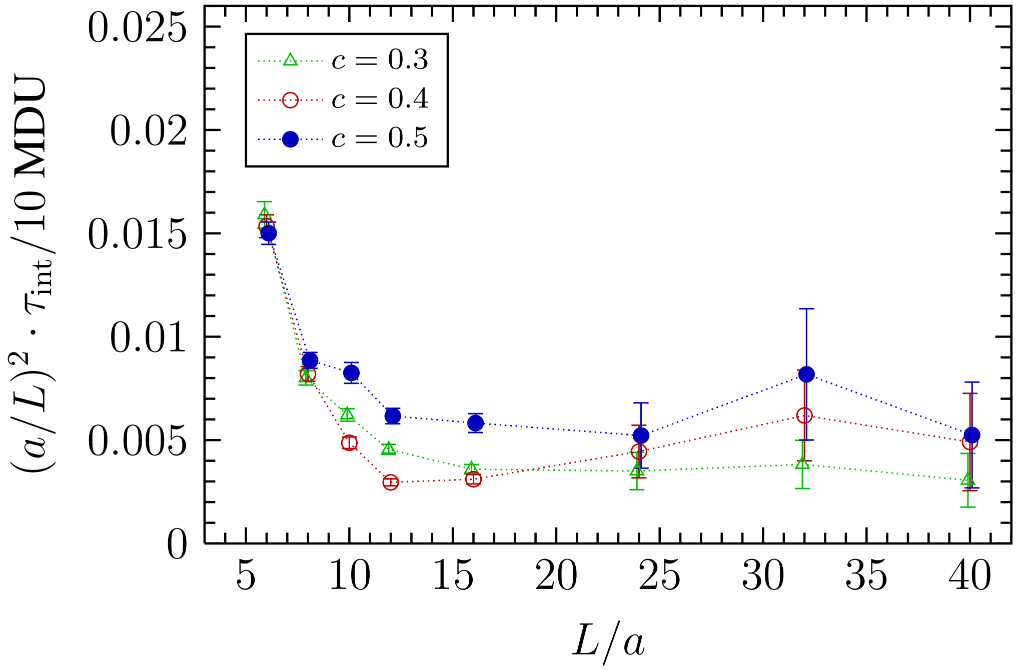


Gradient Flow and SF-coupling

statistical precision

autocorrelations scale as expected: $\tau_{\mathrm{int}} \propto a^{-2}$





Statistical precision is good and theoretically understood.
There will be no surprises on the way to the continuum limit.