グラディエントフローの基礎とその応用

鈴木　博
Hiroshi Suzuki

九州大学
Kyushu University

2019/07/01–07/03 @ 北京大学

Outline

1. Gradient flow in (lattice) gauge theory
   - Yang–Mills gradient flow
   - Perturbative expansion of the gradient flow
   - Brief encounter to the renormalizability of the gradient flow
   - Fermion flow

Outline

1. Gradient flow in (lattice) gauge theory
   - Yang–Mills gradient flow
   - Perturbative expansion of the gradient flow
   - Brief encounter to the renormalizability of the gradient flow
   - Fermion flow
2. Renormalizability proof of the gradient flow
   - $D+1$-dimensional field theoretical representation of the gradient flow
   - Ward–Takahashi relation or the Zinn-Justin equation
   - Fermion (or general matter) fields

Outline

1. Gradient flow in (lattice) gauge theory
   - Yang–Mills gradient flow
   - Perturbative expansion of the gradient flow
   - Brief encounter to the renormalizability of the gradient flow
   - Fermion flow
2. Renormalizability proof of the gradient flow
   - $D + 1$-dimensional field theoretical representation of the gradient flow
   - Ward–Takahashi relation or the Zinn-Justin equation
   - Fermion (or general matter) fields
3. Universal formula for the energy–momentum tensor
   - Energy–momentum tensor on the lattice?
   - EMT in the dimensional regularization
   - Small flow-time expansion
   - Computation of the expansion coefficients

Outline

1. Gradient flow in (lattice) gauge theory
   - Yang–Mills gradient flow
   - Perturbative expansion of the gradient flow
   - Brief encounter to the renormalizability of the gradient flow
   - Fermion flow
2. Renormalizability proof of the gradient flow
   - $D+1$-dimensional field theoretical representation of the gradient flow
   - Ward–Takahashi relation or the Zinn-Justin equation
   - Fermion (or general matter) fields
3. Universal formula for the energy–momentum tensor
   - Energy–momentum tensor on the lattice?
   - EMT in the dimensional regularization
   - Small flow-time expansion
   - Computation of the expansion coefficients
4. Summary and further prospects

Notations

- The spacetime signature is $(+,+,+,+)$ (euclidean) and gamma matrices are all hermitian.
- We normalize the gauge group generator as

$$\mathrm{tr}(T^a T^b) = -\frac{1}{2}\delta^{ab}.$$

The structure constant are defined by

$$[T^a, T^b] = f^{abc} T^c,$$

and quadratic Casimirs are

$$f^{acd} f^{bcd} = C_A \delta^{ab}, \qquad \mathrm{tr}(T^a T^b) = -T\delta^{ab}, \qquad T^a T^a = -C_F \mathbb{1}.$$

- We also use the following abbreviation for the momentum integral:

$$\int_p \equiv \int \frac{d^D p}{(2\pi)^D}.$$

Yang–Mills gradient flow (Lüscher (2009–))

- Yang–Mills gradient flow is an evolution of the gauge field $A_\mu(x)$ along a fictitious time $t \in [0, \infty)$, according to

$$\partial_t B_\mu(t,x) = -g_0^2 \frac{\delta S_{\text{YM}}}{\delta B_\mu(t,x)} = D_\nu G_{\nu\mu}(t,x) = \Delta B_\mu(t,x) + \cdots ,$$

where

$$G_{\mu\nu}(t,x) = \partial_\mu B_\nu(t,x) - \partial_\nu B_\mu(t,x) + [B_\mu(t,x), B_\nu(t,x)], \qquad D_\mu = \partial_\mu + [B_\mu, \cdot]$$

and its initial value is the conventional gauge field

$$B_\mu(t=0,x) = A_\mu(x).$$

Yang–Mills gradient flow (Lüscher (2009–))

- Yang–Mills gradient flow is an evolution of the gauge field $A_\mu(x)$ along a fictitious time $t \in [0, \infty)$, according to

$$\partial_t B_\mu(t,x) = -g_0^2 \frac{\delta S_{\mathrm{YM}}}{\delta B_\mu(t,x)} = D_\nu G_{\nu\mu}(t,x) = \Delta B_\mu(t,x) + \cdots ,$$

where

$$G_{\mu\nu}(t,x) = \partial_\mu B_\nu(t,x) - \partial_\nu B_\mu(t,x) + [B_\mu(t,x), B_\nu(t,x)], \qquad D_\mu = \partial_\mu + [B_\mu, \cdot]$$

and its initial value is the conventional gauge field

$$B_\mu(t=0,x) = A_\mu(x).$$

- RHS is the Yang–Mills equation of motion, the gradient in function space if S_{YM} is regarded as a potential height. So the name of the gradient flow.

Yang–Mills gradient flow (Lüscher (2009–))

- This is a sort of diffusion equation in which the diffusion length is

$$x \sim \sqrt{8t}.$$

Yang–Mills gradient flow (Lüscher (2009–))

- This is a sort of diffusion equation in which the diffusion length is

$$x \sim \sqrt{8t}.$$

- The flow makes the field configuration smooth; it generates the smearing/cooling for a lattice gauge field.

Yang–Mills gradient flow (Lüscher (2009–))

- This is a sort of diffusion equation in which the diffusion length is

$$x \sim \sqrt{8t}.$$

- The flow makes the field configuration smooth; it generates the smearing/cooling for a lattice gauge field.
- A theoretical understanding on its renormalizability (see below) however distinguishes the gradient flow from other smearing/cooling methods.

Yang–Mills gradient flow

- Yang–Mills gradient flow (continuum)

$$\partial_t B_\mu(t,x) = -g_0^2 \frac{\delta S_{\mathrm{YM}}}{\delta B_\mu(t,x)}, \qquad B_\mu(t=0,x) = A_\mu(x).$$

Yang–Mills gradient flow

- Yang–Mills gradient flow (continuum)

$$\partial_t B_\mu(t,x) = -g_0^2 \frac{\delta S_{\mathrm{YM}}}{\delta B_\mu(t,x)}, \qquad B_\mu(t=0,x) = A_\mu(x).$$

- Applications in lattice gauge theory ($\sim$ 490 citations of Lüscher's original paper)
  - Topological charge
  - Scale setting
  - Non-perturbative gauge coupling constant
  - Chiral condensate
  - Energy–momentum tensor
  - Fermion current and density operators
  - Supersymmetric theory (Kikuchi–Onogi, Kadoh–Ukita)
  - Chiral gauge theory (Grabowska–Kaplan)
  - Supersymmetric current (Hieda–Kasai–Maskino–Morikawa–H.S.)
  - ...

Yang–Mills gradient flow

- Yang–Mills gradient flow (continuum)

$$\partial_t B_\mu(t,x) = -g_0^2 \frac{\delta S_{\mathrm{YM}}}{\delta B_\mu(t,x)}, \qquad B_\mu(t=0,x) = A_\mu(x).$$

- Applications in lattice gauge theory ($\sim$ 490 citations of Lüscher's original paper)
  - Topological charge
  - Scale setting
  - Non-perturbative gauge coupling constant
  - Chiral condensate
  - Energy–momentum tensor
  - Fermion current and density operators
  - Supersymmetric theory (Kikuchi–Onogi, Kadoh–Ukita)
  - Chiral gauge theory (Grabowska–Kaplan)
  - Supersymmetric current (Hieda–Kasai–Maskino–Morikawa–H.S.)
  - ...
- The most important property of the gradient flow, which makes these applications possible, is its simple renormalization property.

Perturbative expansion of the gradient flow

- Yang–Mills gradient flow

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x) + \alpha_0 D_\mu \partial_\nu B_\nu(t,x), \qquad B_\mu(t=0,x) = A_\mu(x),$$

where the term with α_0 is introduced to suppress gauge modes.

Perturbative expansion of the gradient flow

- Yang–Mills gradient flow

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x) + \alpha_0 D_\mu \partial_\nu B_\nu(t,x), \qquad B_\mu(t=0,x) = A_\mu(x),$$

  where the term with α_0 is introduced to suppress gauge modes.
- This equation can be formally solved as

$$B_\mu(t,x) = \int d^D y \left[K_t(x-y)_{\mu\nu} A_\nu(y) + \int_0^t ds\, K_{t-s}(x-y)_{\mu\nu} R_\nu(s,y) \right],$$

  by using the heat kernel,

$$K_t(x)_{\mu\nu} = \int_p \frac{e^{ipx}}{p^2} \left[(\delta_{\mu\nu} p^2 - p_\mu p_\nu) e^{-tp^2} + p_\mu p_\nu e^{-\alpha_0 tp^2} \right].$$

Perturbative expansion of the gradient flow

- Yang–Mills gradient flow

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x) + \alpha_0 D_\mu \partial_\nu B_\nu(t,x), \qquad B_\mu(t=0,x) = A_\mu(x),$$

  where the term with α_0 is introduced to suppress gauge modes.
- This equation can be formally solved as

$$B_\mu(t,x) = \int d^D y \left[K_t(x-y)_{\mu\nu} A_\nu(y) + \int_0^t ds\, K_{t-s}(x-y)_{\mu\nu} R_\nu(s,y) \right],$$

  by using the heat kernel,

$$K_t(x)_{\mu\nu} = \int_p \frac{e^{ipx}}{p^2} \left[(\delta_{\mu\nu} p^2 - p_\mu p_\nu) e^{-tp^2} + p_\mu p_\nu e^{-\alpha_0 t p^2} \right].$$

- R is the non-linear terms

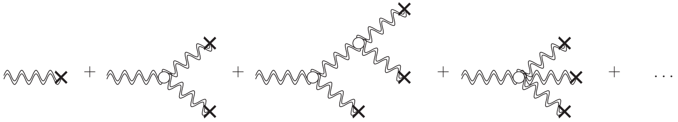
$$R_\mu = 2[B_\nu, \partial_\nu B_\mu] - [B_\nu, \partial_\mu B_\nu] + (\alpha_0 - 1)[B_\mu, \partial_\nu B_\nu] + [B_\nu, [B_\nu, B_\mu]].$$

Perturbative expansion of the gradient flow

- The solution

$$B_\mu(t,x) = \int d^D y \left[K_t(x-y)_{\mu\nu} A_\nu(y) + \int_0^t ds\, K_{t-s}(x-y)_{\mu\nu} R_\nu(s,y) \right],$$

is represented pictorially as (double lines: K, crosses: A_μ, white circles: R),



Justification of the “gauge fixing term”

- Under the infinitesimal gauge transformation

$$B_\mu(t,x) \to B_\mu(t,x) + D_\mu\omega(t,x),$$

the flow equation

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x) + \alpha_0 D_\mu \partial_\nu B_\nu(t,x),$$

changes to

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x) + \alpha_0 D_\mu \partial_\nu B_\nu(t,x) - D_\mu(\partial_t - \alpha_0 D_\nu \partial_\nu)\omega(t,x).$$

Justification of the “gauge fixing term”

- Under the infinitesimal gauge transformation

$$B_\mu(t,x) \to B_\mu(t,x) + D_\mu\omega(t,x),$$

the flow equation

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x) + \alpha_0 D_\mu \partial_\nu B_\nu(t,x),$$

changes to

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x) + \alpha_0 D_\mu \partial_\nu B_\nu(t,x) - D_\mu(\partial_t - \alpha_0 D_\nu\partial_\nu)\omega(t,x).$$

- Choosing $\omega(t,x)$ as

$$(\partial_t - \alpha_0 D_\nu\partial_\nu)\omega(t,x) = -\delta\alpha_0\partial_\nu B_\nu(t,x), \qquad \omega(t=0,x) = 0,$$

α_0 can be changed accordingly

$$\alpha_0 \to \alpha_0 + \delta\alpha_0.$$

Justification of the "gauge fixing term"

- Under the infinitesimal gauge transformation

$$B_\mu(t,x) \to B_\mu(t,x) + D_\mu \omega(t,x),$$

the flow equation

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x) + \alpha_0 D_\mu \partial_\nu B_\nu(t,x),$$

changes to

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x) + \alpha_0 D_\mu \partial_\nu B_\nu(t,x) - D_\mu(\partial_t - \alpha_0 D_\nu \partial_\nu)\omega(t,x).$$

- Choosing $\omega(t,x)$ as

$$(\partial_t - \alpha_0 D_\nu \partial_\nu)\omega(t,x) = -\delta\alpha_0 \partial_\nu B_\nu(t,x), \qquad \omega(t=0,x) = 0,$$

α_0 can be changed accordingly

$$\alpha_0 \to \alpha_0 + \delta\alpha_0.$$

- Thus, any gauge invariant quantity (in usual 4D sense) is independent of α_0, as far as it do not contain the flow time derivative ∂_t.

Quantum correlation functions

- Quantum correlation function of the flowed gauge field is obtained by the functional integral over the initial value $A_\mu(x)$:

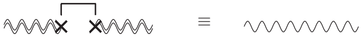
$$
\begin{aligned}
&\langle B_{\mu_1}(t_1, x_1)\cdots B_{\mu_n}(t_n, x_n)\rangle \\
&= \frac{1}{\mathcal{Z}}\int \mathcal{D}A_\mu\, B_{\mu_1}(t_1, x_1)\cdots B_{\mu_n}(t_n, x_n)\, e^{-S_{\mathrm{YM}}-S_{\mathrm{gf}}-S_{c\bar{c}}}.
\end{aligned}
$$

Quantum correlation functions

- Quantum correlation function of the flowed gauge field is obtained by the functional integral over the initial value $A_\mu(x)$:

$$\begin{aligned}&\langle B_{\mu_1}(t_1,x_1)\cdots B_{\mu_n}(t_n,x_n)\rangle\\&=\frac{1}{\mathcal{Z}}\int\mathcal{D}A_\mu\,B_{\mu_1}(t_1,x_1)\cdots B_{\mu_n}(t_n,x_n)\,e^{-S_{\mathrm{YM}}-S_{\mathrm{gf}}-S_{c\bar{c}}}.\end{aligned}$$

- For example, the contraction of two A_μ's

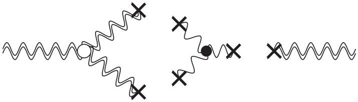


produces the free propagator of the flowed field

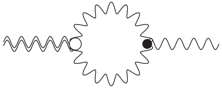
$$\begin{aligned}&\left\langle B_\mu^a(t,x)B_\nu^b(s,y)\right\rangle_0\\&=\delta^{ab}g_0^2\int_p\frac{e^{ip(x-y)}}{(p^2)^2}\left[(\delta_{\mu\nu}p^2-p_\mu p_\nu)e^{-(t+s)p^2}+\frac{1}{\lambda_0}p_\mu p_\nu e^{-\alpha_0(t+s)p^2}\right].\end{aligned}$$

Quantum correlation functions

- Similarly, for (black circle: Yang–Mills vertex)

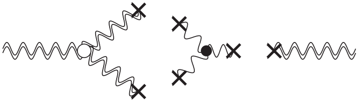


we have the loop flow-line Feynman diagram

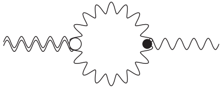


Quantum correlation functions

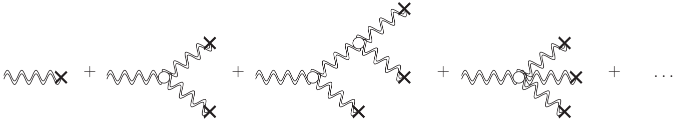
- Similarly, for (black circle: Yang–Mills vertex)



we have the loop flow-line Feynman diagram



- Recall that the flowed gauge field is represented as



Renormalizability of the gradient flow I (Lüscher–Weisz (2011))

- Correlation function of the flowed gauge field

$$\langle B_{\mu_1}(t_1, x_1) \cdots B_{\mu_n}(t_n, x_n) \rangle , \qquad t_1 > 0, \ldots, t_n > 0,$$

when expressed in terms of renormalized parameters, is UV finite without the wave function renormalization.

Renormalizability of the gradient flow I (Lüscher–Weisz (2011))

- Correlation function of the flowed gauge field

$$\langle B_{\mu_1}(t_1, x_1) \cdots B_{\mu_n}(t_n, x_n) \rangle, \qquad t_1 > 0, \dots, t_n > 0,$$

  when expressed in terms of renormalized parameters, is UV finite without the wave function renormalization.
- Two-point function in the tree level (in the Feynman gauge $\lambda_0 = \alpha_0 = 1$)

$$\left\langle B_\mu^a(t, x) B_\nu^b(s, y) \right\rangle_0 = \delta^{ab} g_0^2 \delta_{\mu\nu} \int_p e^{ip(x-y)} \frac{e^{-(t+s)p^2}}{p^2}.$$

Renormalizability of the gradient flow I (Lüscher–Weisz (2011))

- Correlation function of the flowed gauge field

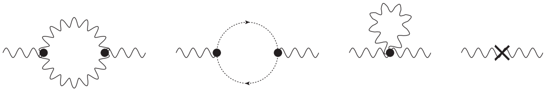
$$\langle B_{\mu_1}(t_1, x_1) \cdots B_{\mu_n}(t_n, x_n) \rangle \,, \qquad t_1 > 0, \ldots, t_n > 0,$$

when expressed in terms of renormalized parameters, is UV finite without the wave function renormalization.

- Two-point function in the tree level (in the Feynman gauge $\lambda_0 = \alpha_0 = 1$)

$$\left\langle B_\mu^a(t,x) B_\nu^b(s,y) \right\rangle_0 = \delta^{ab} g_0^2 \delta_{\mu\nu} \int_p e^{ip(x-y)} \frac{e^{-(t+s)p^2}}{p^2}.$$

- One-loop corrections (consisting only from Yang–Mills vertices)



where the last counter term arises from the parameter renormalization

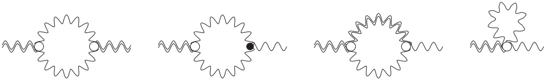
$$g_0^2 = \mu^{2\varepsilon} g^2 Z, \qquad \lambda_0 = \lambda Z_3^{-1}.$$

Renormalizability of the gradient flow I

- Usually, further wave function renormalization ($A_\mu^a = Z^{1/2} Z_3^{1/2} (A_R)_\mu^a$) is required for the two-point function to become UV finite.

Renormalizability of the gradient flow I

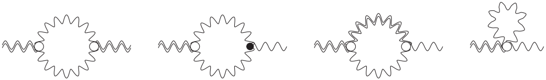
- Usually, further wave function renormalization ($A_\mu^a = Z^{1/2} Z_3^{1/2} (A_R)_\mu^a$) is required for the two-point function to become UV finite.
- In the present flowed system, we also have the white circles (flow vertex)



It turns out that these provide the same effect as the wave function renormalization!

Renormalizability of the gradient flow I

- Usually, further wave function renormalization ($A_\mu^a = Z^{1/2} Z_3^{1/2} (A_R)_\mu^a$) is required for the two-point function to become UV finite.
- In the present flowed system, we also have the white circles (flow vertex)

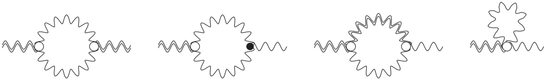


It turns out that these provide the same effect as the wave function renormalization!

- All order proof of this fact, using a local $D + 1$-dimensional field theory, consists the main part of the following lectures.

Renormalizability of the gradient flow I

- Usually, further wave function renormalization ($A_\mu^a = Z^{1/2} Z_3^{1/2} (A_R)_\mu^a$) is required for the two-point function to become UV finite.
- In the present flowed system, we also have the white circles (flow vertex)

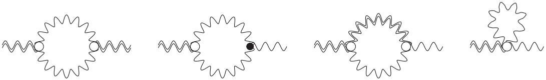


It turns out that these provide the same effect as the wave function renormalization!

- All order proof of this fact, using a local $D+1$-dimensional field theory, consists the main part of the following lectures.
- No bulk ($t > 0$) counterterm: because of the gaussian damping factor $\sim e^{-tp^2}$ in the propagator.

Renormalizability of the gradient flow I

- Usually, further wave function renormalization ($A_\mu^a = Z^{1/2} Z_3^{1/2} (A_R)_\mu^a$) is required for the two-point function to become UV finite.
- In the present flowed system, we also have the white circles (flow vertex)



It turns out that these provide the same effect as the wave function renormalization!

- All order proof of this fact, using a local $D+1$-dimensional field theory, consists the main part of the following lectures.
- No bulk ($t > 0$) counterterm: because of the gaussian damping factor $\sim e^{-tp^2}$ in the propagator.
- No boundary ($t = 0$) counterterm besides Yang–Mills ones: because of a BRS symmetry.

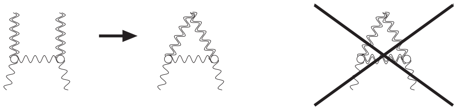
Renormalizability of the gradient flow II

- Correlation function of the flow gauge field

$$\langle B_{\mu_1}(t_1, x_1) B_{\mu_2}(t_2, x_2) \cdots B_{\mu_n}(t_n, x_n) \rangle , \qquad t_1 > 0, \ldots, t_n > 0,$$

remains finite even for the equal-point product

$$t_1 \to t_2, \qquad x_1 \to x_2.$$



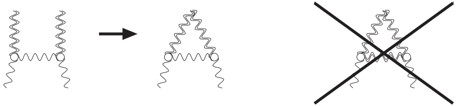
Renormalizability of the gradient flow II

- Correlation function of the flow gauge field

$$\langle B_{\mu_1}(t_1, x_1) B_{\mu_2}(t_2, x_2) \cdots B_{\mu_n}(t_n, x_n) \rangle , \qquad t_1 > 0, \dots, t_n > 0,$$

remains finite even for the equal-point product

$$t_1 \to t_2, \qquad x_1 \to x_2.$$



- The new loop always contains the gaussian damping factor $\sim e^{-tp^2}$ which makes integral finite; no new UV divergences arise.

Renormalizability of the gradient flow II

- Any composite operators of the flowed gauge field $B_\mu(t, x)$ are automatically renormalized UV finite quantities, although the flowed field is a certain combination of the bare gauge field.

Renormalizability of the gradient flow II

- Any composite operators of the flowed gauge field $B_\mu(t,x)$ are automatically renormalized UV finite quantities, although the flowed field is a certain combination of the bare gauge field.
- Such UV finite quantities must be independent of the regularization.

Renormalizability of the gradient flow II

- Any composite operators of the flowed gauge field $B_\mu(t,x)$ are automatically renormalized UV finite quantities, although the flowed field is a certain combination of the bare gauge field.
- Such UV finite quantities must be independent of the regularization.
- $\Rightarrow$ E.g., construction of the energy–momentum tensor in lattice gauge theory.

Flow of fermion fields

- A possible choice (Lüscher (2013))

$$\partial_t\chi(t,x) = [\Delta - \alpha_0\partial_\mu B_\mu(t,x)]\,\chi(t,x), \qquad \chi(t=0,x)=\psi(x),$$
$$\partial_t\bar{\chi}(t,x) = \bar{\chi}(t,x)\left[\overleftarrow{\Delta} + \alpha_0\partial_\mu B_\mu(t,x)\right], \qquad \bar{\chi}(t=0,x)=\bar{\psi}(x),$$

where

$$\Delta = D_\mu D_\mu, \qquad D_\mu = \partial_\mu + B_\mu,$$
$$\overleftarrow{\Delta} = \overleftarrow{D}_\mu\overleftarrow{D}_\mu, \qquad \overleftarrow{D}_\mu \equiv \overleftarrow{\partial}_\mu - B_\mu.$$

Flow of fermion fields

- A possible choice (Lüscher (2013))

$$\partial_t\chi(t,x)=[\Delta-\alpha_0\partial_\mu B_\mu(t,x)]\,\chi(t,x),\qquad \chi(t=0,x)=\psi(x),$$
$$\partial_t\bar\chi(t,x)=\bar\chi(t,x)\left[\overleftarrow{\Delta}+\alpha_0\partial_\mu B_\mu(t,x)\right],\qquad \bar\chi(t=0,x)=\bar\psi(x),$$

where

$$\Delta=D_\mu D_\mu,\qquad D_\mu=\partial_\mu+B_\mu,$$
$$\overleftarrow{\Delta}=\overleftarrow{D}_\mu\overleftarrow{D}_\mu,\qquad \overleftarrow{D}_\mu\equiv\overleftarrow{\partial}_\mu-B_\mu.$$

- It turns out that the flowed fermion field requires the wave function renormalization:

$$\chi_R(t,x)=Z_\chi^{1/2}\chi(t,x),\qquad \bar\chi_R(t,x)=Z_\chi^{1/2}\bar\chi(t,x),$$
$$Z_\chi=1+\frac{g^2}{(4\pi)^2}C_F 3\frac{1}{\varepsilon}+O(g^4).$$

Flow of fermion fields

- A possible choice (Lüscher (2013))

$$\partial_t\chi(t,x) = [\Delta - \alpha_0\partial_\mu B_\mu(t,x)]\,\chi(t,x), \qquad \chi(t=0,x)=\psi(x),$$
$$\partial_t\bar\chi(t,x) = \bar\chi(t,x)\left[\overleftarrow{\Delta} + \alpha_0\partial_\mu B_\mu(t,x)\right], \qquad \bar\chi(t=0,x)=\bar\psi(x),$$

where

$$\Delta = D_\mu D_\mu, \qquad D_\mu = \partial_\mu + B_\mu,$$
$$\overleftarrow{\Delta} = \overleftarrow{D}_\mu\overleftarrow{D}_\mu, \qquad \overleftarrow{D}_\mu \equiv \overleftarrow{\partial}_\mu - B_\mu.$$

- It turns out that the flowed fermion field requires the wave function renormalization:

$$\chi_R(t,x) = Z_\chi^{1/2}\chi(t,x), \qquad \bar\chi_R(t,x) = Z_\chi^{1/2}\bar\chi(t,x),$$
$$Z_\chi = 1 + \frac{g^2}{(4\pi)^2}C_F 3\frac{1}{\varepsilon} + O(g^4).$$

- Still, any composite operators of $\chi_R(t,x)$ are UV finite.

A $D+1$-dimensional local field theory

- The total action

$$S_{\text{tot}} = S + S_{\text{gf}} + S_{c\bar{c}} + S_{\text{fl}} + S_{d\bar{d}}.$$

A $D+1$-dimensional local field theory

- The total action
$$S_{\text{tot}} = S + S_{\text{gf}} + S_{c\bar{c}} + S_{\text{fl}} + S_{d\bar{d}}.$$
- The 4-dimensional part (pure Yang–Mills):
$$S = -\frac{1}{2g_0^2}\int d^D x\, \operatorname{tr}\left[F_{\mu\nu}(x)F_{\mu\nu}(x)\right],$$
$$S_{\text{gf}} + S_{c\bar{c}} = \delta\frac{-2}{g_0^2}\int d^D x\, \operatorname{tr}\left\{\bar{c}(x)\left[\partial_\mu A_\mu(x) - \frac{1}{2\lambda_0}B(x)\right]\right\},$$

where δ is the nilpotent BRS transformation
$$\delta A_\mu(x) = D_\mu c(x), \qquad \delta c(x) = -c(x)^2,$$
$$\delta\bar{c}(x) = B(x), \qquad \delta B(x) = 0.$$

A $D+1$-dimensional local field theory

- The total action

$$S_{\text{tot}} = S + S_{\text{gf}} + S_{c\bar{c}} + S_{\text{fl}} + S_{d\bar{d}}.$$

- The 4-dimensional part (pure Yang–Mills):

$$S = -\frac{1}{2g_0^2}\int d^Dx\,\operatorname{tr}\left[F_{\mu\nu}(x)F_{\mu\nu}(x)\right],$$

$$S_{\text{gf}} + S_{c\bar{c}} = \delta\frac{-2}{g_0^2}\int d^Dx\,\operatorname{tr}\left\{\bar{c}(x)\left[\partial_\mu A_\mu(x) - \frac{1}{2\lambda_0}B(x)\right]\right\},$$

where δ is the nilpotent BRS transformation

$$\delta A_\mu(x) = D_\mu c(x), \qquad \delta c(x) = -c(x)^2,$$
$$\delta\bar{c}(x) = B(x), \qquad \delta B(x) = 0.$$

- So

$$\delta S = 0, \qquad \delta(S_{\text{gf}} + S_{c\bar{c}}) = 0.$$

- The $D + 1$-dimensional part

$$S_{\text{fl}} = -2 \int_0^\infty dt \int d^D x \; \mathrm{tr} \left\{ L_\mu(t,x) \left[\partial_t B_\mu - D_\nu G_{\nu\mu} - \alpha_0 D_\mu \partial_\nu B_\nu \right](t,x) \right\}$$

A $D+1$-dimensional local field theory

- The $D+1$-dimensional part

$$S_{\text{fl}} = -2\int_0^\infty dt \int d^Dx \,\operatorname{tr}\left\{L_\mu(t,x)\left[\partial_t B_\mu - D_\nu G_{\nu\mu} - \alpha_0 D_\mu \partial_\nu B_\nu\right](t,x)\right\}$$

- Integration over the Lagrange multiplier:

$$L_\mu(t,x) = L_\mu^a(t,x)T^a$$

imposes the flow equation

A $D+1$-dimensional local field theory

- The $D+1$-dimensional part

$$S_{\text{fl}} = -2\int_0^\infty dt \int d^D x\ \mathrm{tr}\left\{L_\mu(t,x)\left[\partial_t B_\mu - D_\nu G_{\nu\mu} - \alpha_0 D_\mu \partial_\nu B_\nu\right](t,x)\right\}$$

- Integration over the Lagrange multiplier:

$$L_\mu(t,x) = L_\mu^a(t,x) T^a$$

  imposes the flow equation
- Bulk ghost fields

$$S_{d\bar{d}} = -2\int_0^\infty dt \int d^D x\ \mathrm{tr}\left\{\bar{d}(t,x)\left[\partial_t d - \alpha_0 D_\mu \partial_\mu d\right](t,x)\right\}$$

A $D+1$-dimensional local field theory

- The $D+1$-dimensional part

$$S_{\text{fl}} = -2\int_0^\infty dt \int d^Dx \ \mathrm{tr}\left\{L_\mu(t,x)\left[\partial_t B_\mu - D_\nu G_{\nu\mu} - \alpha_0 D_\mu \partial_\nu B_\nu\right](t,x)\right\}$$

- Integration over the Lagrange multiplier:

$$L_\mu(t,x) = L_\mu^a(t,x)T^a$$

  imposes the flow equation

- Bulk ghost fields

$$S_{d\bar{d}} = -2\int_0^\infty dt \int d^Dx \ \mathrm{tr}\left\{\bar{d}(t,x)\left[\partial_t d - \alpha_0 D_\mu \partial_\mu d\right](t,x)\right\}$$

- Boundary conditions

$$B_\mu(t=0,x) = A_\mu(x), \qquad d(t=0,x) = c(x),$$

  while $L_\mu(t=0,x)$ and $\bar{d}(t=0,x)$ are integrated freely.

Discrete flow time prescription

- We have to make the meaning precise:

$$\int_0^\infty dt \to \epsilon \sum_{t=0}^{\infty}, \qquad \partial_t B_\mu(t,x) \to \frac{1}{\epsilon}\left[B_\mu(t+\epsilon,x) - B_\mu(t,x)\right]$$

Discrete flow time prescription

- We have to make the meaning precise:

$$\int_0^\infty dt \to \epsilon \sum_{t=0}^{\infty}, \qquad \partial_t B_\mu(t,x) \to \frac{1}{\epsilon}\left[B_\mu(t+\epsilon,x) - B_\mu(t,x)\right]$$

- In particular,

$$\partial_t B_\mu(t=0,x) = \frac{1}{\epsilon}\left[B_\mu(t=\epsilon,x) - A_\mu(x)\right],$$

$$\partial_t d(t=0,x) = \frac{1}{\epsilon}\left[d(t=\epsilon,x) - c(x)\right].$$

Discrete flow time prescription

- Heat kernel with the discretized flow time:

$$K_t^\epsilon(x)_{\mu\nu} \equiv \int_p \frac{e^{ipx}}{p^2}\left[(\delta_{\mu\nu}p^2 - p_\mu p_\nu)(1-\epsilon p^2)^{t/\epsilon} + p_\mu p_\nu(1-\epsilon\alpha_0 p^2)^{t/\epsilon}\right].$$

  This fulfills

$$\frac{1}{\epsilon}\left[K_{t+\epsilon}^\epsilon(x)_{\mu\nu} - K_t^\epsilon(x)_{\mu\nu}\right] = \partial_\rho\partial_\rho K_t^\epsilon(x)_{\mu\nu} + (\alpha_0 - 1)\partial_\mu\partial_\rho K_t^\epsilon(x)_{\rho\nu},$$
$$K_0^\epsilon(x)_{\mu\nu} = \delta_{\mu\nu}\delta^D(x).$$

Discrete flow time prescription

- Heat kernel with the discretized flow time:

$$K_t^\epsilon(x)_{\mu\nu} \equiv \int_p \frac{e^{ipx}}{p^2} \left[(\delta_{\mu\nu} p^2 - p_\mu p_\nu)(1 - \epsilon p^2)^{t/\epsilon} + p_\mu p_\nu (1 - \epsilon \alpha_0 p^2)^{t/\epsilon} \right].$$

  This fulfills

$$\frac{1}{\epsilon} \left[K_{t+\epsilon}^\epsilon(x)_{\mu\nu} - K_t^\epsilon(x)_{\mu\nu} \right] = \partial_\rho \partial_\rho K_t^\epsilon(x)_{\mu\nu} + (\alpha_0 - 1) \partial_\mu \partial_\rho K_t^\epsilon(x)_{\rho\nu},$$
$$K_0^\epsilon(x)_{\mu\nu} = \delta_{\mu\nu} \delta^D(x).$$

- We thus change the variable from B_μ to b_μ by

$$B_\mu(t, x) = \int d^D y \, K_t^\epsilon(x - y)_{\mu\nu} A_\nu(y) + b_\mu(t, x),$$

  the boundary condition then becomes

$$b_\mu(t = 0, x) = 0,$$

Discrete flow time prescription

- and the action becomes

$$
\begin{aligned}
S_{\text{fl}} = -2\epsilon \sum_{t=0}^{\infty} \int d^D x \, \operatorname{tr} L_\mu(t,x) \\
\times \biggl\{ \frac{1}{\epsilon} \left[b_\mu(t+\epsilon,x) - b_\mu(t,x) \right] \\
- \partial_\nu \partial_\nu b_\mu(t,x) + (1-\alpha_0) \partial_\mu \partial_\nu b_\nu(t,x) \biggr\} + (\text{interactions}),
\end{aligned}
$$

with the boundary condition

$$
b_\mu(t=0,x) = 0.
$$

Discrete flow time prescription

- and the action becomes

$$
\begin{aligned}
S_{\text{fl}} = -2\epsilon \sum_{t=0}^{\infty} \int d^D x \, \operatorname{tr} L_\mu(t,x) \\
\times \Bigg\{ \frac{1}{\epsilon} [b_\mu(t+\epsilon, x) - b_\mu(t,x)] \\
- \partial_\nu \partial_\nu b_\mu(t,x) + (1-\alpha_0)\partial_\mu \partial_\nu b_\nu(t,x) \Bigg\} + (\text{interactions}),
\end{aligned}
$$

with the boundary condition

$$
b_\mu(t=0,x) = 0.
$$

- It is then straightforward to obtain the tree-level bL propagator:

$$
\left\langle b_\mu^a(t,x) L_\nu^b(s,y) \right\rangle_0 = \delta^{ab} \vartheta(t-s) K_{t-s-\epsilon}^{\epsilon}(x-y)_{\mu\nu},
$$

where $\vartheta(t)$ is a "regularized step function",

$$
\vartheta(t) \equiv \begin{cases} 1, & \text{for } t > 0, \\ 0, & \text{for } t = 0, \\ 0, & \text{for } t < 0. \end{cases}
$$

Addendum: Derivation of the propagator

- Kinetic term (omitting the spacetime coordinates)

$$S_{\mathrm{fl}}^{(2)} = \epsilon \sum_{t=0}^{\infty} L_{\mu}^{a}(t) \left\{ \frac{1}{\epsilon} \left[b_{\mu}^{a}(t+\epsilon) - b_{\mu}^{a}(t) \right] - \partial_{\nu}\partial_{\nu} b_{\mu}^{a}(t) + (1-\alpha_0)\partial_{\mu}\partial_{\nu} b_{\nu}^{a}(t) \right\}.$$

Addendum: Derivation of the propagator

- Kinetic term (omitting the spacetime coordinates)

$$S_{\mathrm{fl}}^{(2)} = \epsilon \sum_{t=0}^{\infty} L_\mu^a(t) \left\{ \frac{1}{\epsilon} \left[b_\mu^a(t+\epsilon) - b_\mu^a(t) \right] - \partial_\nu \partial_\nu b_\mu^a(t) + (1-\alpha_0) \partial_\mu \partial_\nu b_\nu^a(t) \right\} .$$

- Schwinger–Dyson equation

$$\left\langle \frac{1}{\epsilon} \left[b_\mu^a(t+\epsilon) - b_\mu^a(t) \right] L_\nu^b(s) + \left[-\delta_{\mu\rho} \partial_\sigma \partial_\sigma + (1-\alpha_0) \partial_\mu \partial_\rho \right] b_\rho^a(t) L_\nu^b(s) \right\rangle_0$$
$$= \frac{1}{\epsilon} \delta^{ab} \delta_{\mu\nu} \delta_{t,s},$$

with the boundary condition

$$b_\mu(t=0) = 0.$$

Addendum: Derivation of the propagator

- Schwinger–Dyson equation ($b_\mu(t=0)=0$)

$$\begin{aligned}&\left\langle b_\mu^a(t+\epsilon)L_\nu^b(s)\right\rangle_0\\&\quad=\{\delta_{\mu\rho}+\epsilon\left[\delta_{\mu\rho}\partial_\sigma\partial_\sigma-(1-\alpha_0)\partial_\mu\partial_\rho\right]\}\left\langle b_\rho^a(t)L_\nu^b(s)\right\rangle_0\\&\qquad+\delta^{ab}\delta_{\mu\nu}{\color{red}\delta_{t,s}}.\end{aligned}$$

Addendum: Derivation of the propagator

- Schwinger–Dyson equation ($b_\mu(t=0)=0$)

$$\begin{aligned}
&\left\langle b_\mu^a(t+\epsilon)L_\nu^b(s)\right\rangle_0 \\
&\qquad = \{\delta_{\mu\rho}+\epsilon\,[\delta_{\mu\rho}\partial_\sigma\partial_\sigma-(1-\alpha_0)\partial_\mu\partial_\rho]\}\left\langle b_\rho^a(t)L_\nu^b(s)\right\rangle_0 \\
&\qquad\qquad + \delta^{ab}\delta_{\mu\nu}\delta_{t,s}.
\end{aligned}$$

- Step by step solution:

$$\begin{array}{lcr}
t=0, & \left\langle b_\mu^a(0)L_\nu^b(s)\right\rangle_0=0, & \delta_{0,s}=0, \\
t=\epsilon, & \left\langle b_\mu^a(\epsilon)L_\nu^b(s)\right\rangle_0=0, & \delta_{\epsilon,s}=0, \\
t=2\epsilon, & \left\langle b_\mu^a(2\epsilon)L_\nu^b(s)\right\rangle_0=0, & \delta_{2\epsilon,s}=0, \\
t=\cdots & \cdots & \\
t=s-\epsilon, & \left\langle b_\mu^a(s-\epsilon)L_\nu^b(s)\right\rangle_0=0, & \delta_{s-\epsilon,s}=0, \\
t=s, & \left\langle b_\mu^a(s)L_\nu^b(s)\right\rangle_0=0, & \delta_{s,s}=1, \\
t=s+\epsilon, & \left\langle b_\mu^a(s+\epsilon)L_\nu^b(s)\right\rangle_0=\delta^{ab}\delta_{\mu\nu}, & \delta_{s+\epsilon,s}=0, \\
t=\cdots & \cdots &
\end{array}$$

Discrete flow time prescription

- Since the *AL* and *bb* propagators vanish,

$$\left\langle B_\mu^a(t,x)L_\nu^b(s,y)\right\rangle_0=\delta^{ab}\vartheta(t-s)K_{t-s-\epsilon}^{\epsilon}(x-y)_{\mu\nu},$$

$$\left\langle B_\mu^a(t,x)B_\nu^b(s,y)\right\rangle_0=g_0^2\delta^{ab}\int_p\frac{e^{ip(x-y)}}{(p^2)^2}$$

$$\times\left[(\delta_{\mu\nu}p^2-p_\mu p_\nu)(1-\epsilon p^2)^{(t+s)/\epsilon}+\frac{1}{\lambda_0}p_\mu p_\nu(1-\epsilon\alpha_0p^2)^{(t+s)/\epsilon}\right].$$

Discrete flow time prescription

- Since the AL and bb propagators vanish,

$$\left\langle B_\mu^a(t,x) L_\nu^b(s,y) \right\rangle_0 = \delta^{ab} \vartheta(t-s) K_{t-s-\epsilon}^{\epsilon}(x-y)_{\mu\nu},$$

$$\left\langle B_\mu^a(t,x) B_\nu^b(s,y) \right\rangle_0 = g_0^2 \delta^{ab} \int_p \frac{e^{ip(x-y)}}{(p^2)^2}$$
$$\times \left[(\delta_{\mu\nu} p^2 - p_\mu p_\nu)(1-\epsilon p^2)^{(t+s)/\epsilon} + \frac{1}{\lambda_0} p_\mu p_\nu (1-\epsilon\alpha_0 p^2)^{(t+s)/\epsilon} \right].$$

- The former BL-propagator becomes the heat kernel (or flow line) for $\epsilon \to 0$.

Discrete flow time prescription

- Since the *AL* and *bb* propagators vanish,

$$\left\langle B^a_\mu(t,x)L^b_\nu(s,y)\right\rangle_0 = \delta^{ab}\vartheta(t-s)K^\epsilon_{t-s-\epsilon}(x-y)_{\mu\nu},$$

$$\left\langle B^a_\mu(t,x)B^b_\nu(s,y)\right\rangle_0 = g_0^2\delta^{ab}\int_p \frac{e^{ip(x-y)}}{(p^2)^2}$$

$$\times\left[(\delta_{\mu\nu}p^2 - p_\mu p_\nu)(1-\epsilon p^2)^{(t+s)/\epsilon} + \frac{1}{\lambda_0}p_\mu p_\nu(1-\epsilon\alpha_0 p^2)^{(t+s)/\epsilon}\right].$$

- The former *BL*-propagator becomes the heat kernel (or flow line) for $\epsilon \to 0$.
- We note that, in the present prescription,

$$\left\langle B^a_\mu(t,x)L^b_\nu(t,y)\right\rangle_0 = 0,$$

  and thus there is no loop diagram consisting only of the heat kernels (or flow lines).

Discrete flow time prescription

- Since the AL and bb propagators vanish,

$$\left\langle B^a_\mu(t,x)L^b_\nu(s,y)\right\rangle_0 = \delta^{ab}\vartheta(t-s)K^\epsilon_{t-s-\epsilon}(x-y)_{\mu\nu},$$

$$\left\langle B^a_\mu(t,x)B^b_\nu(s,y)\right\rangle_0 = g_0^2\delta^{ab}\int_p \frac{e^{ip(x-y)}}{(p^2)^2}$$
$$\times\left[(\delta_{\mu\nu}p^2 - p_\mu p_\nu)(1-\epsilon p^2)^{(t+s)/\epsilon} + \frac{1}{\lambda_0}p_\mu p_\nu(1-\epsilon\alpha_0 p^2)^{(t+s)/\epsilon}\right].$$

- The former BL-propagator becomes the heat kernel (or flow line) for $\epsilon \to 0$.
- We note that, in the present prescription,

$$\left\langle B^a_\mu(t,x)L^b_\nu(t,y)\right\rangle_0 = 0,$$

  and thus there is no loop diagram consisting only of the heat kernels (or flow lines).
- Similar remark applies also to $d\bar{d}$ system.

BRS invariance of the $D+1$-dimensional system

- Going back to the full system…

BRS invariance of the $D+1$-dimensional system

- Going back to the full system. . .
- The $D+1$-dimensional BRS transformation

$$\delta B_\mu(t,x) = D_\mu d(t,x), \qquad \delta d(t,x) = -d(t,x)^2,$$
$$\delta L_\mu(t,x) = [L_\mu, d](t,x), \qquad \delta \bar{d}(t,x) = D_\mu L_\mu(t,x) - \{d, \bar{d}\}(t,x).$$

This is nilpotent, $\delta^2 = 0$.

BRS invariance of the $D+1$-dimensional system

- BRS invariance of the $D+1$-dimensional part:

$$S_{\mathrm{fl}} + S_{d\bar{d}} = -2\int_0^\infty dt \int d^Dx \ \mathrm{tr}\left[L_\mu(t,x)E_\mu(t,x) + \bar{d}(t,x)e(t,x)\right],$$

where

$$\begin{aligned} E_\mu(t,x) &\equiv \partial_t B_\mu(t,x) - D_\nu G_{\nu\mu}(t,x) - \alpha_0 D_\mu \partial_\nu B_\nu(t,x), \\ e(t,x) &\equiv \partial_t d(t,x) - \alpha_0 D_\mu \partial_\mu d(t,x). \end{aligned}$$

BRS invariance of the $D+1$-dimensional system

- BRS invariance of the $D+1$-dimensional part:

$$S_{\text{fl}} + S_{d\bar{d}} = -2\int_0^\infty dt \int d^Dx\ \mathrm{tr}\left[L_\mu(t,x)E_\mu(t,x) + \bar{d}(t,x)e(t,x)\right],$$

where

$$\begin{aligned}
E_\mu(t,x) &\equiv \partial_t B_\mu(t,x) - D_\nu G_{\nu\mu}(t,x) - \alpha_0 D_\mu \partial_\nu B_\nu(t,x),\\
e(t,x) &\equiv \partial_t d(t,x) - \alpha_0 D_\mu \partial_\mu d(t,x).
\end{aligned}$$

- Under the BRS transformation,

$$\begin{aligned}
\delta E_\mu(t,x) &= [E_\mu, d](t,x) + D_\mu e(t,x),\\
\delta e(t,x) &= -\{e, d\}(t,x),
\end{aligned}$$

and

$$\delta\left(S_{\text{fl}} + S_{d\bar{d}}\right) = 0.$$

BRS invariance of the $D+1$-dimensional system

- Caveat: The fact is that for $\epsilon \neq 0$ the BRS symmetry is broken by $O(\epsilon)$ terms. . .

BRS invariance of the $D+1$-dimensional system

- Caveat: The fact is that for $\epsilon \neq 0$ the BRS symmetry is broken by $O(\epsilon)$ terms...
- It turns out that those breaking terms are harmless.

BRS invariance of the $D + 1$-dimensional system

- Caveat: The fact is that for $\epsilon \neq 0$ the BRS symmetry is broken by $O(\epsilon)$ terms. . .
- It turns out that those breaking terms are harmless.
- For details, see, Kenji Hieda, Hiroki Makino, H. S., “Proof of the renormalizability of the gradient flow”, arXiv:1604.06200 [hep-lat].

Ward–Takahashi relation (or the Zinn-Justin equation)

- To derive the Ward–Takahashi relation, we introduce the source term

$$
\begin{aligned}
S_J = 2 \int d^D x \, \mathrm{tr} \left[J_\mu^A(x) A_\mu(x) + J^c(x) c(x) + J^{\bar{c}}(x) \bar{c}(x) + J^B(x) B(x)\right] \\
+ 2 \int_0^\infty\!\!\!\!\!\!\!\!-\;\; dt \int d^D x \, \mathrm{tr} \left[J_\mu^B(t,x) B_\mu(t,x) + J^d(t,x) d(t,x)\right] \\
+ 2 \int_0^\infty dt \int d^D x \, \mathrm{tr} \left[J_\mu^L(t,x) L_\mu(t,x) + J^{\bar{d}}(t,x) \bar{d}(t,x)\right],
\end{aligned}
$$

where

$$
\int_0^\infty\!\!\!\!\!\!\!\!-\;\; dt \equiv \epsilon \sum_{t=\epsilon}^{\infty}.
$$

- … and

$$
\begin{aligned}
S_K = 2\int d^Dx\,\mathrm{tr}\left[K_\mu^A(x)D_\mu c(x) - K^c(x)c(x)^2\right] \\
+ 2\,\,⨍_0^\infty dt \int d^Dx\,\mathrm{tr}\left[K_\mu^B(t,x)D_\mu d(t,x) - K^d(t,x)d(t,x)^2\right] \\
+ 2\int_0^\infty dt \int d^Dx\,\mathrm{tr}\Big[K_\mu^L(t,x)[L_\mu, d](t,x) \\
+ K^{\bar{d}}(t,x)\left(D_\mu L_\mu - \{d,\bar{d}\}\right)(t,x)\Big],
\end{aligned}
$$

where

$$
⨍_0^\infty dt \equiv \epsilon \sum_{t=\epsilon}^{\infty}.
$$

Ward–Takahashi relation

- Considering the BRS transformation of integration variables,

$$
\begin{aligned}
&\left\langle -2\int d^Dx\,\mathrm{tr}\left[J_\mu^A(x)D_\mu c(x)+J^c(x)c(x)^2-J^{\bar{c}}(x)B(x)\right]\right\rangle\\
&\quad+\left\langle -2\int_0^\infty dt\int d^Dx\,\mathrm{tr}\left[J_\mu^B(t,x)D_\mu d(t,x)+J^d(t,x)d(t,x)^2\right]\right\rangle\\
&\quad\quad+\left\langle -2\int_0^\infty dt\int d^Dx\,\mathrm{tr}\Big[J_\mu^L(t,x)[L_\mu,d](t,x)\right.\\
&\qquad\qquad\qquad\left.-J^{\bar{d}}\left(D_\mu L_\mu-\{d,\bar{d}\}\right)(t,x)\Big]\right\rangle=0.
\end{aligned}
$$

Ward–Takahashi relation

- Considering the BRS transformation of integration variables,

$$\left\langle -2\int d^Dx\,\mathrm{tr}\left[J^A_\mu(x)D_\mu c(x)+J^c(x)c(x)^2-J^{\bar c}(x)B(x)\right]\right\rangle$$
$$+\left\langle -2\int_0^\infty dt\int d^Dx\,\mathrm{tr}\left[J^B_\mu(t,x)D_\mu d(t,x)+J^d(t,x)d(t,x)^2\right]\right\rangle$$
$$+\left\langle -2\int_0^\infty dt\int d^Dx\,\mathrm{tr}\Big[J^L_\mu(t,x)[L_\mu,d](t,x)\right.$$
$$\left.-J^{\bar d}\left(D_\mu L_\mu-\{d,\bar d\}\right)(t,x)\Big]\right\rangle=0.$$

- Introducing the effective action Γ,

$$\int d\varphi\, e^{-S+J\varphi+K\delta\varphi}=e^{-W[J,K]},\qquad \frac{\delta}{\delta J}W[J,K]=-\langle\varphi\rangle=-\phi,$$
$$\Gamma[\phi,K]=W[J,K]+J\phi,\quad \frac{\delta}{\delta\phi}\Gamma[\phi,K]=\pm J,\quad \frac{\delta}{\delta K}\Gamma[\phi,K]=\langle\delta\varphi\rangle\,.$$

- The WT relation reads

$$
\begin{aligned}
&\int d^Dx\left[\frac{\delta\Gamma}{\delta A_\mu^a(x)}\frac{\delta\Gamma}{\delta K_\mu^{Aa}(x)}+\frac{\delta\Gamma}{\delta c^a(x)}\frac{\delta\Gamma}{\delta K^{ca}(x)}-\frac{\delta\Gamma}{\delta\bar{c}^a(x)}B^a(x)\right]\\
&\quad+\int_0^\infty dt\int d^Dx\left[\frac{\delta\Gamma}{\delta B_\mu^a(t,x)}\frac{\delta\Gamma}{\delta K_\mu^{Ba}(t,x)}+\frac{\delta\Gamma}{\delta d^a(t,x)}\frac{\delta\Gamma}{\delta K^{da}(t,x)}\right]\\
&\qquad+\int_0^\infty dt\int d^Dx\left[\frac{\delta\Gamma}{\delta L_\mu^a(t,x)}\frac{\delta\Gamma}{\delta K_\mu^{La}(t,x)}+\frac{\delta\Gamma}{\delta\bar{d}^a(t,x)}\frac{\delta\Gamma}{\delta K^{\bar{d}a}(t,x)}\right]\\
&\qquad\quad=0.
\end{aligned}
$$

Equation of motion of $B(x)$ and $\bar{c}(x)$

- Equation of motion:

$$\left\langle \frac{1}{g_0^2}\left[\partial_\mu A_\mu^a(x) - \frac{1}{\lambda_0} B^a(x)\right]\right\rangle - J^{Ba}(x) = 0,$$

$$\left\langle -\frac{1}{g_0^2}\partial_\mu D_\mu c^a(x)\right\rangle + J^{\bar{c}a}(x) = 0.$$

Equation of motion of $B(x)$ and $\bar{c}(x)$

- Equation of motion:

$$\left\langle \frac{1}{g_0^2}\left[\partial_\mu A_\mu^a(x) - \frac{1}{\lambda_0}B^a(x)\right]\right\rangle - J^{Ba}(x) = 0,$$

$$\left\langle -\frac{1}{g_0^2}\partial_\mu D_\mu c^a(x)\right\rangle + J^{\bar{c}a}(x) = 0.$$

- In terms of the effective action,

$$\frac{\delta\Gamma}{\delta B^a(x)} = \frac{1}{g_0^2}\left[\partial_\mu A_\mu^a(x) - \frac{1}{\lambda_0}B^a(x)\right],$$

$$\frac{\delta\Gamma}{\delta\bar{c}^a(x)} - \frac{1}{g_0^2}\partial_\mu\frac{\delta\Gamma}{\delta K_\mu^{Aa}(x)} = 0.$$

Equation of motion of $B(x)$ and $\bar{c}(x)$

- Equation of motion:

$$\left\langle \frac{1}{g_0^2}\left[\partial_\mu A_\mu^a(x) - \frac{1}{\lambda_0}B^a(x)\right]\right\rangle - J^{Ba}(x) = 0,$$
$$\left\langle -\frac{1}{g_0^2}\partial_\mu D_\mu c^a(x)\right\rangle + J^{\bar{c}a}(x) = 0.$$

- In terms of the effective action,

$$\frac{\delta\Gamma}{\delta B^a(x)} = \frac{1}{g_0^2}\left[\partial_\mu A_\mu^a(x) - \frac{1}{\lambda_0}B^a(x)\right],$$
$$\frac{\delta\Gamma}{\delta\bar{c}^a(x)} - \frac{1}{g_0^2}\partial_\mu\frac{\delta\Gamma}{\delta K_\mu^{Aa}(x)} = 0.$$

- These show, defining $\tilde{K}_\mu^{Aa} \equiv K_\mu^{Aa} - \frac{1}{g_0^2}\partial_\mu\bar{c}^a$,

$$\Gamma = \tilde{\Gamma}[\cancel{B}, \tilde{K}^A, \cancel{\bar{c}}] - \frac{2}{g_0^2}\int d^Dx\, \operatorname{tr}\left[B(x)\left(\partial_\mu A_\mu - \frac{1}{2\lambda_0}B\right)(x)\right].$$

Reduced effective action

- The reduced effective action

$$\tilde{\Gamma} = \tilde{\Gamma}[A_\mu, c, \tilde{K}_\mu^A, K^c; B_\mu, d, L_\mu, \bar{d}, K_\mu^B, K^d, K_\mu^L, K^{\bar{d}}]$$

$$\begin{aligned}
&\int d^D x \left[\frac{\delta\tilde{\Gamma}}{\delta A_\mu^a(x)}\frac{\delta\tilde{\Gamma}}{\delta\tilde{K}_\mu^{Aa}(x)} + \frac{\delta\tilde{\Gamma}}{\delta c^a(x)}\frac{\delta\tilde{\Gamma}}{\delta K^{ca}(x)}\right] \\
&\quad + \int_0^\infty dt \int d^D x \left[\frac{\delta\tilde{\Gamma}}{\delta B_\mu^a(t,x)}\frac{\delta\tilde{\Gamma}}{\delta K_\mu^{Ba}(t,x)} + \frac{\delta\tilde{\Gamma}}{\delta d^a(t,x)}\frac{\delta\tilde{\Gamma}}{\delta K^{da}(t,x)}\right] \\
&\qquad + \int_0^\infty dt \int d^D x \left[\frac{\delta\tilde{\Gamma}}{\delta L_\mu^a(t,x)}\frac{\delta\tilde{\Gamma}}{\delta K_\mu^{La}(t,x)} + \frac{\delta\tilde{\Gamma}}{\delta \bar{d}^a(t,x)}\frac{\delta\tilde{\Gamma}}{\delta K^{\bar{d}a}(t,x)}\right] \\
&\qquad\quad = 0.
\end{aligned}$$

Renormalization constants

- Renormalization ($D = 4 - 2\varepsilon$)

$$
\begin{aligned}
g_0^2 &= \mu^{2\varepsilon} g^2 Z, \\
A_\mu^a &= Z^{1/2} Z_3^{1/2} (A_R)_\mu^a, & \tilde{K}_\mu^{Aa} &= Z^{-1/2} Z_3^{-1/2} (\tilde{K}_R^A)_\mu^a, \\
c^a &= \tilde{Z}_3 Z^{1/2} Z_3^{1/2} c_R^a, & K^{ca} &= \tilde{Z}_3^{-1} Z^{-1/2} Z_3^{-1/2} K_R^{ca},
\end{aligned}
$$

and

$$
\begin{aligned}
\lambda_0 &= \lambda Z_3^{-1}, \\
B^a &= Z^{1/2} Z_3^{-1/2} B_R^a, & \bar{c}^a &= Z^{1/2} Z_3^{-1/2} \bar{c}_R^a.
\end{aligned}
$$

Renormalization constants

- Renormalization ($D = 4 - 2\varepsilon$)

$$
\begin{aligned}
g_0^2 &= \mu^{2\varepsilon} g^2 Z, \\
A_\mu^a &= Z^{1/2} Z_3^{1/2} (A_R)_\mu^a, & \tilde{K}_\mu^{Aa} &= Z^{-1/2} Z_3^{-1/2} (\tilde{K}_R^A)_\mu^a, \\
c^a &= \tilde{Z}_3 Z^{1/2} Z_3^{1/2} c_R^a, & K^{ca} &= \tilde{Z}_3^{-1} Z^{-1/2} Z_3^{-1/2} K_R^{ca},
\end{aligned}
$$

and

$$
\begin{aligned}
\lambda_0 &= \lambda Z_3^{-1}, \\
B^a &= Z^{1/2} Z_3^{-1/2} B_R^a, & \bar{c}^a &= Z^{1/2} Z_3^{-1/2} \bar{c}_R^a.
\end{aligned}
$$

- Note: $(\tilde{K}_R^A)_\mu^a = (K_R^A)_\mu^a - \frac{1}{\mu^{2\varepsilon} g^2} \partial_\mu \bar{c}_R^a$, and

$$
-\frac{2}{g_0^2} \operatorname{tr} \left[B \left(\partial_\mu A_\mu - \frac{1}{2\lambda_0} B \right) \right] = -\frac{2}{\mu^{2\varepsilon} g^2} \operatorname{tr} \left\{ B_R \left[\partial_\mu (A_R)_\mu - \frac{1}{2\lambda} B_R \right] \right\}
$$

Renormalization constants

- Renormalization ($D = 4 - 2\varepsilon$)

$$
\begin{aligned}
g_0^2 &= \mu^{2\varepsilon} g^2 Z, \\
A_\mu^a &= Z^{1/2} Z_3^{1/2} (A_R)_\mu^a, & \tilde{K}_\mu^{Aa} &= Z^{-1/2} Z_3^{-1/2} (\tilde{K}_R^A)_\mu^a, \\
c^a &= \tilde{Z}_3 Z^{1/2} Z_3^{1/2} c_R^a, & K^{ca} &= \tilde{Z}_3^{-1} Z^{-1/2} Z_3^{-1/2} K_R^{ca},
\end{aligned}
$$

and

$$
\begin{aligned}
\lambda_0 &= \lambda Z_3^{-1}, \\
B^a &= Z^{1/2} Z_3^{-1/2} B_R^a, & \bar{c}^a &= Z^{1/2} Z_3^{-1/2} \bar{c}_R^a.
\end{aligned}
$$

- Note: $(\tilde{K}_R^A)_\mu^a = (K_R^A)_\mu^a - \frac{1}{\mu^{2\varepsilon} g^2} \partial_\mu \bar{c}_R^a$, and

$$
-\frac{2}{g_0^2} \operatorname{tr} \left[B \left(\partial_\mu A_\mu - \frac{1}{2\lambda_0} B \right) \right] = -\frac{2}{\mu^{2\varepsilon} g^2} \operatorname{tr} \left\{ B_R \left[\partial_\mu (A_R)_\mu - \frac{1}{2\lambda} B_R \right] \right\}
$$

- We want to show that the above renormalization is enough to make $\tilde{\Gamma}$ finite!

WT relation

- WT relation in terms of renormalized quantity

$$\int d^Dx \left[\frac{\delta\tilde{\Gamma}}{\delta(A_R)^a_\mu(x)}\frac{\delta\tilde{\Gamma}}{\delta(\tilde{K}^A_R)^a_\mu(x)}+\frac{\delta\tilde{\Gamma}}{\delta c^a_R(x)}\frac{\delta\tilde{\Gamma}}{\delta K^{ca}_R(x)}\right]$$
$$+\int_0^\infty dt\int d^Dx\left[\frac{\delta\tilde{\Gamma}}{\delta B^a_\mu(t,x)}\frac{\delta\tilde{\Gamma}}{\delta K^{Ba}_\mu(t,x)}+\frac{\delta\tilde{\Gamma}}{\delta d^a(t,x)}\frac{\delta\tilde{\Gamma}}{\delta K^{da}(t,x)}\right]$$
$$+\int_0^\infty dt\int d^Dx\left[\frac{\delta\tilde{\Gamma}}{\delta L^a_\mu(t,x)}\frac{\delta\tilde{\Gamma}}{\delta K^{La}_\mu(t,x)}+\frac{\delta\tilde{\Gamma}}{\delta\bar{d}^a(t,x)}\frac{\delta\tilde{\Gamma}}{\delta K^{\bar{d}a}(t,x)}\right]$$
$$=0.$$

WT relation

- WT relation in terms of renormalized quantity

$$\begin{aligned}
&\int d^Dx \left[\frac{\delta\tilde{\Gamma}}{\delta(A_R)^a_\mu(x)}\frac{\delta\tilde{\Gamma}}{\delta(\tilde{K}^A_R)^a_\mu(x)}+\frac{\delta\tilde{\Gamma}}{\delta c^a_R(x)}\frac{\delta\tilde{\Gamma}}{\delta K^{ca}_R(x)}\right]\\
&\quad+\int_0^\infty dt\int d^Dx\left[\frac{\delta\tilde{\Gamma}}{\delta B^a_\mu(t,x)}\frac{\delta\tilde{\Gamma}}{\delta K^{Ba}_\mu(t,x)}+\frac{\delta\tilde{\Gamma}}{\delta d^a(t,x)}\frac{\delta\tilde{\Gamma}}{\delta K^{da}(t,x)}\right]\\
&\qquad+\int_0^\infty dt\int d^Dx\left[\frac{\delta\tilde{\Gamma}}{\delta L^a_\mu(t,x)}\frac{\delta\tilde{\Gamma}}{\delta K^{La}_\mu(t,x)}+\frac{\delta\tilde{\Gamma}}{\delta\bar{d}^a(t,x)}\frac{\delta\tilde{\Gamma}}{\delta K^{\bar{d}a}(t,x)}\right]\\
&\qquad\quad=0.
\end{aligned}$$

- Loop expansion in the renormalized perturbation theory

$$\tilde{\Gamma}=\sum_{\ell=0}^{\infty}\tilde{\Gamma}^{(\ell)}$$

WT relation

- WT relation in terms of renormalized quantity

$$\int d^D x \left[\frac{\delta\tilde{\Gamma}}{\delta(A_R)^a_\mu(x)}\frac{\delta\tilde{\Gamma}}{\delta(\tilde{K}^A_R)^a_\mu(x)}+\frac{\delta\tilde{\Gamma}}{\delta c^a_R(x)}\frac{\delta\tilde{\Gamma}}{\delta K^{ca}_R(x)}\right]$$
$$+\int_0^\infty dt \int d^D x \left[\frac{\delta\tilde{\Gamma}}{\delta B^a_\mu(t,x)}\frac{\delta\tilde{\Gamma}}{\delta K^{Ba}_\mu(t,x)}+\frac{\delta\tilde{\Gamma}}{\delta d^a(t,x)}\frac{\delta\tilde{\Gamma}}{\delta K^{da}(t,x)}\right]$$
$$+\int_0^\infty dt \int d^D x \left[\frac{\delta\tilde{\Gamma}}{\delta L^a_\mu(t,x)}\frac{\delta\tilde{\Gamma}}{\delta K^{La}_\mu(t,x)}+\frac{\delta\tilde{\Gamma}}{\delta \bar{d}^a(t,x)}\frac{\delta\tilde{\Gamma}}{\delta K^{\bar{d}a}(t,x)}\right]$$
$$=0.$$

- Loop expansion in the renormalized perturbation theory

$$\tilde{\Gamma}=\sum_{\ell=0}^\infty \tilde{\Gamma}^{(\ell)}$$

- Tree-level effective action in the renormalized perturbation theory

$$\tilde{\Gamma}^{(0)}=(S+S_{\text{fl}}+S_{d\bar{d}}+S_K)\big|_{Z=Z_3=\tilde{Z}_3=1,K^A_\mu\to(\tilde{K}^A_R)_\mu}$$

Renormalizability of the gradient flow

- We want to show that the renormalization constants

$$Z, \qquad Z_3, \qquad \tilde{Z}_3,$$

can be chosen order by order in the loop expansion, such as $\tilde{\Gamma}$ is finite.

Renormalizability of the gradient flow

- We want to show that the renormalization constants

$$Z, \qquad Z_3, \qquad \tilde{Z}_3,$$

can be chosen order by order in the loop expansion, such as $\tilde{\Gamma}$ is finite.

- The corresponding counter term is given by

$$\Delta S = S + S_{\mathrm{fl}} + S_{d\bar{d}} + S_K - (S + S_{\mathrm{fl}} + S_{d\bar{d}} + S_K)|_{Z=Z_3=\tilde{Z}_3=1, K_\mu^A \to (\tilde{K}_R^A)_\mu} .$$

Renormalizability of the gradient flow

- We want to show that the renormalization constants

$$Z, \qquad Z_3, \qquad \tilde{Z}_3,$$

can be chosen order by order in the loop expansion, such as $\tilde{\Gamma}$ is finite.
- The corresponding counter term is given by

$$\Delta S = S + S_{\text{fl}} + S_{d\bar{d}} + S_K - (S + S_{\text{fl}} + S_{d\bar{d}} + S_K)|_{Z=Z_3=\tilde{Z}_3=1, K_\mu^A \to (\tilde{K}_R^A)_\mu}.$$

- This contains in particular the "boundary counter term"

$$\Delta S_{\text{bc}} \equiv 2 \int d^D x \, \text{tr} \left[L_\mu(0,x)(Z^{1/2} Z_3^{1/2} - 1)(A_R)_\mu(x) \right]$$
$$+ 2 \int d^D x \, \text{tr} \left[\bar{d}(0,x)(\tilde{Z}_3^{1/2} Z^{1/2} Z_3^{1/2} - 1) c_R(x) \right].$$

Renormalizability of the gradient flow

- We want to show that the renormalization constants

$$Z, \qquad Z_3, \qquad \tilde{Z}_3,$$

can be chosen order by order in the loop expansion, such as $\tilde{\Gamma}$ is finite.

- The corresponding counter term is given by

$$\Delta S = S + S_{\text{fl}} + S_{d\bar{d}} + S_K - \left(S + S_{\text{fl}} + S_{d\bar{d}} + S_K\right)\big|_{Z=Z_3=\tilde{Z}_3=1, K_\mu^A \to (\tilde{K}_R^A)_\mu} .$$

- This contains in particular the "boundary counter term"

$$\Delta S_{\text{bc}} \equiv 2\int d^Dx\, \mathrm{tr}\left[L_\mu(0,x)(Z^{1/2}Z_3^{1/2}-1)(A_R)_\mu(x)\right] + 2\int d^Dx\, \mathrm{tr}\left[\bar{d}(0,x)(\tilde{Z}_3^{1/2}Z^{1/2}Z_3^{1/2}-1)c_R(x)\right].$$

- In the renormalized perturbation theory, the boundary conditions are taken as

$$B_\mu(t=0,x) = (A_R)_\mu(x), \qquad d(t=0,x) = c_R(x).$$

Renormalizability of the gradient flow

- Mathematical induction: Assume that, to the ℓ-th order Z, Z_3, and $\tilde{Z}_3$ can be chosen so that $\tilde{\Gamma}^{(\ell)}$ is finite.

Renormalizability of the gradient flow

- Mathematical induction: Assume that, to the ℓ-th order Z, Z_3, and $\tilde{Z}_3$ can be chosen so that $\tilde{\Gamma}^{(\ell)}$ is finite.
- Then, writing the divergent part of $\tilde{\Gamma}^{(\ell+1)}$, $\tilde{\Gamma}^{(\ell+1)\mathrm{div}}$, it satisfies the WT relation

$$\tilde{\Gamma}^{(0)} * \tilde{\Gamma}^{(\ell+1)\mathrm{div}} = 0,$$

where

$$\begin{aligned}
\tilde{\Gamma}^{(0)} * \equiv &- \int d^D x \left[\frac{\delta \tilde{\Gamma}^{(0)}}{\delta (A_R)^a_\mu(x)} \frac{\delta}{\delta (\tilde{K}^A_R)^a_\mu(x)} + \frac{\delta \tilde{\Gamma}^{(0)}}{\delta (\tilde{K}^A_R)^a_\mu(x)} \frac{\delta}{\delta (A_R)^a_\mu(x)} \right. \\
&\qquad \left. + \frac{\delta \tilde{\Gamma}^{(0)}}{\delta c^a_R(x)} \frac{\delta}{\delta K^{ca}_R(x)} + \frac{\delta \tilde{\Gamma}^{(0)}}{\delta K^{ca}_R(x)} \frac{\delta}{\delta c^a_R(x)} \right] \\
&- \epsilon \int d^D x \left[\frac{\delta \tilde{\Gamma}^{(0)}}{\delta L^a_\mu(0,x)} \frac{\delta}{\delta K^{La}_\mu(0,x)} + \frac{\delta \tilde{\Gamma}^{(0)}}{\delta K^{La}_\mu(0,x)} \frac{\delta}{\delta L^a_\mu(0,x)} \right. \\
&\qquad \left. + \frac{\delta \tilde{\Gamma}^{(0)}}{\delta \bar{d}^a(0,x)} \frac{\delta}{\delta K^{\bar{d}a}(0,x)} + \frac{\delta \tilde{\Gamma}^{(0)}}{\delta K^{\bar{d}a}(0,x)} \frac{\delta}{\delta \bar{d}^a(0,x)} \right] \\
&+ (\text{derivatives w.r.t. } t \neq 0 \text{ variables})
\end{aligned}$$

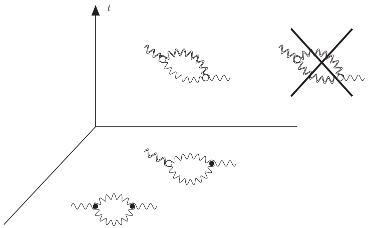
Renormalizability of the gradient flow

- Let us recall

$$
\begin{aligned}
S &= -\frac{1}{2g_0^2}\int d^Dx\,\operatorname{tr}\left[F_{\mu\nu}(x)F_{\mu\nu}(x)\right],\\
S_{\text{fl}} &= -2\int_0^\infty dt\int d^Dx\,\operatorname{tr}\left\{L_\mu(t,x)\left[\partial_t B_\mu - D_\nu G_{\nu\mu} - \alpha_0 D_\mu\partial_\nu B_\nu\right](t,x)\right\},\\
S_{d\bar{d}} &= -2\int_0^\infty dt\int d^Dx\,\operatorname{tr}\left\{\bar{d}(t,x)\left[\partial_t d - \alpha_0 D_\mu\partial_\mu d\right](t,x)\right\},\\
S_K &= 2\int d^Dx\,\operatorname{tr}\left[K_\mu^A(x)D_\mu c(x) - K^c(x)c(x)^2\right]\\
&\quad + 2\int_0^\infty dt\int d^Dx\,\operatorname{tr}\left[K_\mu^B(t,x)D_\mu d(t,x) - K^d(t,x)d(t,x)^2\right]\\
&\quad\quad + 2\int_0^\infty dt\int d^Dx\,\operatorname{tr}\Big[K_\mu^L(t,x)[L_\mu,d](t,x)\\
&\qquad\qquad + K^{\bar{d}}(t,x)\left(D_\mu L_\mu - \{d,\bar{d}\}\right)(t,x)\Big].
\end{aligned}
$$

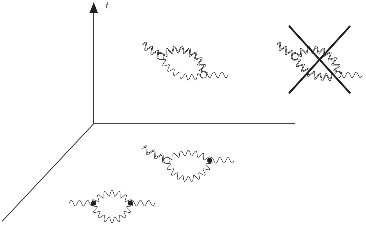
Renormalizability of the gradient flow

- First, we note that there is no “bulk divergence”. This comes from the consideration such as



Renormalizability of the gradient flow

- First, we note that there is no “bulk divergence”. This comes from the consideration such as

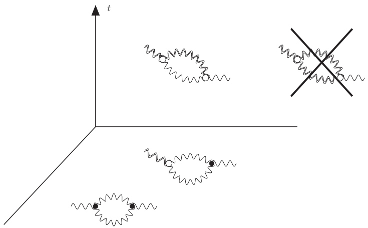


- The divergent part must be of the form,

$$\tilde{\Gamma}^{(\ell+1)\text{div}} = \int d^D x \,(\text{mass dim.} = 4,\ \text{ghost } \# = 0 \text{ local polynomial at } t = 0).$$

Renormalizability of the gradient flow

- First, we note that there is no “bulk divergence”. This comes from the consideration such as



- The divergent part must be of the form,

$$\tilde{\Gamma}^{(\ell+1)\text{div}} = \int d^D x \, (\text{mass dim.} = 4, \text{ghost } \# = 0 \text{ local polynomial at } t = 0).$$

	A_μ	c	$\tilde{K}_\mu^A$	K^c	L_μ	$\bar{d}$	K_μ^L	$K^{\bar{d}}$	B_μ	d	K_μ^B	K^d
dim.	1	1	2	2	3	3	2	2	1	1	4	4
gh. #	0	1	-1	-2	0	-1	-1	0	0	1	-1	-2

Renormalizability of the gradient flow

- $B_\mu(0,x) = (A_R)_\mu(x)$ and $d(0,x) = c_R(x)$ do not appear in $\tilde{\Gamma}^{(\ell+1)\mathrm{div}}$ as independent variables. But

$$-2\int d^Dx\ \mathrm{tr}\left[\partial_t B_\mu(0,x)(A_R)_\mu(x) + \partial_t d(0,x)\bar{c}_R(x)\right],$$

has the correct mass dimension and the ghost number as $\tilde{\Gamma}^{(\ell+1)\mathrm{div}}$.

Renormalizability of the gradient flow

- $B_\mu(0,x) = (A_R)_\mu(x)$ and $d(0,x) = c_R(x)$ do not appear in $\tilde{\Gamma}^{(\ell+1)\mathrm{div}}$ as independent variables. But

$$-2\int d^Dx\ \mathrm{tr}\left[\partial_t B_\mu(0,x)(A_R)_\mu(x) + \partial_t d(0,x)\bar{c}_R(x)\right],$$

  has the correct mass dimension and the ghost number as $\tilde{\Gamma}^{(\ell+1)\mathrm{div}}$.
- This is not the case, however, because a vertex function containing B_μ (or d) necessarily accompanies L_μ (or $\bar{d}$); there is no loop being consist only of the BL (or $d\bar{d}$) propagator.

Renormalizability of the gradient flow

- $B_\mu(0,x) = (A_R)_\mu(x)$ and $d(0,x) = c_R(x)$ do not appear in $\tilde{\Gamma}^{(\ell+1)\mathrm{div}}$ as independent variables. But

$$-2\int d^Dx\ \mathrm{tr}\left[\partial_t B_\mu(0,x)(A_R)_\mu(x) + \partial_t d(0,x)\bar{c}_R(x)\right],$$

  has the correct mass dimension and the ghost number as $\tilde{\Gamma}^{(\ell+1)\mathrm{div}}$.
- This is not the case, however, because a vertex function containing B_μ (or d) necessarily accompanies L_μ (or $\bar{d}$); there is no loop being consist only of the BL (or $d\bar{d}$) propagator.
- Next, we note in $\tilde{\Gamma}^{(0)}*$,

$$\frac{\delta\tilde{\Gamma}^{(0)}}{\delta L_\mu^a(0,x)} \sim \partial_t B_\mu^a(0,x) + \cdots, \qquad \frac{\delta\tilde{\Gamma}^{(0)}}{\delta\bar{d}^a(0,x)} \sim \partial_t d^a(0,x) + \cdots,$$

  and

$$\tilde{\Gamma}^{(0)}* \sim \frac{\delta\tilde{\Gamma}^{(0)}}{\delta L_\mu^a(0,x)}\frac{\delta}{\delta K_\mu^{La}(0,x)} + \frac{\delta\tilde{\Gamma}^{(0)}}{\delta\bar{d}^a(0,x)}\frac{\delta}{\delta K^{\bar{d}a}(0,x)} + \cdots.$$

  This shows that $\tilde{\Gamma}^{(\ell+1)\mathrm{div}}$ contains neither K_μ^L nor $K^{\bar{d}}$.

Renormalizability of the gradient flow

- Now, we see

$$\tilde{\Gamma}^{(\ell+1)\mathrm{div}} = \tilde{\Gamma}^{(\ell+1)\mathrm{div}}[(A_R)_\mu, c_R, (\tilde{K}_R^A)_\mu, K_R^c; L_\mu, \bar{d}].$$

Renormalizability of the gradient flow

- Now, we see
$$\tilde{\Gamma}^{(\ell+1)\mathrm{div}} = \tilde{\Gamma}^{(\ell+1)\mathrm{div}}[(A_R)_\mu, c_R, (\tilde{K}_R^A)_\mu, K_R^c; L_\mu, \bar{d}].$$
- In the equation,
$$\tilde{\Gamma}^{(0)} * \tilde{\Gamma}^{(\ell+1)\mathrm{div}} = 0,$$
we decompose
$$\tilde{\Gamma}^{(0)} * = \tilde{\Gamma}^{(0)}_{4\mathrm{D}} * + \tilde{\Gamma}^{(0)}_{5\mathrm{D}} *,$$
where $\tilde{\Gamma}^{(0)}_{4\mathrm{D}} *$ is the corresponding operator in the original 4D gauge theory:
$$\begin{aligned}\tilde{\Gamma}^{(0)}_{4\mathrm{D}} * \equiv -\int d^D x \Bigg[&\frac{\delta\ \tilde{\Gamma}^{(0)}\big|_{L_\mu=\bar{d}=0}}{\delta (A_R)^a_\mu(x)} \frac{\delta}{\delta (\tilde{K}_R^A)^a_\mu(x)} + \frac{\delta\ \tilde{\Gamma}^{(0)}\big|_{L_\mu=\bar{d}=0}}{\delta (\tilde{K}_R^A)^a_\mu(x)} \frac{\delta}{\delta (A_R)^a_\mu(x)} \\ &+ \frac{\delta\ \tilde{\Gamma}^{(0)}\big|_{L_\mu=\bar{d}=0}}{\delta c_R^a(x)} \frac{\delta}{\delta K_R^{ca}(x)} + \frac{\delta\ \tilde{\Gamma}^{(0)}\big|_{L_\mu=\bar{d}=0}}{\delta K_R^{ca}(x)} \frac{\delta}{\delta c_R^a(x)} \Bigg],\end{aligned}$$
and
$$\tilde{\Gamma}^{(\ell+1)\mathrm{div}} = \tilde{\Gamma}^{(\ell+1)\mathrm{div}}_{4\mathrm{D}}(\cancel{L}_\mu, \cancel{\bar{d}}, \dots) + \tilde{\Gamma}^{(\ell+1)\mathrm{div}}_{5\mathrm{D}}(L_\mu, \bar{d}, \dots)$$

Renormalizability of the gradient flow

- Then
$$\tilde{\Gamma}^{(0)} * \tilde{\Gamma}^{(\ell+1)\mathrm{div}} = 0$$
is decomposed into
$$\begin{aligned}
&\tilde{\Gamma}^{(0)}_{4\mathrm{D}} * \tilde{\Gamma}^{(\ell+1)\mathrm{div}}_{4\mathrm{D}} = 0,\\
&\tilde{\Gamma}^{(0)}_{4\mathrm{D}} * \tilde{\Gamma}^{(\ell+1)\mathrm{div}}_{5\mathrm{D}} + \tilde{\Gamma}^{(0)}_{5\mathrm{D}} * \tilde{\Gamma}^{(\ell+1)\mathrm{div}}_{4\mathrm{D}} + \tilde{\Gamma}^{(0)}_{5\mathrm{D}} * \tilde{\Gamma}^{(\ell+1)\mathrm{div}}_{5\mathrm{D}} = 0.
\end{aligned}$$

- Then

$$\tilde{\Gamma}^{(0)} * \tilde{\Gamma}^{(\ell+1)\text{div}} = 0$$

is decomposed into

$$\tilde{\Gamma}^{(0)}_{4\text{D}} * \tilde{\Gamma}^{(\ell+1)\text{div}}_{4\text{D}} = 0,$$
$$\tilde{\Gamma}^{(0)}_{4\text{D}} * \tilde{\Gamma}^{(\ell+1)\text{div}}_{5\text{D}} + \tilde{\Gamma}^{(0)}_{5\text{D}} * \tilde{\Gamma}^{(\ell+1)\text{div}}_{4\text{D}} + \tilde{\Gamma}^{(0)}_{5\text{D}} * \tilde{\Gamma}^{(\ell+1)\text{div}}_{5\text{D}} = 0.$$

- The general solution to the first equation is known to be

$$\tilde{\Gamma}^{(\ell+1)\text{div}}_{4\text{D}} = -\frac{1}{2\mu^{2\varepsilon} g^2} \int d^D x \operatorname{tr}\left[x_1 (F_R)_{\mu\nu}(x)(F_R)_{\mu\nu}(x)\right] - 2\tilde{\Gamma}^{(0)}_{4\text{D}} * \int d^D x \operatorname{tr} \underbrace{\left[y_1 (\tilde{K}^A_R)_\mu(x)(A_R)_\mu(x) + y_2 K^c_R(x) c_R(x)\right]}_{\text{dim.}=3,\text{ghost }\#=-1},$$

where

$$(F_R)_{\mu\nu}(x) \equiv \partial_\mu (A_R)_\nu(x) - \partial_\nu (A_R)_\mu(x) + [(A_R)_\mu, (A_R)_\nu](x).$$

Renormalizability of the gradient flow

- Thus, we have

$$
\begin{aligned}
\tilde{\Gamma}^{(\ell+1)\mathrm{div}} &= -\frac{1}{2\mu^{2\varepsilon}g^2}\int d^Dx\ \mathrm{tr}\left[x_1(F_R)_{\mu\nu}(x)(F_R)_{\mu\nu}(x)\right] \\
&\qquad - 2\tilde{\Gamma}^{(0)}_{4\mathrm{D}} * \int d^Dx\ \mathrm{tr}\left[y_1(\tilde{K}_R^A)_\mu(x)(A_R)_\mu(x) + y_2 K_R^c(x)c_R(x)\right] \\
&\qquad\quad + \tilde{\Gamma}^{(\ell+1)\mathrm{div}}_{5\mathrm{D}}(L_\mu, \bar{d}, \dots) \\
&= -\frac{1}{2\mu^{2\varepsilon}g^2}\int d^Dx\ \mathrm{tr}\left[x_1(F_R)_{\mu\nu}(x)(F_R)_{\mu\nu}(x)\right] \\
&\qquad - 2\tilde{\Gamma}^{(0)} * \int d^Dx\ \mathrm{tr}\left[y_1(\tilde{K}_R^A)_\mu(x)(A_R)_\mu(x) + y_2 K_R^c(x)c_R(x)\right] \\
&\qquad\quad + \tilde{\tilde{\Gamma}}^{(\ell+1)\mathrm{div}}_{5\mathrm{D}}(L_\mu, \bar{d}, \dots),
\end{aligned}
$$

where

$$
\tilde{\tilde{\Gamma}}^{(\ell+1)\mathrm{div}}_{5\mathrm{D}} \equiv \tilde{\Gamma}^{(\ell+1)\mathrm{div}}_{5\mathrm{D}} + 2\int d^Dx\ \mathrm{tr}\left[y_1 L_\mu(0,x)(A_R)_\mu(x) - y_2\bar{d}(0,x)c_R(x)\right].
$$

Renormalizability of the gradient flow

- The most general form of $\tilde{\tilde{\Gamma}}_{\text{5D}}^{(\ell+1)\text{div}}$ is

$$\tilde{\tilde{\Gamma}}_{\text{5D}}^{(\ell+1)\text{div}} = -2 \int d^D x \ \mathrm{tr}\left[z_1 L_\mu(0,x)(A_R)_\mu(x) + z_2 \bar{d}(0,x) c_R(x)\right].$$

Renormalizability of the gradient flow

- The most general form of $\tilde{\tilde{\Gamma}}_{\text{5D}}^{(\ell+1)\text{div}}$ is

$$\tilde{\tilde{\Gamma}}_{\text{5D}}^{(\ell+1)\text{div}} = -2 \int d^D x \, \operatorname{tr} \left[z_1 L_\mu(0,x)(A_R)_\mu(x) + z_2 \bar{d}(0,x) c_R(x) \right].$$

- Then the WT relation yields,

$$\begin{aligned}
&\tilde{\Gamma}^{(0)} * \tilde{\tilde{\Gamma}}_{\text{5D}}^{(\ell+1)\text{div}} \\
&= -2 \int d^D x \, \operatorname{tr}[(z_1 - z_2) L_\mu(0,x) \partial_\mu c_R(x) \\
&\qquad - z_2 L_\mu(0,x)[(A_R)_\mu, c_R](x) - z_2 \bar{d}(0,x) c_R(x)^2] = 0.
\end{aligned}$$

Renormalizability of the gradient flow

- The most general form of $\tilde{\tilde{\Gamma}}_{5\mathrm{D}}^{(\ell+1)\mathrm{div}}$ is

$$\tilde{\tilde{\Gamma}}_{5\mathrm{D}}^{(\ell+1)\mathrm{div}} = -2\int d^Dx\,\mathrm{tr}\left[z_1 L_\mu(0,x)(A_R)_\mu(x) + z_2\bar{d}(0,x)c_R(x)\right].$$

- Then the WT relation yields,

$$\begin{aligned}
&\tilde{\Gamma}^{(0)} * \tilde{\tilde{\Gamma}}_{5\mathrm{D}}^{(\ell+1)\mathrm{div}} \\
&= -2\int d^Dx\,\mathrm{tr}[(z_1 - z_2)L_\mu(0,x)\partial_\mu c_R(x) \\
&\qquad - z_2 L_\mu(0,x)[(A_R)_\mu, c_R](x) - z_2\bar{d}(0,x)c_R(x)^2] = 0.
\end{aligned}$$

- This shows that

$$\tilde{\tilde{\Gamma}}_{5\mathrm{D}}^{(\ell+1)\mathrm{div}} = 0,$$

and

$$\begin{aligned}
\tilde{\Gamma}^{(\ell+1)\mathrm{div}} &= -\frac{1}{2\mu^{2\varepsilon}g^2}\int d^Dx\,\mathrm{tr}\left[x_1(F_R)_{\mu\nu}(x)(F_R)_{\mu\nu}(x)\right] \\
&\quad - 2\tilde{\Gamma}^{(0)} * \int d^Dx\,\mathrm{tr}\left[y_1(\tilde{K}_R^A)_\mu(x)(A_R)_\mu(x) + y_2 K_R^c(x)c_R(x)\right].
\end{aligned}$$

Renormalizability of the gradient flow

- One can confirm that the above possible $\ell + 1$-th order divergent part,

$$\tilde{\Gamma}^{(\ell+1)\text{div}} = -\frac{1}{2\mu^{2\varepsilon} g^2} \int d^D x \, \text{tr} \left[x_1 (F_R)_{\mu\nu}(x)(F_R)_{\mu\nu}(x) \right] - 2\tilde{\Gamma}^{(0)} * \int d^D x \, \text{tr} \left[y_1 (\tilde{K}_R^A)_\mu(x)(A_R)_\mu(x) + y_2 K_R^c(x) c_R(x) \right],$$

can be canceled by choosing in the $\ell + 1$-th order,

$$Z^{(\ell+1)} = x_1,$$
$$Z_3^{(\ell+1)} = -x_1 + 2y_1,$$
$$\tilde{Z}_3^{(\ell+1)} = -y_1 - y_2$$

in

$$(S + S_{\text{fl}} + S_{d\bar{d}} + S_K)|_{K_\mu^A \to (\tilde{K}_R^A)_\mu}.$$

Renormalizability of the gradient flow

- One can confirm that the above possible $\ell + 1$-th order divergent part,

$$\tilde{\Gamma}^{(\ell+1)\mathrm{div}} = -\frac{1}{2\mu^{2\varepsilon}g^2}\int d^Dx\ \mathrm{tr}\left[x_1(F_R)_{\mu\nu}(x)(F_R)_{\mu\nu}(x)\right]$$
$$- 2\tilde{\Gamma}^{(0)} * \int d^Dx\ \mathrm{tr}\left[y_1(\tilde{K}_R^A)_\mu(x)(A_R)_\mu(x) + y_2K_R^c(x)c_R(x)\right],$$

can be canceled by choosing in the $\ell + 1$-th order,

$$Z^{(\ell+1)} = x_1,$$
$$Z_3^{(\ell+1)} = -x_1 + 2y_1,$$
$$\tilde{Z}_3^{(\ell+1)} = -y_1 - y_2$$

in

$$(S + S_{\mathrm{fl}} + S_{d\bar{d}} + S_K)\big|_{K_\mu^A \to (\tilde{K}_R^A)_\mu}.$$

- This completes the proof that no wave function renormalization of the flowed gauge field is required.

Fermion (or general matter) fields

- How the argument modified when fermion (or general matter) fields are included?

$$\cdots + \int d^D x\, \bar{\psi}(x)(\not{D} + m_0)\psi(x)$$

$$\cdots + \int_0^\infty dt \int d^D x\, \Big[\bar{\lambda}(t,x)(\partial_t - \Delta + \alpha_0 \partial_\nu B_\nu)\chi(t,x)$$

$$+ \bar{\chi}(t,x)(\overleftarrow{\partial}_t - \overleftarrow{\Delta} - \alpha_0 \partial_\nu B_\nu)\lambda(t,x)\Big].$$

Fermion (or general matter) fields

- How the argument modified when fermion (or general matter) fields are included?

$$\cdots + \int d^D x\, \bar{\psi}(x)(\not{D} + m_0)\psi(x)$$

$$\cdots + \int_0^\infty dt \int d^D x\, \big[\bar{\lambda}(t,x)(\partial_t - \Delta + \alpha_0 \partial_\nu B_\nu)\chi(t,x) + \bar{\chi}(t,x)(\overleftarrow{\partial}_t - \overleftarrow{\Delta} - \alpha_0 \partial_\nu B_\nu)\lambda(t,x)\big].$$

- Boundary conditions

$$\chi(t=0,x) = \psi(x), \qquad \bar{\chi}(t=0,x) = \bar{\psi}(x).$$

Fermion (or general matter) fields

- How the argument modified when fermion (or general matter) fields are included?

$$\cdots + \int d^D x\, \bar{\psi}(x)(\not{D} + m_0)\psi(x)$$

$$\cdots + \int_0^\infty dt \int d^D x \left[\bar{\lambda}(t,x)(\partial_t - \Delta + \alpha_0 \partial_\nu B_\nu)\chi(t,x) + \bar{\chi}(t,x)(\overleftarrow{\partial}_t - \overleftarrow{\Delta} - \alpha_0 \partial_\nu B_\nu)\lambda(t,x)\right].$$

- Boundary conditions

$$\chi(t=0,x) = \psi(x), \qquad \bar{\chi}(t=0,x) = \bar{\psi}(x).$$

- BRS transformation

$$\delta\psi(x) = -c(x)\psi(x), \qquad \delta\bar{\psi}(x) = -\bar{\psi}(x)c(x)$$
$$\delta\chi(t,x) = -d(t,x)\chi(t,x), \qquad \delta\bar{\chi}(t,x) = -\bar{\chi}(t,x)d(t,x)$$

and

$$\delta\lambda(t,x) = -d(t,x)\lambda(t,x), \qquad \delta\bar{\lambda}(t,x) = -\bar{\lambda}(t,x)d(t,x).$$

- It turns out that the total action can be made BRS invariant by simply modifying the BRS transformation of $\bar{d}(t,x)$ to

$$\delta\bar{d}(t,x) = \cdots + \bar{\lambda}(t,x)T^a\chi(t,x)T^a - \bar{\chi}(t,x)T^a\lambda(t,x)T^a.$$

Fermion (or general matter) fields

- It turns out that the total action can be made BRS invariant by simply modifying the BRS transformation of $\bar{d}(t,x)$ to

$$\delta\bar{d}(t,x) = \cdots + \bar{\lambda}(t,x)T^a\chi(t,x)T^a - \bar{\chi}(t,x)T^a\lambda(t,x)T^a.$$

- The nilpotency $\delta^2 = 0$ is preserved under this modification.

Fermion (or general matter) fields

- General form of the divergent part containing the new fields is

$$
\begin{aligned}
&\tilde{\Gamma}^{(\ell+1)\mathrm{div}} \\
&= \cdots + \int d^D x\, \left\{ w_1 \bar{\psi}_R(x) \gamma_\mu \left[\partial_\mu + (A_R)_\mu(x) \right] \psi_R(x) + w_2 m_R \bar{\psi}_R(x) \psi_R(x) \right\} \\
&\qquad + \tilde{\Gamma}^{(0)} * \int d^D x\, w_3 \left[\bar{K}_R^{\psi}(x) \psi_R(x) + \bar{\psi}_R(x) K_R^{\psi}(x) \right] \\
&\qquad\quad + \int d^D x\, \xi_1 \left[\bar{\lambda}_R(0,x) \psi_R(x) + \bar{\psi}_R(x) \lambda_R(0,x) \right].
\end{aligned}
$$

Fermion (or general matter) fields

- General form of the divergent part containing the new fields is

$$\tilde{\Gamma}^{(\ell+1)\mathrm{div}}$$

$$= \cdots + \int d^D x \, \left\{ w_1 \bar{\psi}_R(x) \gamma_\mu \left[\partial_\mu + (A_R)_\mu(x)\right] \psi_R(x) + w_2 m_R \bar{\psi}_R(x) \psi_R(x) \right\}$$

$$+ \tilde{\Gamma}^{(0)} * \int d^D x \, w_3 \left[\bar{K}_R^{\psi}(x) \psi_R(x) + \bar{\psi}_R(x) K_R^{\psi}(x) \right]$$

$$+ \int d^D x \, \xi_1 \left[\bar{\lambda}_R(0,x) \psi_R(x) + \bar{\psi}_R(x) \lambda_R(0,x) \right].$$

- These divergences are canceled by

$$Z_\psi^{(\ell+1)} = w_1 + 2w_3, \qquad Z_m^{(\ell+1)} = -w_1 + w_2, \qquad Z_\chi^{(\ell+1)} = w_1 + 4w_3 - 2\xi_1,$$

in $m_0 = Z_m^{-1} m_R$, $\psi = Z_\psi^{-1/2} \psi_R$, and $\bar{\psi} = Z_\psi^{-1/2} \bar{\psi}_R$ and

$$\bar{\lambda} = Z_\chi^{1/2} \bar{\lambda}_R, \qquad \lambda = Z_\chi^{1/2} \lambda_R,$$

$$\chi = Z_\chi^{-1/2} \chi_R, \qquad \bar{\chi} = Z_\chi^{-1/2} \bar{\chi}_R.$$

(Renormalization of Ks is omitted.)

Fermion (or general matter) fields

- Thus, the renormalization of flowed fermion fields is not excluded and an explicit calculation shows

$$Z_\chi = 1 + \frac{g^2}{(4\pi)^2} C_F 3 \frac{1}{\varepsilon} + O(g^4).$$

Fermion (or general matter) fields

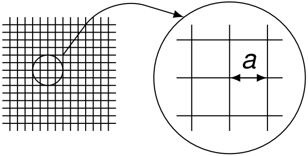
- Thus, the renormalization of flowed fermion fields is not excluded and an explicit calculation shows

$$Z_\chi = 1 + \frac{g^2}{(4\pi)^2} C_F 3 \frac{1}{\varepsilon} + O(g^4).$$

- The situation must be similar for generic matter fields.

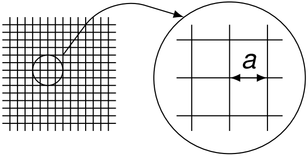
Lattice gauge theory and the energy–momentum tensor (EMT)

- Lattice gauge theory: the most successful non-perturbative formulation of gauge theory. By discretizing the spacetime…



Lattice gauge theory and the energy–momentum tensor (EMT)

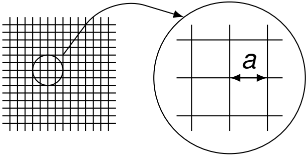
- Lattice gauge theory: the most successful non-perturbative formulation of gauge theory. By discretizing the spacetime…



- internal gauge symmetry is preserved exactly…

Lattice gauge theory and the energy–momentum tensor (EMT)

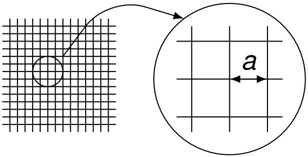
- Lattice gauge theory: the most successful non-perturbative formulation of gauge theory. By discretizing the spacetime...



- internal gauge symmetry is preserved exactly...
- but incompatible with spacetime symmetries (translation, Poincaré, SUSY, conformal, . . .) for $a \neq 0$.

Lattice gauge theory and the energy–momentum tensor (EMT)

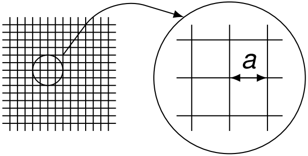
- Lattice gauge theory: the most successful non-perturbative formulation of gauge theory. By discretizing the spacetime...



- internal gauge symmetry is preserved exactly...
- but incompatible with spacetime symmetries (translation, Poincaré, SUSY, conformal, ...) for $a \neq 0$.
- For $a \neq 0$, one cannot define the Noether current associated with the translational invariance, EMT $T_{\mu\nu}(x)$.

Lattice gauge theory and the energy–momentum tensor (EMT)

- Lattice gauge theory: the most successful non-perturbative formulation of gauge theory. By discretizing the spacetime...



- internal gauge symmetry is preserved exactly...
- but incompatible with spacetime symmetries (translation, Poincaré, SUSY, conformal, ...) for $a \neq 0$.
- For $a \neq 0$, one cannot define the Noether current associated with the translational invariance, EMT $T_{\mu\nu}(x)$.
- Even for the continuum limit $a \to 0$, this is difficult, because EMT is a composite operator which generally contains UV divergences:

$$a \times \frac{1}{a} \stackrel{a\to 0}{\to} 1.$$

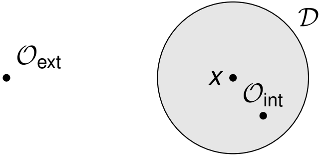
EMT in lattice gauge theory?

- Is it possible to construct EMT on the lattice, which becomes the correct EMT automatically in the continuum limit $a \to 0$?

EMT in lattice gauge theory?

- Is it possible to construct EMT on the lattice, which becomes the correct EMT automatically in the continuum limit $a \to 0$?
- The correct EMT is characterized by the Ward–Takahashi relation

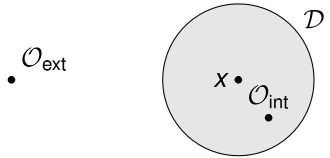
$$\left\langle \mathcal{O}_{\text{ext}} \int_{\mathcal{D}} d^D x \, \partial_\mu T_{\mu\nu}(x) \, \mathcal{O}_{\text{int}} \right\rangle = - \langle \mathcal{O}_{\text{ext}} \, \partial_\nu \mathcal{O}_{\text{int}} \rangle .$$



EMT in lattice gauge theory?

- Is it possible to construct EMT on the lattice, which becomes the correct EMT automatically in the continuum limit $a \to 0$?
- The correct EMT is characterized by the Ward–Takahashi relation

$$\left\langle \mathcal{O}_{\mathrm{ext}} \int_{\mathcal{D}} d^D x\, \partial_\mu T_{\mu\nu}(x)\, \mathcal{O}_{\mathrm{int}} \right\rangle = -\left\langle \mathcal{O}_{\mathrm{ext}}\, \partial_\nu \mathcal{O}_{\mathrm{int}} \right\rangle .$$



- This contains the correct normalization and the conservation law.

EMT in lattice gauge theory?

- If such a construction is possible, we expect wide application to physics related to spacetime symmetries: QCD thermodynamics, transport coefficients in gauge theory, momentum/spin structure of baryons, conformal field theory, dilaton physics, . . .

EMT in lattice gauge theory?

- If such a construction is possible, we expect wide application to physics related to spacetime symmetries: QCD thermodynamics, transport coefficients in gauge theory, momentum/spin structure of baryons, conformal field theory, dilaton physics, . . .
- Also the present work is an attempt to define EMT in quantum field theory in the non-perturbative level.

Conventional approach (Caracciolo et al. (1989–))

- Under the hypercubic symmetry, the operator reproducing the correct EMT of QCD for $a \to 0$ is given by

$$T_{\mu\nu}(x) = \sum_{i=1}^{7} Z_i \mathcal{O}_{i\mu\nu}(x)\big|_{\text{lattice}} - \text{VEV},$$

where

$$\mathcal{O}_{1\mu\nu}(x) \equiv \sum_{\rho} F^a_{\mu\rho}(x) F^a_{\nu\rho}(x), \qquad \mathcal{O}_{2\mu\nu}(x) \equiv \delta_{\mu\nu} \sum_{\rho,\sigma} F^a_{\rho\sigma}(x) F^a_{\rho\sigma}(x),$$

$$\mathcal{O}_{3\mu\nu}(x) \equiv \bar{\psi}(x) \left(\gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu \right) \psi(x), \quad \mathcal{O}_{4\mu\nu}(x) \equiv \delta_{\mu\nu} \bar{\psi}(x) \overleftrightarrow{\not{D}} \psi(x),$$

$$\mathcal{O}_{5\mu\nu}(x) \equiv \delta_{\mu\nu} m_0 \bar{\psi}(x) \psi(x),$$

and, Lorentz non-covariant ones:

$$\mathcal{O}_{6\mu\nu}(x) \equiv \delta_{\mu\nu} \sum_{\rho} F^a_{\mu\rho}(x) F^a_{\mu\rho}(x), \qquad \mathcal{O}_{7\mu\nu}(x) \equiv \delta_{\mu\nu} \bar{\psi}(x) \gamma_\mu \overleftrightarrow{D}_\mu \psi(x)$$

Conventional approach (Caracciolo et al. (1989–))

- Under the hypercubic symmetry, the operator reproducing the correct EMT of QCD for $a \to 0$ is given by

$$T_{\mu\nu}(x) = \sum_{i=1}^{7} Z_i \mathcal{O}_{i\mu\nu}(x)\big|_{\text{lattice}} - \text{VEV},$$

  where

$$\mathcal{O}_{1\mu\nu}(x) \equiv \sum_{\rho} F^a_{\mu\rho}(x) F^a_{\nu\rho}(x), \qquad \mathcal{O}_{2\mu\nu}(x) \equiv \delta_{\mu\nu} \sum_{\rho,\sigma} F^a_{\rho\sigma}(x) F^a_{\rho\sigma}(x),$$

$$\mathcal{O}_{3\mu\nu}(x) \equiv \bar{\psi}(x) \left(\gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu\right) \psi(x), \quad \mathcal{O}_{4\mu\nu}(x) \equiv \delta_{\mu\nu} \bar{\psi}(x) \overleftrightarrow{\not{D}} \psi(x),$$

$$\mathcal{O}_{5\mu\nu}(x) \equiv \delta_{\mu\nu} m_0 \bar{\psi}(x) \psi(x),$$

  and, Lorentz non-covariant ones:

$$\mathcal{O}_{6\mu\nu}(x) \equiv \delta_{\mu\nu} \sum_{\rho} F^a_{\mu\rho}(x) F^a_{\mu\rho}(x), \qquad \mathcal{O}_{7\mu\nu}(x) \equiv \delta_{\mu\nu} \bar{\psi}(x) \gamma_\mu \overleftrightarrow{D}_\mu \psi(x)$$

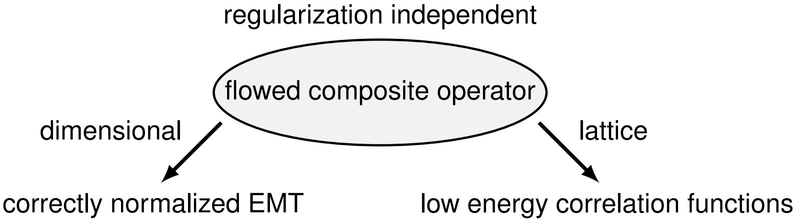
- Seven non-universal coefficients Z_i must be determined by lattice perturbation theory or by a non-perturbative method

Our approach (arXiv:1304.0533)

- We bridge lattice regularization and dimensional regularization which preserves the translational invariance, by using a flowed composite operator as an intermediate tool.

Our approach (arXiv:1304.0533)

- We bridge lattice regularization and dimensional regularization which preserves the translational invariance, by using a flowed composite operator as an intermediate tool.
- Schematically,



EMT in the dimensional regularization

- The action

$$S = -\frac{1}{2g_0^2}\int d^Dx\,\mathrm{tr}\,[F_{\mu\nu}(x)F_{\mu\nu}(x)] + \int d^Dx\,\bar{\psi}(x)(\not{D} + m_0)\psi(x).$$

EMT in the dimensional regularization

- The action

$$S = -\frac{1}{2g_0^2}\int d^Dx\,\mathrm{tr}\left[F_{\mu\nu}(x)F_{\mu\nu}(x)\right] + \int d^Dx\,\bar{\psi}(x)(\not{D} + m_0)\psi(x).$$

- Under the localized translation (plus the gauge transformation),

$$\delta A_\mu(x) = \xi_\nu(x)F_{\nu\mu}(x),$$

$$\delta\psi(x) = \xi(x)_\mu D_\mu\psi(x), \qquad \delta\bar{\psi}(x) = \xi(x)_\mu\bar{\psi}(x)\overleftarrow{D}_\mu,$$

we have

$$\delta S = -\int d^Dx\,\xi_\nu(x)\partial_\mu\left[T_{\mu\nu}(x) + A_{\mu\nu}(x)\right],$$

where the anti-symmetric part,

$$A_{\mu\nu}(x) = \frac{1}{4}\bar{\psi}(x)\sigma_{\mu\nu}(\not{D} + m_0)\psi(x) + \frac{1}{4}\bar{\psi}(x)(\overleftarrow{\not{D}} - m_0)\sigma_{\mu\nu}\psi(x),$$

is proportional to the EoM and can be neglected…

EMT in dimensional regularization

- … and $T_{\mu\nu}(x)$ is the symmetric EMT:

$$T_{\mu\nu}(x) = \frac{1}{g_0^2}\left\{\mathcal{O}_{1\mu\nu}(x) - \frac{1}{4}\mathcal{O}_{2\mu\nu}(x)\right\} + \frac{1}{4}\mathcal{O}_{3\mu\nu}(x) - \frac{1}{2}\mathcal{O}_{4\mu\nu}(x) - \mathcal{O}_{5\mu\nu}(x) - \mathrm{VEV},$$

where

$$\mathcal{O}_{1\mu\nu}(x) \equiv \sum_{\rho} F^a_{\mu\rho}(x)F^a_{\nu\rho}(x), \qquad \mathcal{O}_{2\mu\nu}(x) \equiv \delta_{\mu\nu}\sum_{\rho,\sigma} F^a_{\rho\sigma}(x)F^a_{\rho\sigma}(x),$$

$$\mathcal{O}_{3\mu\nu}(x) \equiv \bar{\psi}(x)\left(\gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu\right)\psi(x), \quad \mathcal{O}_{4\mu\nu}(x) \equiv \delta_{\mu\nu}\bar{\psi}(x)\overleftrightarrow{\not{D}}\psi(x),$$

$$\mathcal{O}_{5\mu\nu}(x) \equiv \delta_{\mu\nu} m_0 \bar{\psi}(x)\psi(x).$$

EMT in dimensional regularization

- … and $T_{\mu\nu}(x)$ is the symmetric EMT:

$$
\begin{aligned}
&T_{\mu\nu}(x)\\
&=\frac{1}{g_0^2}\left\{\mathcal{O}_{1\mu\nu}(x)-\frac{1}{4}\mathcal{O}_{2\mu\nu}(x)\right\}+\frac{1}{4}\mathcal{O}_{3\mu\nu}(x)-\frac{1}{2}\mathcal{O}_{4\mu\nu}(x)-\mathcal{O}_{5\mu\nu}(x)-\mathrm{VEV},
\end{aligned}
$$

where

$$
\begin{aligned}
&\mathcal{O}_{1\mu\nu}(x)\equiv\sum_\rho F^a_{\mu\rho}(x)F^a_{\nu\rho}(x), &&\mathcal{O}_{2\mu\nu}(x)\equiv\delta_{\mu\nu}\sum_{\rho,\sigma}F^a_{\rho\sigma}(x)F^a_{\rho\sigma}(x),\\
&\mathcal{O}_{3\mu\nu}(x)\equiv\bar{\psi}(x)\left(\gamma_\mu\overleftrightarrow{D}_\nu+\gamma_\nu\overleftrightarrow{D}_\mu\right)\psi(x), &&\mathcal{O}_{4\mu\nu}(x)\equiv\delta_{\mu\nu}\bar{\psi}(x)\overleftrightarrow{\not{D}}\psi(x),\\
&\mathcal{O}_{5\mu\nu}(x)\equiv\delta_{\mu\nu}m_0\bar{\psi}(x)\psi(x).
\end{aligned}
$$

- We define the renormalized EMT by subtracting its possibly divergent vacuum expectation value.

EMT in dimensional regularization

- … and $T_{\mu\nu}(x)$ is the symmetric EMT:

$$T_{\mu\nu}(x) = \frac{1}{g_0^2}\left\{\mathcal{O}_{1\mu\nu}(x) - \frac{1}{4}\mathcal{O}_{2\mu\nu}(x)\right\} + \frac{1}{4}\mathcal{O}_{3\mu\nu}(x) - \frac{1}{2}\mathcal{O}_{4\mu\nu}(x) - \mathcal{O}_{5\mu\nu}(x) - \mathrm{VEV},$$

where

$$\mathcal{O}_{1\mu\nu}(x) \equiv \sum_\rho F^a_{\mu\rho}(x)F^a_{\nu\rho}(x), \qquad \mathcal{O}_{2\mu\nu}(x) \equiv \delta_{\mu\nu}\sum_{\rho,\sigma} F^a_{\rho\sigma}(x)F^a_{\rho\sigma}(x),$$

$$\mathcal{O}_{3\mu\nu}(x) \equiv \bar{\psi}(x)\left(\gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu\right)\psi(x), \quad \mathcal{O}_{4\mu\nu}(x) \equiv \delta_{\mu\nu}\bar{\psi}(x)\overleftrightarrow{\not{D}}\psi(x),$$

$$\mathcal{O}_{5\mu\nu}(x) \equiv \delta_{\mu\nu} m_0 \bar{\psi}(x)\psi(x).$$

- We define the renormalized EMT by subtracting its possibly divergent vacuum expectation value.
- Under the dimensional regularization, this is the correct EMT.

Small flow-time expansion

- We thus want to find a composite operator of the flowed fields which reduces to the EMT under the dimensional regularization.

Small flow-time expansion

- We thus want to find a composite operator of the flowed fields which reduces to the EMT under the dimensional regularization.
- But how?

Small flow-time expansion

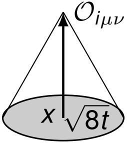
- We thus want to find a composite operator of the flowed fields which reduces to the EMT under the dimensional regularization.
- But how?
- In general, the relation between composite operators in $t > 0$ (heaven) and in 4D (the earth) is not obvious at all...

Small flow-time expansion

- We thus want to find a composite operator of the flowed fields which reduces to the EMT under the dimensional regularization.
- But how?
- In general, the relation between composite operators in $t > 0$ (heaven) and in 4D (the earth) is not obvious at all...
- The relation becomes tractable, in the limit in which the flow time becomes small $t \to 0$.

Small flow-time expansion

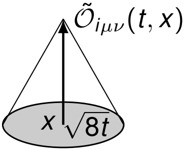
- We thus want to find a composite operator of the flowed fields which reduces to the EMT under the dimensional regularization.
- But how?
- In general, the relation between composite operators in $t > 0$ (heaven) and in 4D (the earth) is not obvious at all. . .
- The relation becomes tractable, in the limit in which the flow time becomes small $t \to 0$.
- Small flow-time expansion (Lüscher–Weisz (2011)):



$$\tilde{\mathcal{O}}_{i\mu\nu}(t,x) = \left\langle \tilde{\mathcal{O}}_{i\mu\nu}(t,x) \right\rangle \mathbb{1} + \sum_j \zeta_{ij}(t) \left[\mathcal{O}_{j\mu\nu}(x) - \mathrm{VEV} \right] + O(t).$$

Small flow-time expansion

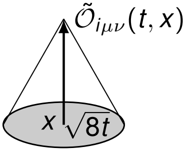
- Small flow-time expansion:



$$\tilde{\mathcal{O}}_{i\mu\nu}(t,x) = \left\langle \tilde{\mathcal{O}}_{i\mu\nu}(t,x) \right\rangle \mathbb{1} + \sum_j \zeta_{ij}(t) \left[\mathcal{O}_{j\mu\nu}(x) - \mathrm{VEV} \right] + O(t).$$

Small flow-time expansion

- Small flow-time expansion:



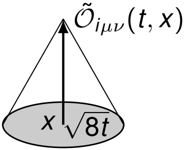
$$\tilde{\mathcal{O}}_{i\mu\nu}(t,x) = \left\langle \tilde{\mathcal{O}}_{i\mu\nu}(t,x) \right\rangle \mathbb{1} + \sum_j \zeta_{ij}(t) \left[\mathcal{O}_{j\mu\nu}(x) - \mathrm{VEV} \right] + O(t).$$

- Inverting this relation,

$$\mathcal{O}_{i\mu\nu}(x) - \mathrm{VEV} = \lim_{t\to 0} \left\{ \sum_j \left(\zeta^{-1}\right)_{ij}(t) \left[\tilde{\mathcal{O}}_{j\mu\nu}(t,x) - \left\langle \tilde{\mathcal{O}}_{j\mu\nu}(t,x) \right\rangle \mathbb{1} \right] \right\}.$$

Small flow-time expansion

- Small flow-time expansion:



$$\tilde{\mathcal{O}}_{i\mu\nu}(t,x) = \left\langle \tilde{\mathcal{O}}_{i\mu\nu}(t,x) \right\rangle \mathbb{1} + \sum_j \zeta_{ij}(t) \left[\mathcal{O}_{j\mu\nu}(x) - \mathrm{VEV} \right] + O(t).$$

- Inverting this relation,

$$\mathcal{O}_{i\mu\nu}(x) - \mathrm{VEV} = \lim_{t\to 0} \left\{ \sum_j \left(\zeta^{-1}\right)_{ij}(t) \left[\tilde{\mathcal{O}}_{j\mu\nu}(t,x) - \left\langle \tilde{\mathcal{O}}_{j\mu\nu}(t,x) \right\rangle \mathbb{1} \right] \right\}.$$

- So, if we know the $t \to 0$ behavior of the coefficients $\zeta_{ij}(t)$, the 4D operator in the LHS can be extracted as the $t \to 0$ limit.

Renormalization group argument

- We are interested in the $t \to 0$ behavior of the coefficients $\zeta_{ij}(t)$ in

$$\tilde{\mathcal{O}}_{i\mu\nu}(t,x) = \left\langle \tilde{\mathcal{O}}_{i\mu\nu}(t,x) \right\rangle \mathbb{1} + \sum_j \zeta_{ij}(t) \left[\mathcal{O}_{j\mu\nu}(x) - \langle \mathcal{O}_{j\mu\nu}(x) \rangle \mathbb{1} \right] + O(t).$$

Renormalization group argument

- We are interested in the $t \to 0$ behavior of the coefficients $\zeta_{ij}(t)$ in

$$\tilde{\mathcal{O}}_{i\mu\nu}(t,x) = \left\langle \tilde{\mathcal{O}}_{i\mu\nu}(t,x) \right\rangle \mathbb{1} + \sum_j \zeta_{ij}(t) \left[\mathcal{O}_{j\mu\nu}(x) - \langle \mathcal{O}_{j\mu\nu}(x) \rangle \mathbb{1} \right] + O(t).$$

- If all the composite operators in this relation are made out from bare quantities,

$$\left(\mu \frac{\partial}{\partial \mu} \right)_0 \zeta_{ij}(t) = 0,$$

  and $\zeta_{ij}(t)$ are indep. of the renormalization scale μ, when expressed in terms of running parameters. We can set $\mu = c/\sqrt{t}$, and

$$\zeta_{ij}(t)\,[g, m; \mu] = \zeta_{ij}(t) \left[g(c/\sqrt{t}), m(c/\sqrt{t}); c/\sqrt{t} \right].$$

Renormalization group argument

- We are interested in the $t \to 0$ behavior of the coefficients $\zeta_{ij}(t)$ in

$$\tilde{\mathcal{O}}_{i\mu\nu}(t,x) = \left\langle \tilde{\mathcal{O}}_{i\mu\nu}(t,x) \right\rangle \mathbb{1} + \sum_j \zeta_{ij}(t) \left[\mathcal{O}_{j\mu\nu}(x) - \langle \mathcal{O}_{j\mu\nu}(x) \rangle \mathbb{1} \right] + O(t).$$

- If all the composite operators in this relation are made out from bare quantities,

$$\left(\mu \frac{\partial}{\partial \mu} \right)_0 \zeta_{ij}(t) = 0,$$

  and $\zeta_{ij}(t)$ are indep. of the renormalization scale μ, when expressed in terms of running parameters. We can set $\mu = c/\sqrt{t}$, and

$$\zeta_{ij}(t)\,[g, m; \mu] = \zeta_{ij}(t) \left[g(c/\sqrt{t}), m(c/\sqrt{t}); c/\sqrt{t} \right].$$

- For $t \to 0$, $g(c/\sqrt{8t}) \to 0$ because of the asymptotic freedom; use of perturbation theory is thus justified!

Ringed fermion fields

- Recall that the flowed fermion field requires the wave function renormalization:

$$\chi_R(t,x) = Z_\chi^{1/2}\chi(t,x), \qquad \bar{\chi}_R(t,x) = Z_\chi^{1/2}\bar{\chi}(t,x),$$

although composite operators of $\chi_R(t,x)$ are UV finite.

Ringed fermion fields

- Recall that the flowed fermion field requires the wave function renormalization:

$$\chi_R(t,x) = Z_\chi^{1/2}\chi(t,x), \qquad \bar{\chi}_R(t,x) = Z_\chi^{1/2}\bar{\chi}(t,x),$$

  although composite operators of $\chi_R(t,x)$ are UV finite.
- To avoid the complication associated with this, we introduce

$$\mathring{\chi}(t,x) = \mathcal{C}\frac{\chi(t,x)}{\sqrt{t^2\left\langle\bar{\chi}(t,x)\overleftrightarrow{\not{D}}\chi(t,x)\right\rangle}} = \chi_R(t,x) + O(g^2),$$

  where

$$\mathcal{C} \equiv \sqrt{\frac{-2\dim(R)}{(4\pi)^2}},$$

  and similarly for $\bar{\chi}(t,x)$.

Ringed fermion fields

- Recall that the flowed fermion field requires the wave function renormalization:

$$\chi_R(t,x) = Z_\chi^{1/2}\chi(t,x), \qquad \bar{\chi}_R(t,x) = Z_\chi^{1/2}\bar{\chi}(t,x),$$

  although composite operators of $\chi_R(t,x)$ are UV finite.
- To avoid the complication associated with this, we introduce

$$\mathring{\chi}(t,x) = \mathcal{C}\frac{\chi(t,x)}{\sqrt{t^2\left\langle\bar{\chi}(t,x)\overleftrightarrow{\not{D}}\chi(t,x)\right\rangle}} = \chi_R(t,x) + O(g^2),$$

  where

$$\mathcal{C} \equiv \sqrt{\frac{-2\dim(R)}{(4\pi)^2}},$$

  and similarly for $\bar{\chi}(t,x)$.
- Since Z_χ is canceled out in $\mathring{\chi}(t,x)$, any composite operators of $\mathring{\chi}(t,x)$ and $\mathring{\bar{\chi}}(t,x)$ are UV finite.

EMT from the gradient flow

- We take following composite operators of flowed fields:

$$\tilde{\mathcal{O}}_{1\mu\nu}(t,x) \equiv G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x),$$
$$\tilde{\mathcal{O}}_{2\mu\nu}(t,x) \equiv \delta_{\mu\nu}G^a_{\rho\sigma}(t,x)G^a_{\rho\sigma}(t,x),$$
$$\tilde{\mathcal{O}}_{3\mu\nu}(t,x) \equiv \mathring{\bar{\chi}}(t,x)\left(\gamma_\mu\overleftrightarrow{D}_\nu + \gamma_\nu\overleftrightarrow{D}_\mu\right)\mathring{\chi}(t,x),$$
$$\tilde{\mathcal{O}}_{4\mu\nu}(t,x) \equiv \delta_{\mu\nu}\mathring{\bar{\chi}}(t,x)\overleftrightarrow{\not{D}}\mathring{\chi}(t,x),$$
$$\tilde{\mathcal{O}}_{5\mu\nu}(t,x) \equiv \delta_{\mu\nu}m\mathring{\bar{\chi}}(t,x)\mathring{\chi}(t,x),$$

and then set the small flow-time expansion:

$$\tilde{\mathcal{O}}_{i\mu\nu}(t,x) = \left\langle\tilde{\mathcal{O}}_{i\mu\nu}(t,x)\right\rangle\mathbb{1} + \sum_j \zeta_{ij}(t)\left[\mathcal{O}_{j\mu\nu}(x) - \left\langle\mathcal{O}_{j\mu\nu}(x)\right\rangle\mathbb{1}\right] + O(t).$$

EMT from the gradient flow

- We take following composite operators of flowed fields:

$$\tilde{\mathcal{O}}_{1\mu\nu}(t,x) \equiv G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x),$$
$$\tilde{\mathcal{O}}_{2\mu\nu}(t,x) \equiv \delta_{\mu\nu}G^a_{\rho\sigma}(t,x)G^a_{\rho\sigma}(t,x),$$
$$\tilde{\mathcal{O}}_{3\mu\nu}(t,x) \equiv \mathring{\bar{\chi}}(t,x)\left(\gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu\right)\mathring{\chi}(t,x),$$
$$\tilde{\mathcal{O}}_{4\mu\nu}(t,x) \equiv \delta_{\mu\nu}\mathring{\bar{\chi}}(t,x)\overleftrightarrow{\not{D}}\mathring{\chi}(t,x),$$
$$\tilde{\mathcal{O}}_{5\mu\nu}(t,x) \equiv \delta_{\mu\nu}m\mathring{\bar{\chi}}(t,x)\mathring{\chi}(t,x),$$

  and then set the small flow-time expansion:

$$\tilde{\mathcal{O}}_{i\mu\nu}(t,x) = \left\langle\tilde{\mathcal{O}}_{i\mu\nu}(t,x)\right\rangle \mathbb{1} + \sum_j \zeta_{ij}(t)\left[\mathcal{O}_{j\mu\nu}(x) - \langle\mathcal{O}_{j\mu\nu}(x)\rangle \mathbb{1}\right] + O(t).$$

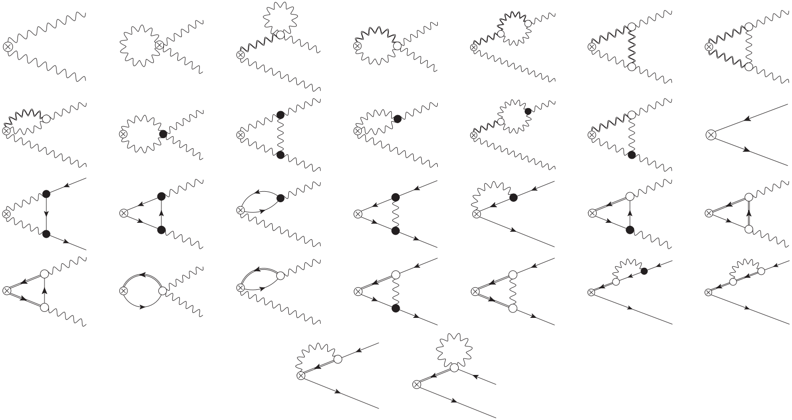
- We compute $\zeta_{ij}(t)$ to the one-loop order and substitute

$$\mathcal{O}_{i\mu\nu}(x) - \langle\mathcal{O}_{i\mu\nu}(x)\rangle \mathbb{1} = \lim_{t\to 0}\left\{\sum_j \left(\zeta^{-1}\right)_{ij}(t)\left[\tilde{\mathcal{O}}_{j\mu\nu}(t,x) - \left\langle\tilde{\mathcal{O}}_{j\mu\nu}(t,x)\right\rangle \mathbb{1}\right]\right\},$$

  in the expression of the EMT in the dimensional regularization

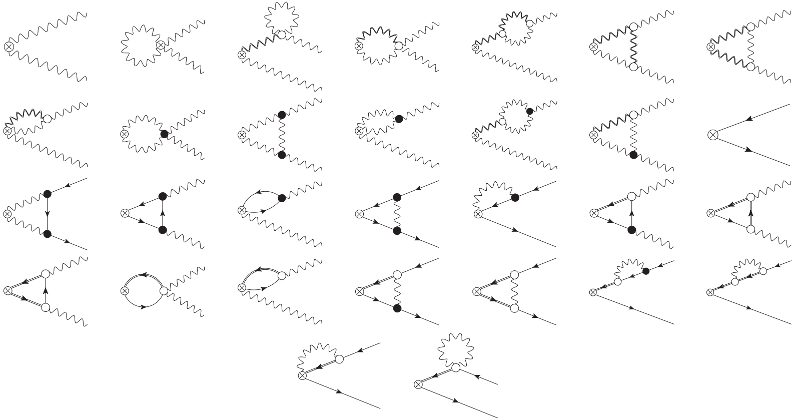
Computation of expansion coefficients $\zeta_{ij}(t)$

- To the one-loop order, we have to evaluate following flow-line Feynman diagrams:



Computation of expansion coefficients $\zeta_{ij}(t)$

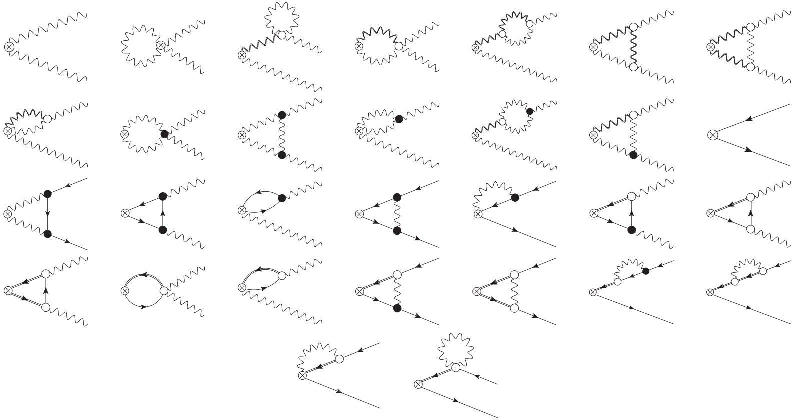
- To the one-loop order, we have to evaluate following flow-line Feynman diagrams:



- Even to write down correct set of diagrams is tedious. . .

Computation of expansion coefficients $\zeta_{ij}(t)$

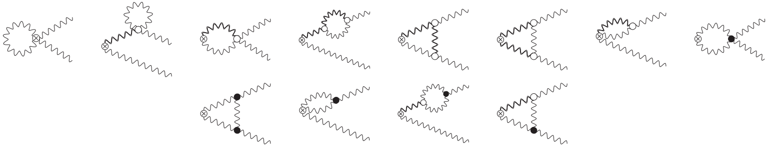
- To the one-loop order, we have to evaluate following flow-line Feynman diagrams:



- Even to write down correct set of diagrams is tedious…
- … and it is very easy to make mistakes in the loop calculation, as I actually did!

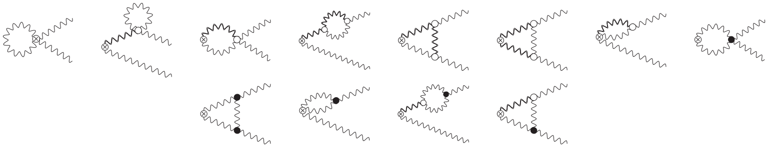
A very simple and quick computational scheme (arXiv:1507.02360)

- Let us consider the pure Yang–Mills theory for which the conventional approach would require



A very simple and quick computational scheme (arXiv:1507.02360)

- Let us consider the pure Yang–Mills theory for which the conventional approach would require



- The small flow-time expansion reads

$$
\begin{aligned}
&G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x)\\
&\overset{t\to 0}{\sim} \left\langle G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x)\right\rangle \mathbb{1}\\
&\qquad + \zeta_{11}(t)F^a_{\mu\rho}(x)F^a_{\nu\rho}(x) + \zeta_{12}(t)\delta_{\mu\nu}F^a_{\rho\sigma}(x)F^a_{\rho\sigma}(x) + O(t).
\end{aligned}
$$

A very simple and quick computational scheme

- The EMT can then be given by

$$T_{\mu\nu} = \lim_{t\to 0}\left\{ c_1(t)\left[G^a_{\mu\rho}G^a_{\nu\rho} - \frac{1}{4}\delta_{\mu\nu}G^a_{\rho\sigma}G^a_{\rho\sigma}\right] + c_2(t)\left[\delta_{\mu\nu}G^a_{\rho\sigma}G^a_{\rho\sigma} - \left\langle\delta_{\mu\nu}G^a_{\rho\sigma}G^a_{\rho\sigma}\right\rangle \mathbb{1}\right]\right\},$$

where

$$c_1(t) = \lim_{\varepsilon\to 0}\frac{1}{g_0^2}\left[2 - \zeta_{11}(t)\right], \qquad c_2(t) = \lim_{\varepsilon\to 0}\frac{1}{g_0^2}\left[-\frac{1}{2}\varepsilon\zeta_{12}(t)\right].$$

Background field method in the gradient flow

- The "gauge fixing term" (Lüscher)

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x) + \alpha_0 D_\mu \partial_\nu B_\nu(t,x),$$

leads to

$$\langle B_\mu B_\nu \rangle_0 \sim \frac{1}{(p^2)^2}\left[(\delta_{\mu\nu}p^2 - p_\mu p_\nu)e^{-(t+s)p^2} + \frac{1}{\lambda_0}p_\mu p_\nu e^{-\alpha_0(t+s)p^2}\right].$$

Background field method in the gradient flow

- The "gauge fixing term" (Lüscher)

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x) + \alpha_0 D_\mu \partial_\nu B_\nu(t,x),$$

leads to

$$\langle B_\mu B_\nu \rangle_0 \sim \frac{1}{(p^2)^2}\left[(\delta_{\mu\nu}p^2 - p_\mu p_\nu)e^{-(t+s)p^2} + \frac{1}{\lambda_0} p_\mu p_\nu e^{-\alpha_0(t+s)p^2}\right].$$

- Here, we adopt instead

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x) + \alpha_0 D_\mu \hat{D}_\nu b_\nu(t,x),$$

where we have set

$$B_\mu(t,x) = \underbrace{\hat{B}_\mu(t,x)}_{\text{background}} + \underbrace{b_\mu(t,x)}_{\text{quantum}}$$

and

$$\hat{D}_\mu = \partial_\mu + [\underbrace{\hat{B}_\mu(t,x)}_{\text{background}}, \cdot\,]$$

is the covariant derivative wrt the background field.

Background field method in the gradient flow

- Our "gauge fixing term" breaks the covariance under the full gauge transformation, but preserves covariance under the background gauge transformation:

$$\hat{B}_\mu \to \hat{B}_\mu + \hat{D}_\mu \omega(x), \qquad b_\mu \to b_\mu + [b_\mu, \omega(x)].$$

Background field method in the gradient flow

- Our "gauge fixing term" breaks the covariance under the full gauge transformation, but preserves covariance under the background gauge transformation:

$$\hat{B}_\mu \to \hat{B}_\mu + \hat{D}_\mu \omega(x), \qquad b_\mu \to b_\mu + [b_\mu, \omega(x)].$$

- We also postulate that the background field obeys its own flow equation:

$$\partial_t \hat{B}_\mu(t,x) = \hat{D}_\nu \hat{G}_{\nu\mu}(t,x), \qquad \hat{B}_\mu(0,x) = \hat{A}_\mu(x),$$

  and assume furthermore that

$$\hat{D}_\nu \hat{F}_{\nu\mu}(x) = 0,$$

  for simplicity.

Background field method in the gradient flow

- Our "gauge fixing term" breaks the covariance under the full gauge transformation, but preserves covariance under the background gauge transformation:

$$\hat{B}_\mu \to \hat{B}_\mu + \hat{D}_\mu \omega(x), \qquad b_\mu \to b_\mu + [b_\mu, \omega(x)].$$

- We also postulate that the background field obeys its own flow equation:

$$\partial_t \hat{B}_\mu(t,x) = \hat{D}_\nu \hat{G}_{\nu\mu}(t,x), \qquad \hat{B}_\mu(0,x) = \hat{A}_\mu(x),$$

  and assume furthermore that

$$\hat{D}_\nu \hat{F}_{\nu\mu}(x) = 0,$$

  for simplicity.

- Then the background field does not flow:

$$\hat{B}_\mu(t,x) = \hat{A}_\mu(x).$$

The background field method ('t Hooft, DeWitt, Boulware, Abbott, Omote–Ichinose, . . .)

- Background–quantum splitting

$$A_\mu(x) = \underbrace{\hat{A}_\mu(x)}_{\text{background}} + \underbrace{a_\mu(x)}_{\text{quantum}},$$

The background field method ('t Hooft, DeWitt, Boulware, Abbott, Omote–Ichinose, . . .)

- Background–quantum splitting

$$A_\mu(x) = \underbrace{\hat{A}_\mu(x)}_{\text{background}} + \underbrace{a_\mu(x)}_{\text{quantum}},$$

- Background gauge fixing term

$$S_{\text{gf}} = -\frac{\lambda_0}{g_0^2} \int d^D x \, \text{tr} \left[\hat{D}_\mu a_\mu(x) \hat{D}_\nu a_\nu(x) \right],$$

and the corresponding Faddeev–Popov ghost action

$$S_{c\bar{c}} = \frac{2}{g_0^2} \int d^D x \, \text{tr} \left[\bar{c}(x) \hat{D}_\mu D_\mu c(x) \right],$$

preserve invariance under the background gauge transformation

$$\hat{A}_\mu(x) \to \hat{A}_\mu(x) + \hat{D}_\mu \omega(x), \qquad a_\mu(x) \to a_\mu(x) + [a_\mu(x), \omega(x)].$$

The background field method ('t Hooft, DeWitt, Boulware, Abbott, Omote–Ichinose, . . .)

- Background–quantum splitting

$$A_\mu(x) = \underbrace{\hat{A}_\mu(x)}_{\text{background}} + \underbrace{a_\mu(x)}_{\text{quantum}},$$

- Background gauge fixing term

$$S_{\text{gf}} = -\frac{\lambda_0}{g_0^2} \int d^D x \, \operatorname{tr}\left[\hat{D}_\mu a_\mu(x) \hat{D}_\nu a_\nu(x)\right],$$

  and the corresponding Faddeev–Popov ghost action

$$S_{c\bar{c}} = \frac{2}{g_0^2} \int d^D x \, \operatorname{tr}\left[\bar{c}(x) \hat{D}_\mu D_\mu c(x)\right],$$

  preserve invariance under the background gauge transformation

$$\hat{A}_\mu(x) \to \hat{A}_\mu(x) + \hat{D}_\mu \omega(x), \qquad a_\mu(x) \to a_\mu(x) + [a_\mu(x), \omega(x)].$$

- This greatly simplifies perturbative calculations (of renormalization constants, for example).

With our gauge fixing term in the flow equation

- Any gauge invariant quantity (that does not contain ∂_t) is independent of α_0

With our gauge fixing term in the flow equation

- Any gauge invariant quantity (that does not contain ∂_t) is independent of α_0
- Manifestly background gauge covariant expressions

With our gauge fixing term in the flow equation

- Any gauge invariant quantity (that does not contain ∂_t) is independent of α_0
- Manifestly background gauge covariant expressions
- Tree-level propagator in the presence of the background field (in the "Feynman gauge" $\lambda_0 = \alpha_0 = 1$) is

$$\left\langle b^a_\mu(t,x) b^b_\nu(s,y) \right\rangle_{\hat{A},0} = g_0^2 \left(e^{(t+s)\hat{\Delta}_x} \frac{1}{-\hat{\Delta}_x} \right)^{ab}_{\mu\nu} \delta(x-y),$$

where

$$\hat{\Delta}^{ab}_{\mu\nu} = (\hat{\mathcal{D}}^2)^{ab} \delta_{\mu\nu} + 2\hat{\mathcal{F}}^{ab}_{\mu\nu}, \qquad (\hat{\mathcal{D}}^2)^{ab} = \hat{\mathcal{D}}^{ac}_\mu \hat{\mathcal{D}}^{cb}_\mu,$$

and

$$\hat{\mathcal{D}}^{ab}_\mu \equiv \delta^{ab}\partial_\mu + \hat{A}^c_\mu(x) f^{acb}, \qquad \hat{\mathcal{F}}^{ab}_{\mu\nu}(x) \equiv \hat{F}^c_{\mu\nu}(x) f^{acb}.$$

Small flow time expansion by the background field method

- In the small flow time expansion,

$$G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x)$$
$$\stackrel{t\to0}{\sim} \text{VEV}\times\mathbb{1}+\zeta_{11}(t)F^a_{\mu\rho}(x)F^a_{\nu\rho}(x)+\zeta_{12}(t)\delta_{\mu\nu}F^a_{\rho\sigma}(x)F^a_{\rho\sigma}(x)+O(t),$$

we substitute the background-quantum decomposition as

$$G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x)$$
$$=\hat{F}^a_{\mu\rho}(x)\hat{F}^a_{\nu\rho}(x)+\left.G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x)\right|_{O(b^1)}+\left.G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x)\right|_{O(b^2)}+O(b^3$$

and

$$F^a_{\mu\rho}(x)F^a_{\nu\rho}(x)$$
$$=\hat{F}^a_{\mu\rho}(x)\hat{F}^a_{\nu\rho}(x)+\left.F^a_{\mu\rho}(x)F^a_{\nu\rho}(x)\right|_{O(a^1)}+\left.F^a_{\mu\rho}(x)F^a_{\nu\rho}(x)\right|_{O(a^2)}+O(a^3),$$

Small flow time expansion by the background field method

- In the small flow time expansion,

$$G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x)$$
$$\stackrel{t\to 0}{\sim} \text{VEV} \times \mathbb{1} + \zeta_{11}(t)F^a_{\mu\rho}(x)F^a_{\nu\rho}(x) + \zeta_{12}(t)\delta_{\mu\nu}F^a_{\rho\sigma}(x)F^a_{\rho\sigma}(x) + O(t),$$

we substitute the background-quantum decomposition as

$$G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x)$$
$$= \hat{F}^a_{\mu\rho}(x)\hat{F}^a_{\nu\rho}(x) + \left. G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x)\right|_{O(b^1)} + \left. G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x)\right|_{O(b^2)} + O(b^3$$

and

$$F^a_{\mu\rho}(x)F^a_{\nu\rho}(x)$$
$$= \hat{F}^a_{\mu\rho}(x)\hat{F}^a_{\nu\rho}(x) + \left. F^a_{\mu\rho}(x)F^a_{\nu\rho}(x)\right|_{O(a^1)} + \left. F^a_{\mu\rho}(x)F^a_{\nu\rho}(x)\right|_{O(a^2)} + O(a^3),$$

- Then, we take the expectation value in the presence of the background field noting $\langle \mathbb{1} \rangle_{\hat{A}} = 1$.

Small flow time expansion by the background field method

- Then we have

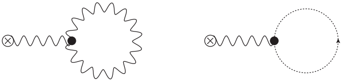
$$\left\langle G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x)\Big|_{O(b^1)} - F^a_{\mu\rho}(x)F^a_{\nu\rho}(x)\Big|_{O(a^1)}\right\rangle_{\hat{A}}$$
$$+\left\langle G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x)\Big|_{O(b^2)} - F^a_{\mu\rho}(x)F^a_{\nu\rho}(x)\Big|_{O(a^2)}\right\rangle_{\hat{A}}$$
$$\overset{t\to 0}{\sim} \text{VEV} \times 1 + [\zeta_{11}(t) - 1]\,\hat{F}^a_{\mu\rho}(x)\hat{F}^a_{\nu\rho}(x) + \zeta_{12}(t)\delta_{\mu\nu}\hat{F}^a_{\rho\sigma}(x)\hat{F}^a_{\rho\sigma}(x) + O(t)$$
$$+ \text{(2-loop quantities)},$$

where the first line reads

$$\left\langle G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x)\Big|_{O(b^1)} - F^a_{\mu\rho}(x)F^a_{\nu\rho}(x)\Big|_{O(a^1)}\right\rangle_{\hat{A}}$$
$$= \hat{F}^a_{\mu\rho}\left[\hat{D}_\nu\left\langle b_\rho(t,x) - a_\rho(x)\right\rangle_{\hat{A}} - \hat{D}_\rho\left\langle b_\nu(t,x) - a_\nu(x)\right\rangle_{\hat{A}}\right]^a$$
$$+\left[\hat{D}_\mu\left\langle b_\rho(t,x) - a_\rho(x)\right\rangle_{\hat{A}} - \hat{D}_\rho\left\langle b_\mu(t,x) - a_\mu(x)\right\rangle_{\hat{A}}\right]^a\hat{F}^a_{\nu\rho}.$$

Tadpoles

- First type (Yang–Mills vertex only):

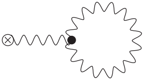


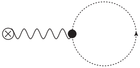
These identically vanish (!) under the dimensional regularization for which

$$\int_p \frac{1}{(p^2)^\alpha} = 0.$$

Tadpoles

- First type (Yang–Mills vertex only):

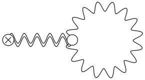




These identically vanish (!) under the dimensional regularization for which

$$\int_p \frac{1}{(p^2)^\alpha} = 0.$$

- Second type (flow vertex):



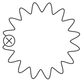
By the background gauge covariance,

$$\left\langle b^a_\mu(t,x)\right\rangle_{\hat{A}} \sim t\hat{D}_\nu \hat{F}^a_{\nu\mu}(x) + O(t^2).$$

This is higher order in t and does not contribute. . .

“1PI” diagram

- The remaining 1-loop diagram is:



Oh! There is only a single diagram!!!

$$
\left\langle G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x)\Big|_{O(b^2)} - F^a_{\mu\rho}(x)F^a_{\nu\rho}(x)\Big|_{O(a^2)}\right\rangle_{\hat{A}}
$$

$$
= 2g_0^2\int_0^t d\xi\,\lim_{y\to x}\operatorname{tr}\Bigg[\mathcal{P}_{\mu\alpha,\nu\delta,\beta\gamma}\hat{\mathcal{D}}_\alpha\left(e^{2\xi\hat{\Delta}}\right)_{\beta\gamma}\hat{\mathcal{D}}_\delta
$$

$$
+\hat{\mathcal{F}}_{\mu\rho}(x)\left(e^{2\xi\hat{\Delta}}\right)_{\rho\nu}+\hat{\mathcal{F}}_{\nu\rho}(x)\left(e^{2\xi\hat{\Delta}}\right)_{\rho\mu}\Bigg]\delta(x-y),
$$

where $\mathcal{P}_{\mu\alpha,\nu\delta,\beta\gamma}\equiv\delta_{\mu\alpha}\delta_{\nu\delta}\delta_{\beta\gamma}-\delta_{\mu\alpha}\delta_{\nu\gamma}\delta_{\beta\delta}-\delta_{\mu\beta}\delta_{\nu\delta}\delta_{\alpha\gamma}+\delta_{\mu\beta}\delta_{\nu\gamma}\delta_{\alpha\delta}$.

- The remaining 1-loop diagram is:

$$
\begin{aligned}
&\left\langle G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x)\Big|_{O(b^2)} - F^a_{\mu\rho}(x)F^a_{\nu\rho}(x)\Big|_{O(a^2)}\right\rangle_{\hat{A}} \\
&= 2g_0^2\int_0^t d\xi\,\lim_{y\to x}\operatorname{tr}\Bigg[\mathcal{P}_{\mu\alpha,\nu\delta,\beta\gamma}\hat{\mathcal{D}}_\alpha\left(e^{2\xi\hat{\Delta}}\right)_{\beta\gamma}\hat{\mathcal{D}}_\delta \\
&\qquad\qquad + \hat{\mathcal{F}}_{\mu\rho}(x)\left(e^{2\xi\hat{\Delta}}\right)_{\rho\nu} + \hat{\mathcal{F}}_{\nu\rho}(x)\left(e^{2\xi\hat{\Delta}}\right)_{\rho\mu}\Bigg]\delta(x-y).
\end{aligned}
$$

“1PI” diagram

- The remaining 1-loop diagram is:

$$\left\langle G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x)\big|_{O(b^2)} - F^a_{\mu\rho}(x)F^a_{\nu\rho}(x)\big|_{O(a^2)}\right\rangle_{\hat{A}}$$
$$= 2g_0^2\int_0^t d\xi\,\lim_{y\to x}\operatorname{tr}\Big[\mathcal{P}_{\mu\alpha,\nu\delta,\beta\gamma}\hat{\mathcal{D}}_\alpha\left(e^{2\xi\hat{\Delta}}\right)_{\beta\gamma}\hat{\mathcal{D}}_\delta$$
$$+\hat{\mathcal{F}}_{\mu\rho}(x)\left(e^{2\xi\hat{\Delta}}\right)_{\rho\nu}+\hat{\mathcal{F}}_{\nu\rho}(x)\left(e^{2\xi\hat{\Delta}}\right)_{\rho\mu}\Big]\delta(x-y).$$

- This is evaluated by setting

$$\delta(x-y)=\int_p e^{ipx}e^{-ipy}$$

and noting

$$\hat{\mathcal{D}}_\mu e^{ipx}=e^{ipx}(ip_\mu+\hat{\mathcal{D}}_\mu).$$

- Then

$$
\begin{aligned}
&= 2g_0^2 \int_0^t d\xi\, \xi^{-D/2} \int_p e^{-2p^2} \\
&\quad \times \mathrm{tr}\Big[\xi^{-1}\mathcal{P}_{\mu\alpha,\nu\delta,\beta\gamma}\left(ip_\alpha + \sqrt{\xi}\hat{\mathcal{D}}_\alpha\right)\left(e^{4i\sqrt{\xi}p\cdot\hat{\mathcal{D}}+2\xi\hat{\Delta}}\right)_{\beta\gamma}\left(ip_\delta + \sqrt{\xi}\hat{\mathcal{D}}_\delta\right) \\
&\qquad + \hat{\mathcal{F}}_{\mu\rho}(x)\left(e^{4i\sqrt{\xi}p\cdot\hat{\mathcal{D}}+2\xi\hat{\Delta}}\right)_{\rho\nu} + \hat{\mathcal{F}}_{\nu\rho}(x)\left(e^{4i\sqrt{\xi}p\cdot\hat{\mathcal{D}}+2\xi\hat{\Delta}}\right)_{\rho\mu}\Big].
\end{aligned}
$$

“1PI” diagram

- Then

$$
\begin{aligned}
&= 2g_0^2 \int_0^t d\xi\, \xi^{-D/2} \int_p e^{-2p^2} \\
&\quad \times \operatorname{tr}\Biggl[\xi^{-1} \mathcal{P}_{\mu\alpha,\nu\delta,\beta\gamma} \left(ip_\alpha + \sqrt{\xi}\hat{\mathcal{D}}_\alpha \right) \left(e^{4i\sqrt{\xi}p\cdot\hat{\mathcal{D}}+2\xi\hat{\Delta}} \right)_{\beta\gamma} \left(ip_\delta + \sqrt{\xi}\hat{\mathcal{D}}_\delta \right) \\
&\qquad + \hat{\mathcal{F}}_{\mu\rho}(x) \left(e^{4i\sqrt{\xi}p\cdot\hat{\mathcal{D}}+2\xi\hat{\Delta}} \right)_{\rho\nu} + \hat{\mathcal{F}}_{\nu\rho}(x) \left(e^{4i\sqrt{\xi}p\cdot\hat{\mathcal{D}}+2\xi\hat{\Delta}} \right)_{\rho\mu} \Biggr].
\end{aligned}
$$

- The second term is

$$
\begin{aligned}
&2g_0^2 \int_0^t d\xi\, \xi^{-D/2} \int_p e^{-2p^2} \operatorname{tr}\left[\hat{\mathcal{F}}_{\mu\rho}(x) \left(e^{4i\sqrt{\xi}p\cdot\hat{\mathcal{D}}+2\xi\hat{\Delta}} \right)_{\rho\nu} \right] \\
&= 8g_0^2 \int_0^t d\xi\, \xi^{-D/2+1} \int_p e^{-2p^2} \operatorname{tr}\left[\hat{\mathcal{F}}_{\mu\rho}(x) \hat{\mathcal{F}}_{\rho\nu}(x) \right] + O(t^{3-D/2}) \\
&= \frac{g_0^2}{(4\pi)^2} \frac{(8\pi t)^\varepsilon}{\varepsilon} 2 \operatorname{tr}\left[\hat{\mathcal{F}}(x)^2 \right]_{\mu\nu} + O(t^{1+\varepsilon}).
\end{aligned}
$$

The third term gives rise to the same contribution.

"1PI" diagram

- The first term (without $\mathcal{P}_{\mu\alpha,\nu\delta,\beta\gamma}$) is,

$$
\begin{aligned}
&2g_0^2 \int_0^t d\xi\, \xi^{-D/2-1} \int_p e^{-2p^2} \\
&\qquad \times \operatorname{tr}\left[\left(ip_\alpha + \sqrt{\xi}\hat{\mathcal{D}}_\alpha\right)\left(e^{4i\sqrt{\xi}p\cdot\hat{\mathcal{D}}+2\xi\hat{\Delta}}\right)_{\beta\gamma}\left(ip_\delta + \sqrt{\xi}\hat{\mathcal{D}}_\delta\right)\right].
\end{aligned}
$$

"1PI" diagram

- The first term (without $\mathcal{P}_{\mu\alpha,\nu\delta,\beta\gamma}$) is,

$$2g_0^2\int_0^t d\xi\,\xi^{-D/2-1}\int_p e^{-2p^2}$$
$$\times\operatorname{tr}\left[\left(ip_\alpha+\sqrt{\xi}\hat{\mathcal{D}}_\alpha\right)\left(e^{4i\sqrt{\xi}p\cdot\hat{\mathcal{D}}+2\xi\hat{\Delta}}\right)_{\beta\gamma}\left(ip_\delta+\sqrt{\xi}\hat{\mathcal{D}}_\delta\right)\right].$$

- After the expansion wrt $\sqrt{\xi}$ and integrations,

$$\frac{g_0^2}{(4\pi)^2}\frac{(8\pi t)^\varepsilon}{1-\varepsilon/2}\frac{1}{16t^2}\delta_{\alpha\delta}\delta_{\beta\gamma}\dim G$$
$$+\frac{g_0^2}{(4\pi)^2}\frac{(8\pi t)^\varepsilon}{\varepsilon}\operatorname{tr}\Bigl\{-\delta_{\alpha\delta}\hat{\mathcal{F}}(x)^2_{\beta\gamma}-[\hat{\mathcal{D}}_\alpha,\hat{\mathcal{D}}_\delta]\hat{\mathcal{F}}(x)_{\beta\gamma}$$
$$+\frac{1}{12}\delta_{\beta\gamma}\Bigl[\hat{\mathcal{D}}_\alpha\hat{\mathcal{D}}_\varepsilon\hat{\mathcal{D}}_\delta\hat{\mathcal{D}}_\varepsilon+\hat{\mathcal{D}}_\delta\hat{\mathcal{D}}_\varepsilon\hat{\mathcal{D}}_\alpha\hat{\mathcal{D}}_\varepsilon-\hat{\mathcal{D}}_\alpha\hat{\mathcal{D}}^2\hat{\mathcal{D}}_\delta-\hat{\mathcal{D}}_\delta\hat{\mathcal{D}}^2\hat{\mathcal{D}}_\alpha$$
$$-\hat{\mathcal{D}}_\varepsilon\hat{\mathcal{D}}_\alpha\hat{\mathcal{D}}_\delta\hat{\mathcal{D}}_\varepsilon-\hat{\mathcal{D}}_\varepsilon\hat{\mathcal{D}}_\delta\hat{\mathcal{D}}_\alpha\hat{\mathcal{D}}_\varepsilon+\hat{\mathcal{D}}_\varepsilon\hat{\mathcal{D}}_\alpha\hat{\mathcal{D}}_\varepsilon\hat{\mathcal{D}}_\delta+\hat{\mathcal{D}}_\varepsilon\hat{\mathcal{D}}_\delta\hat{\mathcal{D}}_\varepsilon\hat{\mathcal{D}}_\alpha$$
$$-\delta_{\alpha\delta}\hat{\mathcal{D}}_\varepsilon\hat{\mathcal{D}}_\varphi\hat{\mathcal{D}}_\varepsilon\hat{\mathcal{D}}_\varphi+\delta_{\alpha\delta}\hat{\mathcal{D}}_\varepsilon\hat{\mathcal{D}}^2\hat{\mathcal{D}}_\varepsilon\Bigr]\Bigr\}+O(t^{1+\varepsilon}).$$

“1PI” diagram

- We then use

$$[\hat{\mathcal{D}}_\mu, \hat{\mathcal{D}}_\nu] = \hat{\mathcal{F}}_{\mu\nu},$$

to yield

$$\begin{aligned}
&\frac{g_0^2}{(4\pi)^2}\frac{(8\pi t)^\varepsilon}{1-\varepsilon/2}\frac{1}{16t^2}\delta_{\alpha\delta}\delta_{\beta\gamma}\dim G \\
&\quad + \frac{g_0^2}{(4\pi)^2}\frac{(8\pi t)^\varepsilon}{\varepsilon}\operatorname{tr}\Big[-\delta_{\alpha\delta}\hat{\mathcal{F}}(x)^2_{\beta\gamma} - \hat{\mathcal{F}}(x)_{\alpha\delta}\hat{\mathcal{F}}(x)_{\beta\gamma} \\
&\qquad\qquad - \frac{1}{6}\delta_{\beta\gamma}\hat{\mathcal{F}}(x)^2_{\alpha\delta} + \frac{1}{24}\delta_{\alpha\delta}\delta_{\beta\gamma}\hat{\mathcal{F}}(x)^2_{\rho\rho}\Big] + O(t^{1+\varepsilon}),
\end{aligned}$$

"1PI" diagram

- Finally, taking the contraction with $\mathcal{P}_{\mu\alpha,\nu\delta,\beta\gamma}$,

$$\begin{aligned}&\left\langle G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x)\Big|_{O(b^2)} - F^a_{\mu\rho}(x)F^a_{\nu\rho}(x)\Big|_{O(a^2)}\right\rangle_{\hat{A}} \\ &= \frac{g_0^2}{(4\pi)^2}\,\dim(G)\frac{3}{8t^2}\delta_{\mu\nu} \\ &\qquad + \frac{g_0^2}{(4\pi)^2}\left[\frac{11}{3}\varepsilon(t)^{-1}+\frac{7}{3}\right]C_A\hat{F}^a_{\mu\rho}(x)\hat{F}^a_{\nu\rho}(x) \\ &\qquad\qquad + \frac{g_0^2}{(4\pi)^2}\left[-\frac{11}{12}\varepsilon(t)^{-1}-\frac{1}{6}\right]C_A\delta_{\mu\nu}\hat{F}^a_{\rho\sigma}(x)\hat{F}^a_{\rho\sigma}(x) \\ &\qquad\qquad\qquad + O(t),\end{aligned}$$

where

$$\varepsilon(t)^{-1} \equiv \frac{1}{\varepsilon} + \ln(8\pi t).$$

"1PI" diagram

- Finally, taking the contraction with $\mathcal{P}_{\mu\alpha,\nu\delta,\beta\gamma}$,

$$
\begin{aligned}
&\left\langle G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x)\Big|_{O(b^2)} - F^a_{\mu\rho}(x)F^a_{\nu\rho}(x)\Big|_{O(a^2)}\right\rangle_{\hat{A}} \\
&= \frac{g_0^2}{(4\pi)^2}\,\mathrm{dim}(G)\frac{3}{8t^2}\delta_{\mu\nu} \leftarrow \text{VEV} \\
&\qquad + \frac{g_0^2}{(4\pi)^2}\left[\frac{11}{3}\varepsilon(t)^{-1} + \frac{7}{3}\right] C_A\hat{F}^a_{\mu\rho}(x)\hat{F}^a_{\nu\rho}(x) \\
&\qquad\qquad + \frac{g_0^2}{(4\pi)^2}\left[-\frac{11}{12}\varepsilon(t)^{-1} - \frac{1}{6}\right] C_A\delta_{\mu\nu}\hat{F}^a_{\rho\sigma}(x)\hat{F}^a_{\rho\sigma}(x) \\
&\qquad\qquad\quad + O(t),
\end{aligned}
$$

where

$$
\varepsilon(t)^{-1} \equiv \frac{1}{\varepsilon} + \ln(8\pi t).
$$

“1PI” diagram

- Finally, taking the contraction with $\mathcal{P}_{\mu\alpha,\nu\delta,\beta\gamma}$,

$$
\begin{aligned}
&\left\langle G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x)\Big|_{O(b^2)} - F^a_{\mu\rho}(x)F^a_{\nu\rho}(x)\Big|_{O(a^2)}\right\rangle_{\hat{A}} \\
&= \frac{g_0^2}{(4\pi)^2}\dim(G)\frac{3}{8t^2}\delta_{\mu\nu} \leftarrow \text{VEV} \\
&\qquad + \frac{g_0^2}{(4\pi)^2}\left[\frac{11}{3}\varepsilon(t)^{-1} + \frac{7}{3}\right] C_A\hat{F}^a_{\mu\rho}(x)\hat{F}^a_{\nu\rho}(x) \leftarrow [\zeta_{11}(t) - 1] \\
&\qquad\qquad + \frac{g_0^2}{(4\pi)^2}\left[-\frac{11}{12}\varepsilon(t)^{-1} - \frac{1}{6}\right] C_A\delta_{\mu\nu}\hat{F}^a_{\rho\sigma}(x)\hat{F}^a_{\rho\sigma}(x) \\
&\qquad\qquad\quad + O(t),
\end{aligned}
$$

where

$$
\varepsilon(t)^{-1} \equiv \frac{1}{\varepsilon} + \ln(8\pi t).
$$

- Finally, taking the contraction with $\mathcal{P}_{\mu\alpha,\nu\delta,\beta\gamma}$,

$$
\begin{aligned}
&\left\langle G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x)\Big|_{O(b^2)} - F^a_{\mu\rho}(x)F^a_{\nu\rho}(x)\Big|_{O(a^2)}\right\rangle_{\hat{A}} \\
&= \frac{g_0^2}{(4\pi)^2}\dim(G)\frac{3}{8t^2}\delta_{\mu\nu} \leftarrow \text{VEV} \\
&\quad + \frac{g_0^2}{(4\pi)^2}\left[\frac{11}{3}\varepsilon(t)^{-1} + \frac{7}{3}\right] C_A \hat{F}^a_{\mu\rho}(x)\hat{F}^a_{\nu\rho}(x) \leftarrow [\zeta_{11}(t) - 1] \\
&\qquad + \frac{g_0^2}{(4\pi)^2}\left[-\frac{11}{12}\varepsilon(t)^{-1} - \frac{1}{6}\right] C_A \delta_{\mu\nu}\hat{F}^a_{\rho\sigma}(x)\hat{F}^a_{\rho\sigma}(x) \leftarrow \zeta_{12}(t) \\
&\qquad\quad + O(t),
\end{aligned}
$$

where

$$
\varepsilon(t)^{-1} \equiv \frac{1}{\varepsilon} + \ln(8\pi t).
$$

Small flow time expansion relevant to EMT

- To the one-loop level, we thus have

$$\mathrm{VEV} = \frac{g_0^2}{(4\pi)^2} \dim(G) \frac{3}{8t^2} \delta_{\mu\nu},$$

$$\zeta_{11}(t) = 1 + \frac{g_0^2}{(4\pi)^2} C_A \left[\frac{11}{3} \varepsilon(t)^{-1} + \frac{7}{3} \right],$$

$$\zeta_{12}(t) = \frac{g_0^2}{(4\pi)^2} C_A \left[-\frac{11}{12} \varepsilon(t)^{-1} - \frac{1}{6} \right].$$

Small flow time expansion relevant to EMT

- To the one-loop level, we thus have

$$\begin{aligned}
\text{VEV} &= \frac{g_0^2}{(4\pi)^2}\dim(G)\frac{3}{8t^2}\delta_{\mu\nu},\\
\zeta_{11}(t) &= 1 + \frac{g_0^2}{(4\pi)^2}C_A\left[\frac{11}{3}\varepsilon(t)^{-1} + \frac{7}{3}\right],\\
\zeta_{12}(t) &= \frac{g_0^2}{(4\pi)^2}C_A\left[-\frac{11}{12}\varepsilon(t)^{-1} - \frac{1}{6}\right].
\end{aligned}$$

- Plugging these into

$$c_1(t) = \lim_{\varepsilon\to 0}\frac{1}{g_0^2}\left[2 - \zeta_{11}(t)\right], \qquad c_2(t) = \lim_{\varepsilon\to 0}\frac{1}{g_0^2}\left[-\frac{1}{2}\varepsilon\zeta_{12}(t)\right],$$

and making the renormalization in the $\overline{\text{MS}}$ scheme ($\mu \propto 1/\sqrt{t}$),

$$c_1(t) = \frac{1}{g(\mu)^2}\left\{1 + \frac{g(\mu)^2}{(4\pi)^2}\left[-\beta_0 L(\mu,t) - \frac{7}{3}C_A\right]\right\}, \qquad c_2(t) = \frac{1}{(4\pi)^2}\frac{\beta_0}{8},$$

$$\beta_0 = \frac{11}{3}C_A, \qquad L(\mu,t) \equiv \ln(2\mu^2 t) + \gamma_E.$$

Universal formula for EMT

- For the system containing fermions (with Makino, arXiv:1403.4772),

$$
\begin{aligned}
T_{\mu\nu}(x) = \lim_{t\to 0}\Big\{ & c_1(t)\left[\tilde{\mathcal{O}}_{1,\mu\nu}(t,x) - \frac{1}{4}\tilde{\mathcal{O}}_{2,\mu\nu}(t,x)\right] + c_2(t)\tilde{\mathcal{O}}_{2,\mu\nu}(t,x) \\
& + c_3(t)\left[\tilde{\mathcal{O}}_{3,\mu\nu}(t,x) - 2\tilde{\mathcal{O}}_{4,\mu\nu}(t,x)\right] \\
& + c_4(t)\tilde{\mathcal{O}}_{4,\mu\nu}(t,x) + c_5(t)\tilde{\mathcal{O}}_{5,\mu\nu}(t,x) - \mathrm{VEV}\Big\},
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{\mathcal{O}}_{1\mu\nu}(t,x) &\equiv G^a_{\mu\rho}(t,x)G^a_{\nu\rho}(t,x), \\
\tilde{\mathcal{O}}_{2\mu\nu}(t,x) &\equiv \delta_{\mu\nu}G^a_{\rho\sigma}(t,x)G^a_{\rho\sigma}(t,x), \\
\tilde{\mathcal{O}}_{3\mu\nu}(t,x) &\equiv \mathring{\bar{\chi}}(t,x)\left(\gamma_\mu\overleftrightarrow{D}_\nu + \gamma_\nu\overleftrightarrow{D}_\mu\right)\mathring{\chi}(t,x), \\
\tilde{\mathcal{O}}_{4\mu\nu}(t,x) &\equiv \delta_{\mu\nu}\mathring{\bar{\chi}}(t,x)\overleftrightarrow{\not{D}}\mathring{\chi}(t,x), \\
\tilde{\mathcal{O}}_{5\mu\nu}(t,x) &\equiv \delta_{\mu\nu}m\mathring{\bar{\chi}}(t,x)\mathring{\chi}(t,x),
\end{aligned}
$$

Universal formula for EMT

- and

$$c_{1,2}(t) = \frac{1}{g(\mu)^2} \sum_{\ell=0}^{\infty} k_{1,2}^{(\ell)} \left[\frac{g(\mu)^2}{(4\pi)^2} \right]^{\ell}, \qquad c_{3,4,5}(t) = \sum_{\ell=0}^{\infty} k_{3,4,5}^{(\ell)} \left[\frac{g(\mu)^2}{(4\pi)^2} \right]^{\ell},$$

and to the one-loop order ($T_F = T n_f$)

$$\begin{aligned}
k_1^{(0)} &= 1, & k_1^{(1)} &= -\beta_0 L(\mu, t) - \frac{7}{3} C_A + \frac{3}{2} T_F, \\
k_2^{(0)} &= 0, & k_2^{(1)} &= \frac{1}{4} \left(\frac{11}{6} C_A + \frac{11}{6} T_F \right), \\
k_3^{(0)} &= \frac{1}{4}, & k_3^{(1)} &= \frac{1}{4} \left(\frac{3}{2} + \ln 432 \right) C_F, \\
k_4^{(0)} &= 0, & k_4^{(1)} &= \frac{3}{4} C_F, \\
k_5^{(0)} &= -1, & k_5^{(1)} &= - \left[3L(\mu, t) + \frac{7}{2} + \ln 432 \right] C_F.
\end{aligned}$$

Universal formula for EMT

- The coefficients $c_i(t)$ are universal, i.e., indep. of the regularization.

Universal formula for EMT

- The coefficients $c_i(t)$ are universal, i.e., indep. of the regularization.
- Correlation functions of the RHS of the formula can be computed non-perturbatively by using the lattice regularization.

Universal formula for EMT

- The coefficients $c_i(t)$ are universal, i.e., indep. of the regularization.
- Correlation functions of the RHS of the formula can be computed non-perturbatively by using the lattice regularization.
- “Universality” holds however only when one removes the regulator.

Universal formula for EMT

- The coefficients $c_i(t)$ are universal, i.e., indep. of the regularization.
- Correlation functions of the RHS of the formula can be computed non-perturbatively by using the lattice regularization.
- "Universality" holds however only when one removes the regulator.
- Thus, we have to take first the continuum limit $a \to 0$ and then the small flow time limit $t \to 0$.

Universal formula for EMT

- The coefficients $c_i(t)$ are universal, i.e., indep. of the regularization.
- Correlation functions of the RHS of the formula can be computed non-perturbatively by using the lattice regularization.
- "Universality" holds however only when one removes the regulator.
- Thus, we have to take first the continuum limit $a \to 0$ and then the small flow time limit $t \to 0$.
- Practically, we cannot simply take $a \to 0$ and may take t as small as possible in the fiducial window,

$$a \ll \sqrt{8t} \ll \frac{1}{\Lambda}.$$

The usefulness with presently-accessible lattice parameters is not obvious a priori...

Universal formula for EMT

- The coefficients $c_i(t)$ are universal, i.e., indep. of the regularization.
- Correlation functions of the RHS of the formula can be computed non-perturbatively by using the lattice regularization.
- "Universality" holds however only when one removes the regulator.
- Thus, we have to take first the continuum limit $a \to 0$ and then the small flow time limit $t \to 0$.
- Practically, we cannot simply take $a \to 0$ and may take t as small as possible in the fiducial window,

  $$a \ll \sqrt{8t} \ll \frac{1}{\Lambda}.$$

  The usefulness with presently-accessible lattice parameters is not obvious a priori. . .
- Quite recently, a two-loop computation of $c_i(t)$!!!
  (Harlander–Kluth–Lange, arXiv:1808.09837)

Summary and further prospects

- We observed that the gradient flow possesses a very simple renormalization property.

Summary and further prospects

- We observed that the gradient flow possesses a very simple renormalization property.
- This property, combined with the small flow time expansion, provides a versatile method to define renormalized quantities in gauge theory in a regularization independent way.

Summary and further prospects

- We observed that the gradient flow possesses a very simple renormalization property.
- This property, combined with the small flow time expansion, provides a versatile method to define renormalized quantities in gauge theory in a regularization independent way.
- Here, we presented a construction of a universal formula of the EMT, which may be used in lattice Monte Carlo simulations.

Summary and further prospects

- We observed that the gradient flow possesses a very simple renormalization property.
- This property, combined with the small flow time expansion, provides a versatile method to define renormalized quantities in gauge theory in a regularization independent way.
- Here, we presented a construction of a universal formula of the EMT, which may be used in lattice Monte Carlo simulations.
- Possible obstacle would be

$$a \ll \sqrt{8t}.$$

Summary and further prospects

- We observed that the gradient flow possesses a very simple renormalization property.
- This property, combined with the small flow time expansion, provides a versatile method to define renormalized quantities in gauge theory in a regularization independent way.
- Here, we presented a construction of a universal formula of the EMT, which may be used in lattice Monte Carlo simulations.
- Possible obstacle would be

$$a \ll \sqrt{8t}.$$

- Nevertheless, bulk thermodynamics (one-point function) shows encouraging results; the method appears promising even practically (see below).

Summary and further prospects

- We observed that the gradient flow possesses a very simple renormalization property.
- This property, combined with the small flow time expansion, provides a versatile method to define renormalized quantities in gauge theory in a regularization independent way.
- Here, we presented a construction of a universal formula of the EMT, which may be used in lattice Monte Carlo simulations.
- Possible obstacle would be
$$a \ll \sqrt{8t}.$$
- Nevertheless, bulk thermodynamics (one-point function) shows encouraging results; the method appears promising even practically (see below).
- Further applications to other Noether currents, such as vector/chiral currents and the supercurrent have been considered. . . (Makino–Kasai–Endo–Hieda–Miura–Morikawa–H.S.)