Lattice calculation for muon $g - 2$

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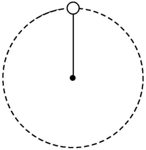
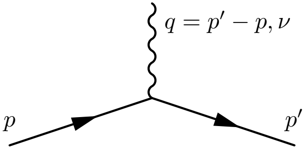
Outline

Spin, magnetic moment, and g factor

Gyromagnetic ratio g factor		$\vec{\mu} = g\frac{e}{2m}\vec{L}$ $\vec{\mu} = \frac{1}{2}\int \vec{r}\times\vec{j}\, d^3\vec{r}$	
Circular motion		$\mu = \frac{1}{2}Rev$ $L = Rmv$ $\mu/L = e/(2m)$	$g = 1$
Charged rotating blackhole	The Kerr-Newman metric		$g = 2$
Charged leptons e, μ, τ	Dirac Equation $(\gamma^\mu(i\partial_\mu - eA_\mu) - m)\,\psi = 0$		$g = 2$

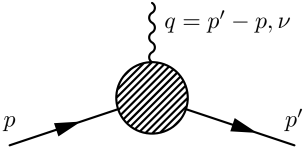
- To provide a systematic quantization for both **electron** and electromagnetic field (**photon**), we need **quantum field theory**. In this case, the theory is **quantum electrodynamics** - **QED**.
- **Electron magnetic moment** is one of its early successful applications.



Dirac equation implies:

$$\bar{u}(p')\gamma_\nu u(p)$$

$$g = 2$$



$$\bar{u}(p')\left(F_1(q^2)\gamma_\nu + i\frac{F_2(q^2)[\gamma_\nu, \gamma_\rho]q_\rho}{4m}\right)u(p)$$

(Euclidean space time)

$$a = F_2(q^2 = 0) = \frac{g-2}{2}$$

- The quantity a is called the anomalous magnetic moments.
- Its value comes from quantum correction.

[4] Glendenin, Nucleonics, in press for January, 1948.
[5] Marshall and Ward, Can. J. Research 15, 29 (1939).
[6] This result is in good agreement with a value of 250 kev, given in *Radioisotopes, Catalog and Price List No. 2*, revised September, 1947, distributed by Isotopes Branch, United States Atomic Energy Commission. Unfortunately, the Atomic Energy Commission's result is not supported by any published experimental evidence.

On Quantum-Electrodynamics and the Magnetic Moment of the Electron

JULIAN SCHWINGER
Harvard University, Cambridge, Massachusetts
December 30, 1947

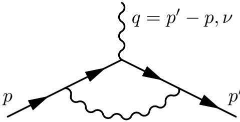
ATTEMPTS to evaluate radiative corrections to electron phenomena have heretofore been beset by divergence difficulties, attributable to self-energy and vacuum polarization effects. Electrodynamics unquestionably requires revision at ultra-relativistic energies, but is presumably accurate at moderate relativistic energies. It would be desirable, therefore, to isolate those aspects of the current theory that essentially involve high energies, and are subject to modification by a more satisfactory theory, from aspects that involve only moderate energies and are thus relatively trustworthy. This goal has been achieved by transforming the Hamiltonian of current hole theory electrodynamics to exhibit explicitly the logarithmically divergent self-energy of a free electron, which arises from

The simplest example of a radiative correction is that for the energy of an electron in an external magnetic field. The detailed application of the theory shows that the radiative correction to the magnetic interaction energy corresponds to an additional magnetic moment associated with the electron spin, of magnitude $\delta\mu/\mu=(\frac{1}{2}\pi)e^2/\hbar c=0.001162$. It is indeed gratifying that recently acquired experimental data confirm this prediction. Measurements on the hyperfine splitting of the ground states of atomic hydrogen and deuterium[1] have yielded values that are definitely larger than those to be expected from the directly measured nuclear moments and an electron moment of one Bohr magneton. These discrepancies can be accounted for by a small additional electron spin magnetic moment.[2] Recalling that the nuclear moments have been calibrated in terms of the electron moment, we find the additional moment necessary to account for the measured hydrogen and deuterium hyperfine structures to be $\delta\mu/\mu=0.00126\pm0.00019$ and $\delta\mu/\mu=0.00131\pm0.00025$, respectively. These values are not in disagreement with the theoretical prediction. More precise conformation is provided by measurement of the g values for the $^2S_{\frac{1}{2}}$, $^2P_{\frac{1}{2}}$, and $^2P_{3/2}$ states of sodium and gallium.[3] To account for these results, it is necessary to ascribe the following additional spin magnetic moment to the electron, $\delta\mu/\mu=0.00118\pm0.00003$.

Electron $g - 2$

1-loop: $a^{(1)} = 0.5 \left(\frac{\alpha}{\pi}\right)$

Schwinger 1948



2-loop: $a^{(2)} = -0.328478444\ldots \left(\frac{\alpha}{\pi}\right)^2$

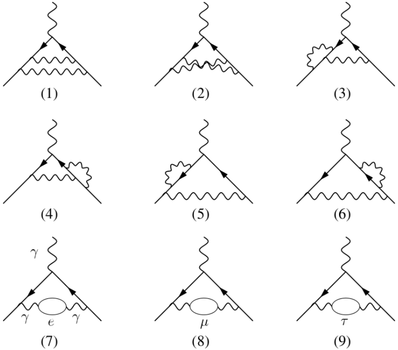
Petermann 1957, Sommerfield 1958

3-loop: $a^{(3)} = 1.181234017\ldots \left(\frac{\alpha}{\pi}\right)^3$

Laporta and Remiddi 1996

4-loop: $a^{(4)} = -1.911321\ldots \left(\frac{\alpha}{\pi}\right)^4$

Laporta 2017



5-loop: $a^{(5)} = 6.671(192) \left(\frac{\alpha}{\pi}\right)^5$

Aoyama, Kinoshita, and Nio 2018

- Finally, include weak and hadronic contribution, one obtain:

$$a_e^{theory} = 1159652181.606(11)_{QED}(12)_{QCD}(229)_\alpha \times 10^{-12}$$

- This can be compared with experimental result:

$$a_e^{exp} = 1159652180.72(28) \times 10^{-12}$$

- The difference is tiny (2.4σ):

$$a_e^{exp} - a_e^{theory} = (-0.88 \pm 0.36) \times 10^{-12}$$

- **QED is very successful in this example.**
- However, does this 2.4σ difference indicate something?

Accurate Determination of the μ^+ Magnetic Moment*

R. L. GARWIN,† D. P. HUTCHINSON, S. PENMAN,‡ AND G. SHAPIRO§
Columbia University, New York, New York

(Received August 4, 1959)

Using a precession technique, the magnetic moment of the positive mu meson is determined to an accuracy of 0.007%. Muons are brought to rest in a bromoform target situated in a homogeneous magnetic field, oriented at right angles to the initial muon spin direction. The precession of the spin about the field direction, together with the asymmetric decay of the muon, produces a periodic time variation in the probability distribution of electrons emitted in a fixed laboratory direction. The period of this variation is compared with that of a reference oscillator by means of phase measurements of the "beat note" between the two. The magnetic field at which the precession and reference frequencies coincide is measured with reference to a proton nuclear magnetic resonance magnetometer. The ratio of the muon precession frequency to that of the proton in the same magnetic field is thus determined to be 3.1834 ± 0.0002. Using a re-evaluated lower limit to the muon mass, this is shown to yield a lower limit on the muon g factor of $2(1.00122\pm0.00008)$, in agreement with the predictions of quantum electrodynamics.

I. INTRODUCTION

RECENT developments in the theory of weak interactions[1] make it appear that many of the properties of the mu meson can be accounted for on the assumption that it enters into interactions in the same way as the electron but has a much larger mass. The electromagnetic properties of the muon, therefore, acquire increased interest as a further test of the identity of the interactions of the two particles.

Quantum electrodynamics[2] makes the prediction that the magnetic moment of a spin $\frac{1}{2}$ Dirac particle is 1.00116[3] times that predicted by Dirac theory, $e\hbar/mc$. The anomalous magnetic moment of the electron does indeed agree with this prediction.[4] Application of

of detecting the direction of polarization via their asymmetric decay[6] made possible the measurement of the muon magnetic moment. In the original experiment it was found necessary, to obtain agreement with the asymmetry curve, to assume a value of the moment close to the Dirac prediction. In this way the value was determined to an accuracy of 1%. The Liverpool group,[7] using an analog time-to-height converter to record the distribution in time of the emitted electrons, achieved an accuracy of 0.7%. A resonance technique, in which the muons were stopped in a large static magnetic field oriented parallel (or antiparallel) to the direction of initial polarization, and were then re-oriented by an rf oscillating field perpendicular to it, was employed at this laboratory.[8] The reversal in polarization was de-

Measurement of the Negative Muon Anomalous Magnetic Moment to 0.7 ppm

G.W. Bennett,[2] B. Bousquet,[9] H. N. Brown,[2] G. Bunce,[2] R. M. Carey,[1] P. Cushman,[9] G. T. Danby,[2] P.T. Debevec,[7] M. Deile,[11] H. Deng,[11] S. K. Dhawan,[11] V. P. Druzhinin,[3] L. Duong,[9] F. J. M. Farley,[11] G.V. Fedotovich,[3] F. E. Gray,[7] D. Grigoriev,[3] M. Grosse-Perdekamp,[11] A. Grossmann,[6] M. F. Hare,[1] D.W. Hertzog,[7] X. Huang,[1] V.W. Hughes,[11,*] M. Iwasaki,[10] K. Jungmann,[5] D. Kawall,[11] B. I. Khazin,[3] F. Krienen,[1] I. Kronkvist,[9] A. Lam,[1] R. Larsen,[2] Y.Y. Lee,[2] I. Logashenko,[1,3] R. McNabb,[9] W. Meng,[2] J. P. Miller,[1] W. M. Morse,[2] D. Nikas,[2] C. J. G. Onderwater,[7] Y. Orlov,[4] C. S. Özben,[2,7] J. M. Paley,[1] Q. Peng,[1] C. C. Polly,[7] J. Pretz,[11] R. Prigl,[2] G. zu Putlitz,[6] T. Qian,[9] S. I. Redin,[3,11] O. Rind,[1] B. L. Roberts,[1] N. Ryskulov,[3] Y. K. Semertzidis,[2] P. Shagin,[9] Yu. M. Shatunov,[3] E. P. Sichtermann,[11] E. Solodov,[3] M. Sossong,[7] L. R. Sulak,[1] A. Trofimov,[1] P. von Walter,[6] and A. Yamamoto[8]

(Muon $(g-2)$ Collaboration)

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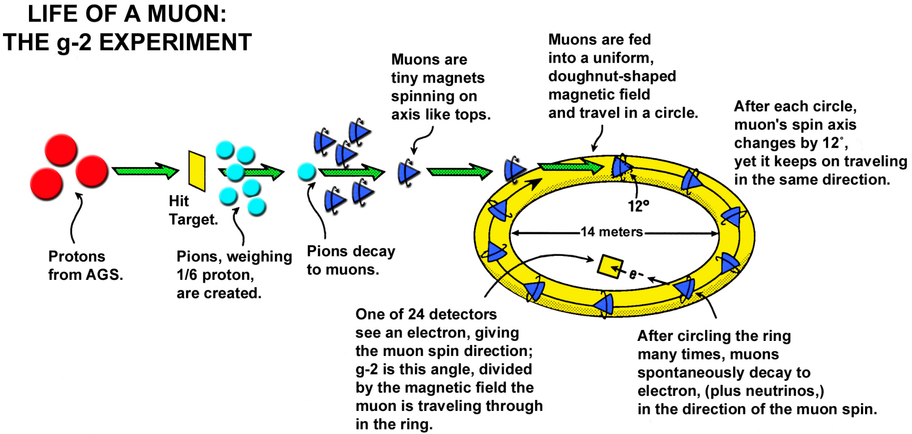
(Received 10 January 2004; published 23 April 2004)

The anomalous magnetic moment of the negative muon has been measured to a precision of 0.7 ppm (ppm) at the Brookhaven Alternating Gradient Synchrotron. This result is based on data collected in 2001, and is over an order of magnitude more precise than the previous measurement for the negative muon. The result $a_{\mu^-} = 11\,659\,214(8)(3) \times 10^{-10}$ (0.7 ppm), where the first uncertainty is statistical and the second is systematic, is consistent with previous measurements of the anomaly for the positive and the negative muon. The average of the measurements of the muon anomaly is



$$a_\mu = 11659208.9(6.3) \times 10^{-10}$$

$$a_e = 1159652180.72(28) \times 10^{-12}$$

$$\text{sensitivity to new physics ratio} = \frac{m_\mu^2}{m_e^2}\frac{\sigma_{a_e}}{\sigma_{a_\mu}} = 206.8^2 \times \frac{1}{2250} = 19$$

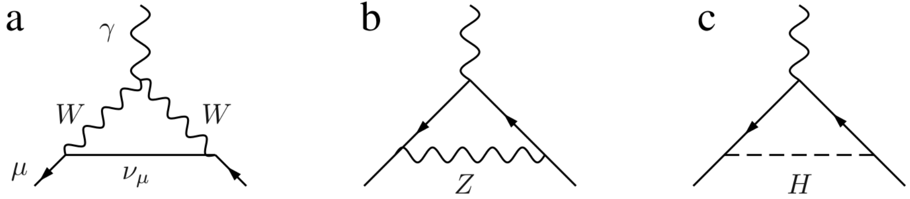
Muon $g-2$: Fermilab E989

Standard Model	11659181.3 ± 4.0
BNL E821 Exp	11659208.9 ± 6.3
Diff (Exp − SM)	27.6 ± 7.5

3.7σ deviations

New Physics?

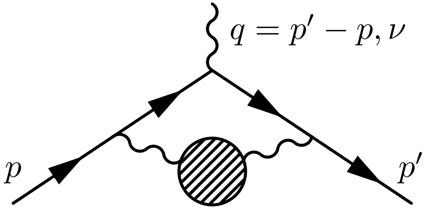
$$
\begin{aligned}
a_\mu^{\mathrm{QED}} &= 0.5 \times \left(\frac{\alpha}{\pi}\right) + 0.765\,857\,425 \underbrace{(17)}_{m_\mu/m_{e,\tau}} \times \left(\frac{\alpha}{\pi}\right)^2 \\
&\quad + 24.050\,509\,96 \underbrace{(32)}_{m_\mu/m_{e,\tau}} \times \left(\frac{\alpha}{\pi}\right)^3 + 130.8796 \underbrace{(63)}_{\text{num. int.}} \times \left(\frac{\alpha}{\pi}\right)^4 \\
&\quad + 753.29 \underbrace{(1.04)}_{\text{num. int.}} \times \left(\frac{\alpha}{\pi}\right)^5 \\
&= 116\,584\,718.853 \underbrace{(9)}_{m_\mu/m_{e,\tau}} \underbrace{(19)}_{c_4} \underbrace{(7)}_{c_5} \underbrace{(29)}_{\alpha(a_e)} [36] \times 10^{-11}
\end{aligned}
$$



Leading weak contribution. $a = 38.87$; $b = -19.39$; $c = 0.00$ [in units 10^{-10}]

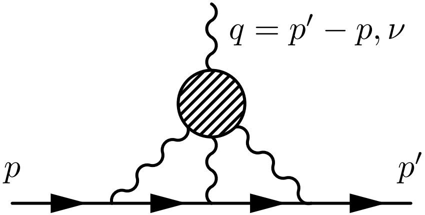
	Value $\pm$ Error	Reference
QED incl. 5-loops	11658471.8853 ± 0.0036	Aoyama, et al, 2012
Weak incl. 2-loops	15.36 ± 0.10	Gnendiger et al, 2013

We will be using the unit 10^{-10} by default.



HVP (LO)

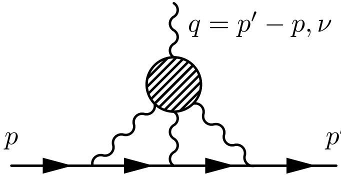
Hadronic Vacuum Polarization



HLbL

Hadronic Light-by-Light

HVP (LO)	692.5 ± 2.7	RBC-UKQCD and FJ17 combined
	693.26 ± 2.46	KNT18
HLbL	10.3 ± 2.9	Fred Jegerlehner, 2017
	10.5 ± 2.6	Glasgow Consensus, 2007
	$7.41 \pm 6.32_{\text{stat}} \pm 0.32_{\text{sys},a^2}$	RBC-UKQCD prelim (QED_L)
	$11.40 \pm 1.27_{\text{stat}} \pm ???_{\text{sys}}$	RBC-UKQCD prelim (QED_∞ & LMD)



Various contributions to $a_\mu^{\mathrm{HLbL}} \times 10^{10}$

	PdRV09 (Glasgow consensus)	JN09	FJ17
π^0, η, η'	11.4 ± 1.3	9.9 ± 1.6	9.5 ± 1.2
π, K loops	-1.9 ± 1.9	-1.9 ± 1.3	-2.0 ± 0.5
axial-vector	1.5 ± 1.0	2.2 ± 0.5	0.8 ± 0.3
scalar	-0.7 ± 0.7	-0.7 ± 0.2	-0.6 ± 0.1
quark loops	0.2 (charm)	2.1 ± 0.3	2.2 ± 0.4
tensor	-	-	0.1 ± 0.0
NLO	-	-	0.3 ± 0.2
Total	10.5 ± 4.9 10.5 ± 2.6 (quadrature)	11.6 ± 3.9	10.3 ± 2.9

Muon $g-2$: Summary

QED 5-loops	11658471.8853 ± 0.0036	Aoyama, et al, 2012
Weak 2-loops	15.36 ± 0.10	Gnendiger et al, 2013
HVP (LO)	692.5 ± 2.7	RBC-UKQCD and FJ17 combined
	693.26 ± 2.46	KNT18
HVP (NLO)	-9.93 ± 0.07	Fred Jegerlehner, 2017
HVP (NNLO)	1.22 ± 0.01	Fred Jegerlehner, 2017
HLbL	10.3 ± 2.9	Fred Jegerlehner, 2017
	10.5 ± 2.6	Glasgow Consensus, 2007
	$7.41 \pm 6.32_{\mathrm{stat}} \pm 0.32_{\mathrm{sys},a^2}$	RBC-UKQCD prelim (QED_L)
	$11.40 \pm 1.27_{\mathrm{stat}} \pm ???_{\mathrm{sys}}$	RBC-UKQCD prelim (QED_∞ & LMD)
SM Theory	11659181.3 ± 4.0	
BNL E821 Exp	11659208.9 ± 6.3	
Exp − SM	27.6 ± 7.5	

Outline

The RBC & UKQCD collaborations

BNL and BNL/RBRC

Yasumichi Aoki (KEK)
Taku Izubuchi
Yong-Chull Jang
Chulwoo Jung
Meifeng Lin
Aaron Meyer
Hiroshi Ohki
Shigemi Ohta (KEK)
Amarjit Soni

UC Boulder

Oliver Witzel

CERN

Mattia Bruno

Columbia University

Ryan Abbot
Norman Christ
Duo Guo
Christopher Kelly
Bob Mawhinney
Masaaki Tomii
Jiqun Tu
Bigeng Wang
Tianle Wang
Yidi Zhao

University of Connecticut

Tom Blum
Dan Hoying (BNL)
Luchang Jin (RBRC)
Cheng Tu

Edinburgh University

Peter Boyle
Luigi Del Debbio
Felix Erben
Vera Gülpers
Tadeusz Janowski
Julia Kettle
Michael Marshall
Fionn Ó hÓgáin
Antonin Portelli
Tobias Tsang
Andrew Yong
Azusa Yamaguchi

Masashi Hayakawa (Nagoya)

KEK

Julien Frison

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Nicolas Garron

MIT

David Murphy

Peking University

Xu Feng

University of Regensburg

Christoph Lehner (BNL)

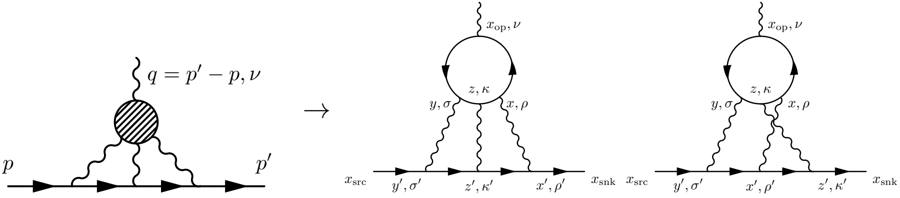
University of Southampton

Nils Asmussen
Jonathan Flynn
Ryan Hill
Andreas Jüttner
James Richings
Chris Sachrajda

Stony Brook University

Jun-Sik Yoo
Sergey Syritsyn (RBRC)

HLbL: diagrams

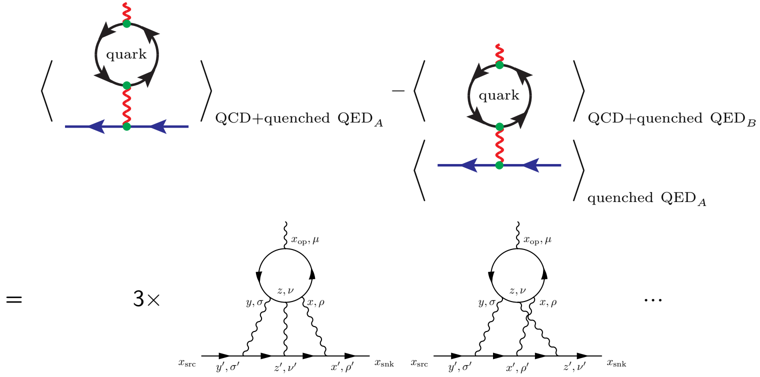


- There are additional four different permutations of photons not shown.
- There are quark loop and muon line.
- Gluons and sea quark loops (not directly connected to photons) are included automatically to all orders!
- The photons can be connected to different quark loops, will be discussed later.

Outline

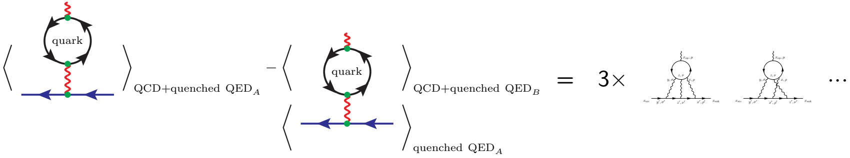
HLbL: subtraction method

- Introduced in LATTICE 2005.
- **PoS LAT2005 (2006) 353. hep-lat/0509016.**
- T. Blum, S. Chowdhury, M. Hayakawa, T. Izubuchi.



- Evalutate the quark and muon propagators in the background quenched QED fields, generating all kinds of diagrams.

- QED field subscript "A" and "B" correspond to the same set of QED fields.
- Both "A" and "B" QED fields are averaged independently.
- Factor "3" is related with the special photon already explicitly included.



- After subtraction, the lowest order signal remains is $\mathcal{O}(e^6)$ which is exact LbL diagram.
- Solved the 3-loop problem. Now we only need to compute point source propagators in the background of QED fields.
- Unwanted higher order effects. In practice, one normally choose $e = 1$.
- Lower order noise problem. The signal after subtraction is $\mathcal{O}(e^6)$. But even after charge conjugation average on the muon line, the noise is still $\mathcal{O}(e^4)$.
- "Disconnect diagram" problem. Noise will likely increase in larger volume.

Hadronic Light-by-Light Scattering Contribution to the Muon Anomalous Magnetic Moment from Lattice QCD

Thomas Blum,[1,2] Saumitra Chowdhury,[1] Masashi Hayakawa,[3,4] and Taku Izubuchi[5,2]
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(Received 18 July 2014; published 7 January 2015)

The most compelling possibility for a new law of nature beyond the four fundamental forces comprising the standard model of high-energy physics is the discrepancy between measurements and calculations of the muon anomalous magnetic moment. Until now a key part of the calculation, the hadronic light-by-light contribution, has only been accessible from models of QCD, the quantum description of the strong force, whose accuracy at the required level may be questioned. A first principles calculation with systematically improvable errors is needed, along with the upcoming experiments, to decisively settle the matter. For the first time, the form factor that yields the light-by-light scattering contribution to the muon anomalous magnetic moment is computed in such a framework, lattice QCD + QED and QED. A nonperturbative treatment of QED is used and checked against perturbation theory. The hadronic contribution is calculated for unphysical quark and muon masses, and only the diagram with a single quark loop is computed for which statistically significant signals are obtained. Initial results are promising, and the prospect for a complete calculation with physical masses and controlled errors is discussed.

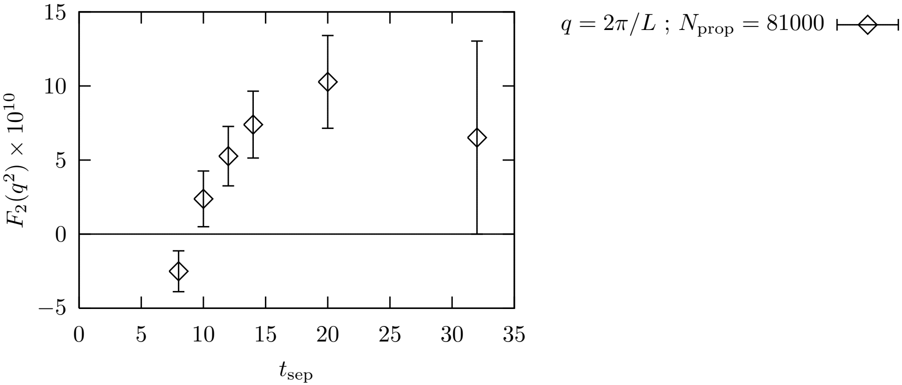
DOI: 10.1103/PhysRevLett.114.012001 PACS numbers: 12.38.Gc, 12.20.-m, 12.38.-t, 13.40.Em

Introduction.—The muon anomalous magnetic moment, or anomaly $a_\mu = (g_\mu - 2)/2$, provides one of the most stringent tests of the standard model because it has been measured to great accuracy (0.54 ppm) [1] and calculated to

scattering (HLbL) contribution (Fig. 1), a_μ(HLbL), from experimental data and a dispersion relation [9,10]. So far, only model calculations have been done. The uncertainty quoted in Table I was estimated by the "Glasgow con-

HLbL: subtraction method works!

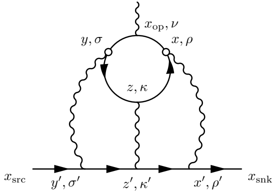
- Ten years after the method is proposed.
- **Phys.Rev.Lett. 114 (2015) 1, 012001. arXiv:1407.2923**
- T. Blum, S. Chowdhury, M. Hayakawa, T. Izubuchi.



- RBC/UKQCD $24^3 \times 64$ DWF, with $a^{-1} = 1.785\,\mathrm{GeV}$, $m_\pi = 342\,\mathrm{MeV}$. $m_\mu = 178.5\,\mathrm{MeV}$.
- Only **connected diagrams** is calculated.

Outline

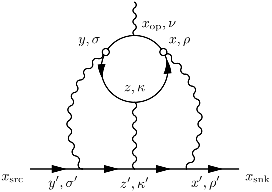
HLbL: Exact photon propagator



- We can use two point source photons at x and y, which are chosen randomly. It is a very standard 8-dimentional Monte Carlo integral over two space-time points.
- Major contribution comes from the region where x and y are not far separated. Importance sampling is needed. In fact, we can evaluate all possible (upto discrete symmetries) relative positions for distance less than a certain value $r_{\rm max}$, which is normally set to be 5 lattice units.

Improvement over the previous method:

- The muon line is not suffered from long distance noise of a stochastic QED field anymore.
- Two points of the four point function are exactly summed over, previously only one.
- Benefit from the fact that QCD has a mass gap => importance sampling.



$$i\,\mathcal{M}_\nu(\vec{q}) = \sum_{x,y,z} i\mathcal{F}_\nu(\vec{q};x,y,z,x_{\rm op}=0)$$

$$\Downarrow$$

$$i\,\mathcal{M}_\nu(\vec{q}) = \sum_{r,z,x_{\rm op}} e^{i\vec{q}\cdot\vec{x}_{\rm op}}\, i\mathcal{F}_\nu(\vec{q};x,y,z,x_{\rm op})$$

$$x = x_{\rm ref} + r/2 \qquad y = x_{\rm ref} - r/2$$

$$\mathcal{F}_\nu(\vec{q};x,y,z,x_{\rm op}) = (-ie)^6\,\mathcal{G}_{\rho,\sigma,\kappa}(\vec{q};x,y,z)\mathcal{H}_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\rm op}) \tag{2}$$

$$\begin{aligned} & i^4\,\mathcal{H}_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\rm op}) \\ = & \sum_{q=u,d,s} (e_q/e)^4 \left\langle {\rm tr}\left[-i\gamma_\rho S_q(x,z) i\gamma_\kappa S_q(z,y) i\gamma_\sigma S_q(y,x_{\rm op}) i\gamma_\nu S_q(x_{\rm op},x)\right]\right\rangle_{\rm QCD} \end{aligned} \tag{3}$$

$$\begin{aligned} & i^3\,\mathcal{G}_{\rho,\sigma,\kappa}(\vec{q};x,y,z) \\ = & \; e^{\sqrt{m_\mu^2+\vec{q}^2/4}(t_{\rm snk}-t_{\rm src})} \sum_{x',y',z'} G_{\rho,\rho'}(x,x')G_{\sigma,\sigma'}(y,y')G_{\kappa,\kappa'}(z,z') \\ & \times \sum_{\vec{x}_{\rm snk},\vec{x}_{\rm src}} e^{-i\vec{q}/2\cdot(\vec{x}_{\rm snk}+\vec{x}_{\rm src})} S_\mu(x_{\rm snk},x') i\gamma_{\rho'} S_\mu(x',z') i\gamma_{\kappa'} S_\mu(z',y') i\gamma_{\sigma'} S_\mu(y',x_{\rm src}) \\ + & \; \text{other 5 permutations} \end{aligned} \tag{4}$$

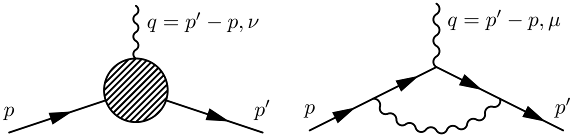


Figure 2. (L) Muon Vertex Function Diagram (R) Schwinger Term Diagram.

$$\langle \vec{p}', s' | j_\nu(\vec{x}_{\rm op} = \vec{0}) | \vec{p}, s \rangle = \left\langle \vec{p}', s' \left| \sum_f q_f \bar{\psi}_f(\vec{x}_{\rm op} = 0)\, \gamma_\nu \psi_f(\vec{x}_{\rm op} = 0) \right| \vec{p}, s \right\rangle$$

$$= -e\, \bar{u}_{s'}(\vec{p}') \left[F_1(q^2)\gamma_\nu + i\frac{F_2(q^2)}{4m}[\gamma_\nu, \gamma_\rho] q_\rho \right] u_s(\vec{p}) \quad (5)$$

$$= -e\, \bar{u}_{s'}(\vec{p}')\, \mathcal{M}_\nu(p', p)\, u_s(\vec{p}) \quad (6)$$

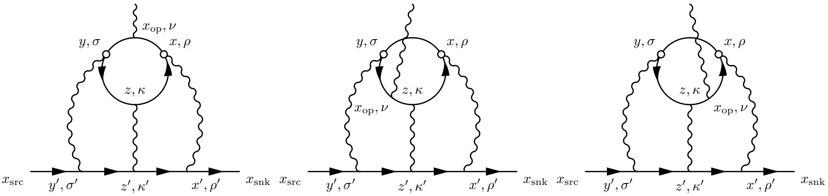
$$\vec{\mu} = -g\frac{e}{2m}\vec{s} = -(F_1(0) + F_2(0))\,\frac{e}{m}\,\vec{s} \quad (7)$$

$$F_1(0) = 1 \quad (8)$$

$$F_2(0) = \frac{g-2}{2} \equiv a \quad (9)$$

Outline

HLbL diagram (2: more permutations)

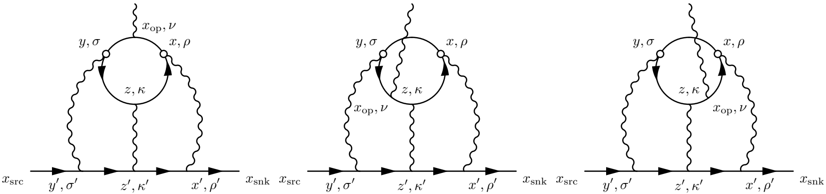


$$
\begin{aligned}
i\,\mathcal{M}_\nu(\vec{q}) &= \sum_{x,y,z} i\,\mathcal{F}_\nu(\vec{q}; x, y, z, x_{\rm op}) \\
\text{(1: shift coordinates)} &= \sum_{r,z,x_{\rm op}} e^{i\vec{q}\cdot\vec{x}_{\rm op}}\, i\,\mathcal{F}_\nu(\vec{q}; x, y, z, x_{\rm op}) \\
\text{(2: more permuations)} &= \sum_{r,z,x_{\rm op}} e^{i\vec{q}\cdot\vec{x}_{\rm op}}\, i\,\mathcal{F}^C_\nu(\vec{q}; x, y, z, x_{\rm op})
\end{aligned}
$$

$$
\mathcal{F}^C_\nu(\vec{q}; x, y, z, x_{\rm op}) = (-ie)^6\, \mathcal{G}_{\rho,\sigma,\kappa}(\vec{q}; x, y, z)\mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x, y, z, x_{\rm op}) \tag{10}
$$

$$
\begin{aligned}
& i^4\,\mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x, y, z, x_{\rm op}) \qquad (11) \\
= & \sum_{q=u,d,s} \frac{(e_q/e)^4}{6} \left\langle \mathrm{tr}\left[-i\gamma_\rho S_q(x,z) i\gamma_\kappa S_q(z,y) i\gamma_\sigma S_q(y,x_{\rm op}) i\gamma_\nu S_q(x_{\rm op},x) \right] \right\rangle_{\rm QCD} \\
+ & \ \text{other 5 permutations}
\end{aligned}
$$

HLbL diagram (3: subtract "1")



$$
\begin{aligned}
i\,\mathcal{M}_\nu(\vec{q}) &= \sum_{x,y,z} i\,\mathcal{F}_\nu(\vec{q}; x, y, z, x_{\rm op}) \\
\text{(1: shift coordinates)} &= \sum_{r,z,x_{\rm op}} e^{i\vec{q}\cdot\vec{x}_{\rm op}}\, i\,\mathcal{F}_\nu(\vec{q}; x, y, z, x_{\rm op}) \\
\text{(2: more permuations)} &= \sum_{r,z,x_{\rm op}} e^{i\vec{q}\cdot\vec{x}_{\rm op}}\, i\,\mathcal{F}^C_\nu(\vec{q}; x, y, z, x_{\rm op}) \\
\text{(3: subtract “1”)} &= \sum_{r,z,x_{\rm op}} e^{i\vec{q}\cdot\vec{x}_{\rm ref}} \left(e^{i\vec{q}\cdot(\vec{x}_{\rm op}-\vec{x}_{\rm ref})} - 1\right) i\,\mathcal{F}^C_\nu(\vec{q}; x, y, z, x_{\rm op})
\end{aligned}
$$

Consequence of current conservation

Consider a vector field $J_\rho(x)$. It satisfies two conditions:

- $\partial_\rho J_\rho(x) = 0$.
- $J_\rho(x) = 0$ if $|x|$ is large.

We can conclude (the result is a little bit unexpected, but actually correct):

$$\int d^4x\, J_\rho(x) \;=\; \int d^4x\, \partial_\sigma(x_\rho\, J_\sigma(x)) = 0 \tag{12}$$

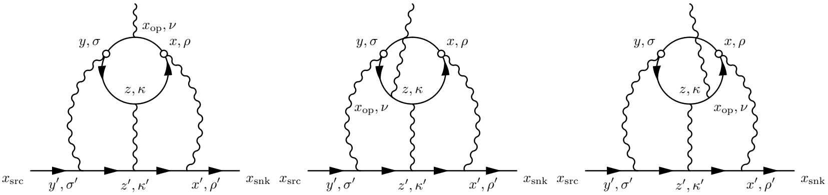
In three dimension, this result have a consequence which is well-known.

Consider a finite size system with stationary current. We then have

- $\vec{\nabla}\cdot\vec{j}(\vec{x}) = 0$, because of current conservation.
- $\vec{j}(\vec{x}) = 0$ if $|\vec{x}|$ large, because the system if of finite size.

Within a constant external magnetic field $\vec{B}$, the total magnetic force should be

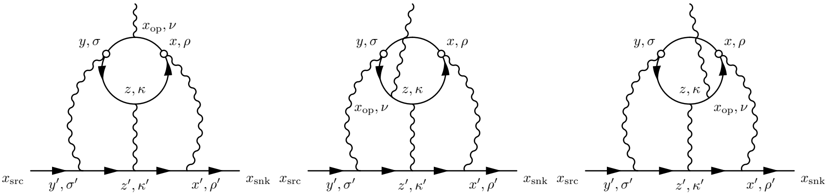
$$\int\, [\,\vec{j}(\vec{x}) \times \vec{B}\,]\, d^3x \;=\; \left[\int \vec{j}(\vec{x})\, d^3x\right] \times \vec{B} = 0 \tag{13}$$



$$
\begin{aligned}
i\,\mathcal{M}_\nu(\vec{q}) &= \sum_{x,y,z} i\,\mathcal{F}_\nu(\vec{q}; x, y, z, x_{\rm op}) \\
\text{(1: shift coordinates)} &= \sum_{r,z,x_{\rm op}} e^{i\vec{q}\cdot\vec{x}_{\rm op}}\, i\,\mathcal{F}_\nu(\vec{q}; x, y, z, x_{\rm op}) \\
\text{(2: more permuations)} &= \sum_{r,z,x_{\rm op}} e^{i\vec{q}\cdot\vec{x}_{\rm op}}\, i\,\mathcal{F}^C_\nu(\vec{q}; x, y, z, x_{\rm op}) \\
\text{(3: subtract ``1'')} &= \sum_{r,z,x_{\rm op}} e^{i\vec{q}\cdot\vec{x}_{\rm ref}} \left(e^{i\vec{q}\cdot(\vec{x}_{\rm op}-\vec{x}_{\rm ref})} - 1\right) i\,\mathcal{F}^C_\nu(\vec{q}; x, y, z, x_{\rm op}) \\
\text{(4: } q \to 0 \text{ limit)} &= \sum_{r,z,x_{\rm op}} i\,\vec{q}\cdot(\vec{x}_{\rm op} - \vec{x}_{\rm ref})\, i\,\mathcal{F}^C_\nu(\vec{0}; x, y, z, x_{\rm op})
\end{aligned}
$$

$$
\bar{u}_{s'}(0)\, i\mathcal{M}_\nu(\vec{q})\, u_s(\vec{0}) = i\,\bar{u}_{s'}(\vec{0}) \left[i\frac{F_2(q^2)}{4m} [\gamma_\nu, \gamma_\rho] q_\rho \right] u_s(\vec{0}) \qquad (14)
$$

HLbL diagram (match with F_2)



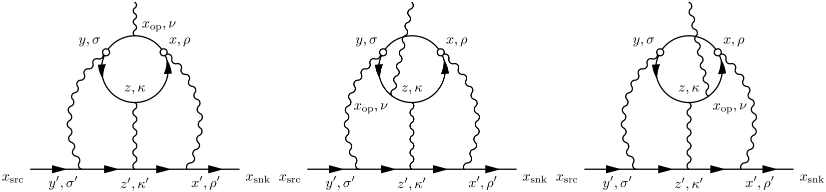
$$\bar{u}_{s'}(0)\, i\mathcal{M}_\nu(\vec{q})\, u_s(\vec{0}) \;=\; i\,\bar{u}_{s'}(\vec{0})\left[i\frac{F_2(q^2)}{4m}[\gamma_\nu,\gamma_\rho]q_\rho\right]u_s(\vec{0})$$

$$i\,\mathcal{M}_\nu(\vec{q}) \;=\; \sum_{r,z,x_{\rm op}} i\,\vec{q}\cdot(\vec{x}_{\rm op}-\vec{x}_{\rm ref})\, i\,\mathcal{F}^C_\nu(\vec{0};x,y,z,x_{\rm op})$$

$$\bar{u}_{s'}(\vec{0})\left[i\frac{F_2(q^2)}{4m}[\gamma_k,\gamma_j]\right]u_s(\vec{0}) \;=\; \sum_{r,z,x_{\rm op}} (\vec{x}_{\rm op}-\vec{x}_{\rm ref})_j\, i\,\mathcal{F}^C_k(\vec{0};x,y,z,x_{\rm op})$$

$$\bar{u}_{s'}(\vec{0})\left[i\frac{F_2(q^2)}{4m}\,\frac{1}{2}\epsilon_{i,j,k}[\gamma_k,\gamma_j]\right]u_s(\vec{0}) \;=\; \sum_{r,z,x_{\rm op}} \frac{1}{2}\,\epsilon_{i,j,k}\,(\vec{x}_{\rm op}-\vec{x}_{\rm ref})_j\, i\,\mathcal{F}^C_k(\vec{0};x,y,z,x_{\rm op})$$

- Recall $x = x_{\rm ref} + r/2$ and $y = x_{\rm ref} - r/2$. Also $\Sigma_i = \frac{1}{4i}\epsilon_{i,j,k}\,[\gamma_j,\gamma_k]$.



$$\bar{u}_{s'}(\vec{0})\left[i\frac{F_2(q^2)}{4m}\,\frac{1}{2}\epsilon_{i,j,k}[\gamma_k,\gamma_j]\right]u_s(\vec{0}) \;=\; \sum_{r,z,x_{\rm op}} \frac{1}{2}\,\epsilon_{i,j,k}\,(\vec{x}_{\rm op}-\vec{x}_{\rm ref})_j\, i\,\mathcal{F}^C_k\big(\vec{0};x,y,z,x_{\rm op}\big)$$

$$\frac{F_2(0)}{m}\,\bar{u}_{s'}(\vec{0})\frac{\vec{\Sigma}}{2}u_s(\vec{0}) \;=\; \sum_{r,z,x_{\rm op}} \frac{1}{2}\,(\vec{x}_{\rm op}-\vec{x}_{\rm ref})\times \bar{u}_{s'}(\vec{0})\, i\vec{\mathcal{F}}^C\big(\vec{0};x,y,z,x_{\rm op}\big)\,u_s(\vec{0})$$

- Recall $x = x_{\rm ref} + r/2$ and $y = x_{\rm ref} - r/2$. Also $\Sigma_i = \frac{1}{4i}\epsilon_{i,j,k}\,[\gamma_j,\gamma_k] = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}$.

Classicaly, magnetic moment is simply

$$\vec{\mu} \;=\; \int \frac{1}{2}\vec{x} \times \vec{j}\, d^3x \tag{15}$$

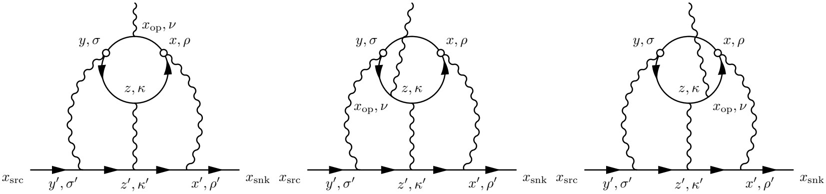
- This formula is not correct in Quantum Mechanics, because the magnetic moment result from the spin is not included.
- In Quantum Field Thoery, Dirac equation automatically predict fermion spin, so the naive equation is correct again!

$$\langle \vec{\mu} \rangle \;=\; \left\langle \psi \middle| \int \frac{1}{2}\, \vec{x}_{\mathrm{op}} \times i\, \vec{j}(\vec{x}_{\mathrm{op}})\, d^3x_{\mathrm{op}} \middle| \psi \right\rangle \tag{16}$$

- $i\, \vec{j}(\vec{x}_{\mathrm{op}})$ is the conventional Minkovski spatial current, because of our γ matrix convention.
- The right hand generate the total magnetic moment for the entire system, including magnetic moment from spin.
- Above formula applies to **normalizable state** with zero total current. Not practical on lattice because it need extremely large volume to evaluate.

$$L \;\gg\; \Delta x_{\mathrm{op}} \sim 1/\Delta p \tag{17}$$

HLbL: moment method



$$\frac{F_2(0)}{m}\,\bar{u}_{s'}(\vec{0})\frac{\vec{\Sigma}}{2}u_s(\vec{0}) \;=\; \sum_{r=x-y}\left[\sum_{z,x_{\rm op}}\frac{1}{2}\,(\vec{x}_{\rm op}-\vec{x}_{\rm ref})\times\bar{u}_{s'}(\vec{0})\,i\vec{\mathcal{F}}^C(\vec{0};x,y,z,x_{\rm op})\,u_s(\vec{0})\right]$$

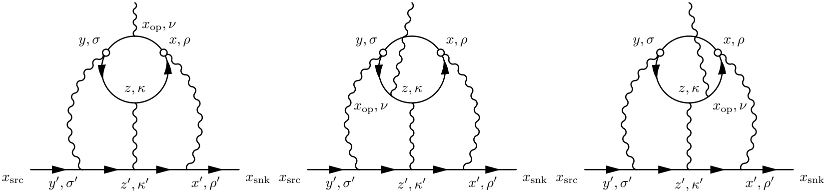
- The initial and final muon states are plane waves instead of properly normalized states.
- The time coordinate of the current, $(x_{\rm op})_0$ is integrated instead of being held fixed.
- For x and y, only $r = x - y$ is summed over, instead of both x and y.

These features allow us to perform the lattice simulation efficiently in finite volume.

Note that we set the reference point the average of the two sampled points.

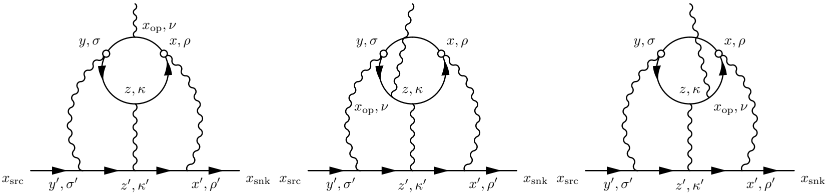
$$x_{\rm ref} \;=\; (x+y)/2 \tag{18}$$

HLbL: reorder the summation



- The points x, y, z are equivalent, we are free to re-label them.
- Since we sum over z, but sample over $r = y - x$. It is beneficial to keep r small, where the fluctuation is small and sampling can be complete.
- So, when we sum over z, we only sum the region where z is far from x, y compare with the distance between x and y.
- This way, we **move** most of the contribution into the small r region, where the fluctuation is small and sampling can be complete.

HLbL: reorder the summation



$$\frac{F_2(0)}{m}\,\bar{u}_{s'}(\vec{0})\,\frac{\vec{\Sigma}}{2}u_s(\vec{0}) \;=\; \sum_{r,z}\mathfrak{Z}\,(x,y,z)\sum_{x_{\rm op}}\frac{1}{2}\,(\vec{x}_{\rm op}-\vec{x}_{\rm ref})_j\times\bar{u}_{s'}(\vec{0})\,i\,\vec{\mathcal{F}}^C(x,y,z,x_{\rm op})u_s(\vec{0})$$

$$\mathfrak{Z}(x,y,z) \;=\; \begin{cases} 3 & \text{if } |x-y|<|x-z| \text{ and } |x-y|<|y-z| \\ 3/2 & \text{if } |x-y|=|x-z|<|y-z| \text{ or } |x-y|=|y-z|<|x-z| \\ 1 & \text{if } |x-y|=|x-z|=|y-z| \\ 0 & \text{otherwise} \end{cases} \qquad (68)$$

- Here $x=x_{\rm ref}+r/2$ and $y=x_{\rm ref}-r/2$.

PHYSICAL REVIEW D **93,** 014503 (2016)

Lattice calculation of hadronic light-by-light contribution to the muon anomalous magnetic moment

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The quark-connected part of the hadronic light-by-light scattering contribution to the muon's anomalous magnetic moment is computed using lattice QCD with chiral fermions. We report several significant algorithmic improvements and demonstrate their effectiveness through specific calculations which show a reduction in statistical errors by more than an order of magnitude. The most realistic of these calculations is performed with a near-physical 171 MeV pion mass on a $(4.6\ \mathrm{fm})^3$ spatial volume using the $32^3 \times 64$ Iwasaki + DSDR gauge ensemble of the RBC/UKQCD Collaboration.

DOI: 10.1103/PhysRevD.93.014503

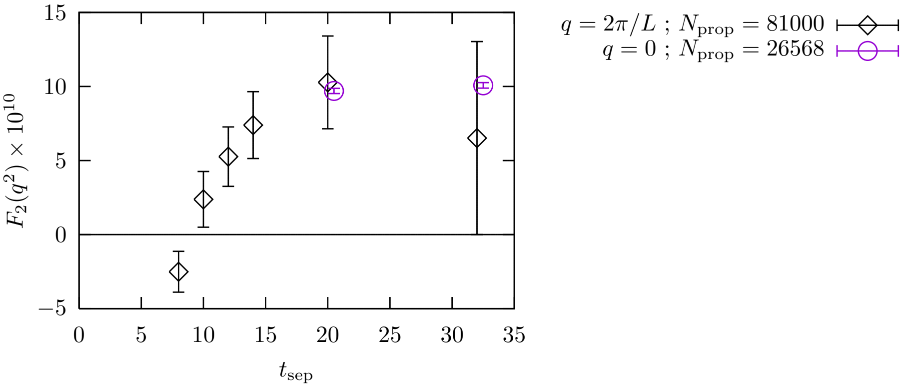
I. INTRODUCTION

New particles and interactions which occur at a very large energy scale Λ, above the reach of present-day accelerators, may be first discovered through their indirect effects at low energy. A particularly promising low-energy quantity that may reveal such effects is the anomalous moment of the muon. This "anomalous" difference $g_\mu - 2$ between the muon's gyromagnetic ratio g_μ and the Dirac value of 2 for a noninteracting particle can receive contributions from such

planned at Fermilab (E989) and J-PARC (E34) with a targeted precision as small as 0.14 ppm, and for a reduction in the theoretical errors.

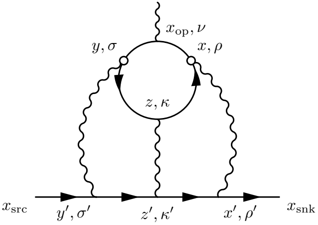
The two components of the theoretical calculation with the largest errors involve couplings to the up, down and strange quarks: the hadronic vacuum polarization (HVP) and hadronic light-by-light scattering (HLbL). These are the first cases in which the effects of the strong interaction enter the determination of $g_\mu - 2$. The HVP effects enter

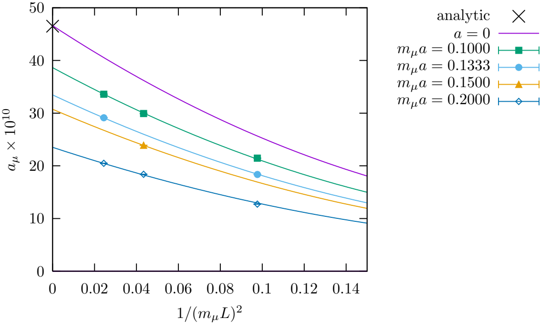
- **Phys.Rev. D93 (2016) no.1, 014503.**
- T. Blum, N. Christ, M. Hayakawa, T. Izubuchi, L. Jin, C. Lehner.



- RBC/UKQCD $24^3 \times 64$ DWF, with $a^{-1} = 1.785\,\mathrm{GeV}$, $m_\pi = 342\,\mathrm{MeV}$. $m_\mu = 178.5\,\mathrm{MeV}$.
- Only **connected diagrams** is calculated.
- Statistical error significantly reduced with less cost. Error do not increase with increasing $t_{\rm sep}$. In the future, we will always compute with $t_{\rm sep} = T/2$.

- We test our setup by computing **muon leptonic light by light** contribution to muon $g-2$.





$$F_2(a, L) = F_2\left(1 - \frac{c_1}{(m_\mu L)^2} + \frac{c_1'}{(m_\mu L)^4}\right)(1 - c_2 a^2 + c_2' a^4) \;\to\; F_2 = 46.6(2) \times 10^{-10} \quad (19)$$

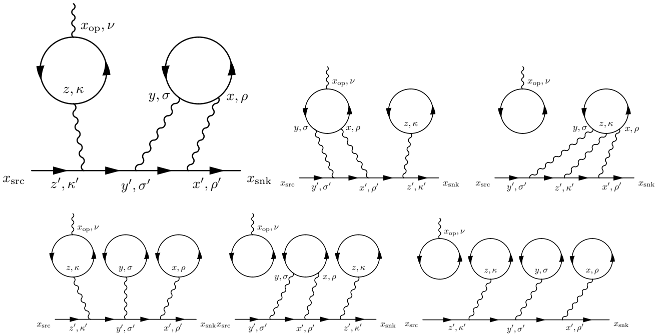
- **Pure QED computation**. Muon leptonic light by light contribution to muon $g-2$. Phys.Rev. D93 (2016) 1, 014503. arXiv:1510.07100.
- Analytic results: $0.371 \times (\alpha/\pi)^3 = 46.5 \times 10^{-10}$.
- $\mathcal{O}(1/L^2)$ finite volume effect, because the photons are emitted from a conserved loop.

Outline

1. Introduction
2. HLbL: QED_L approach
   - Subtraction method
   - Exact photon propagator
   - The moment method
   - **Disconnected diagrams**
   - Results
3. HLbL: QED_∞ approach
4. HVP

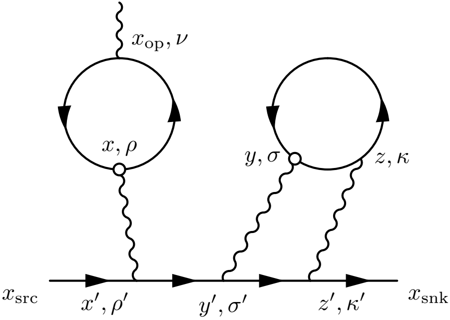
HLbL: disconnected diagrams

- One diagram (the biggest diagram below) do not vanish even in the SU(3) limit.
- We extend the method and computed this leading disconnected diagram as well.



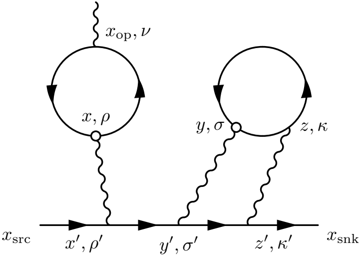
- Permutations of the three internal photons are not shown.
- **Gluons exchange between and within the quark loops are not drawn.**
- We need to make sure that the loops are connected by gluons by "vacuum" subtraction. So the diagrams are 1-particle irreducible.

HLbL: disconnected formula



- Point x is used as the reference point for the moment method.
- We can use two point source photons at x and y, which are chosen randomly. The points $x_{\rm op}$ and z are summed over exactly on lattice.
- Only point source quark propagators are needed. We compute M point source propagators and all M^2 combinations of them are used to perform the stochastic sum over $r = x - y$.

HLbL: disconnected formula

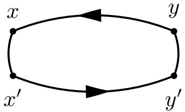


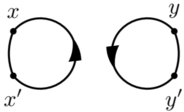
$$\frac{F_2^{\mathrm{dHLbL}}(0)}{m}\frac{(\sigma_{s',s})_i}{2} = \sum_{r=x-y,z}\sum_{x_{\mathrm{op}}}\frac{1}{2}\,\epsilon_{i,j,k}(x_{\mathrm{op}}-x)_j\cdot i\,\bar{u}_{s'}(\vec{0})\,\mathcal{F}_k^D(x,y,z,x_{\mathrm{op}})u_s(\vec{0}) \quad (20)$$

$$\mathcal{F}_\nu^D(x,y,z,x_{\mathrm{op}}) = (-ie)^6\,\mathcal{G}_{\rho,\sigma,\kappa}(x,y,z)\mathcal{H}^D_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\mathrm{op}}) \quad (21)$$

$$\mathcal{H}^D_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\mathrm{op}}) = \left\langle\frac{1}{2}\,\Pi_{\nu,\rho}(x_{\mathrm{op}},x)\left[\Pi_{\sigma,\kappa}(y,z)-\Pi^{\mathrm{avg}}_{\sigma,\kappa}(y-z)\right]\right\rangle_{\mathrm{QCD}} \quad (22)$$

$$\Pi_{\sigma,\kappa}(y,z) = -\sum_q (e_q/e)^2\,\mathrm{Tr}[\gamma_\sigma\,S_q(y,z)\,\gamma_\kappa\,S_q(z,y)]. \quad (23)$$





For the four-point-function, when its two ends, x and y, are far separated, but x' is close to x and y' is close to y, the four-point-function is dominated by π^0 exchange.

Both the connected and the disconnected diagram will contribute in these region. We can find a connection between the connnected diagram and the disconnected diagram by first investigating the η correlation function.

$$\langle \bar{u}\gamma_5 u(x)(\bar{u}\gamma_5 u + \bar{d}\gamma_5 d)(y)\rangle \sim e^{-m_\eta |x-y|} \tag{24}$$

$$\langle \bar{u}\gamma_5 u(x)(\bar{u}\gamma_5 u - \bar{d}\gamma_5 d)(y)\rangle + 2\langle \bar{u}\gamma_5 u(x)\bar{d}\gamma_5 d(y)\rangle \sim e^{-m_\eta |x-y|} \tag{25}$$

That is

$$\langle \bar{u}\gamma_5 u(x)\bar{d}\gamma_5 d(y)\rangle = -\frac{1}{2}\langle \bar{u}\gamma_5 u(x)(\bar{u}\gamma_5 u - \bar{d}\gamma_5 d)(y)\rangle + \mathcal{O}\big(e^{-m_\eta |x-y|}\big) \tag{26}$$

Above is a relation between disconnected diagram π^0 exchange (left hand side) and connected diagram π^0 exchange (right hand side).

The nearby two current operater can be viewed as an interpolating operator for π^0, just like $\bar{u}\gamma_5 u$ or $\bar{d}\gamma_5 d$ with appropriate charge factors.

Multiplied by appropriate charge factors:

$$\text{Connected contribution} \quad \left[\left(\frac{2}{3}\right)^4 + \left(-\frac{1}{3}\right)^4\right] = \frac{17}{81} \tag{27}$$

$$\text{Disconnected contribution} \quad \left[\left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2\right]^2 \left(-\frac{1}{2}\right) = \frac{25}{81}\left(-\frac{1}{2}\right) \tag{28}$$

$$\text{Connected} : \text{Disconnected} = 34 : -25 \tag{29}$$

Different approach by J. Bijnens and J. Relefors: **JHEP 1609 (2016) 113**.

Outline

Connected and Leading Disconnected Hadronic Light-by-Light Contribution to the Muon Anomalous Magnetic Moment with a Physical Pion Mass

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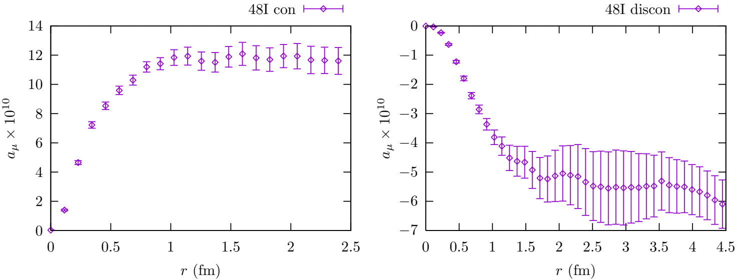
We report a lattice QCD calculation of the hadronic light-by-light contribution to the muon anomalous magnetic moment at a physical pion mass. The calculation includes the connected diagrams and the leading, quark-line-disconnected diagrams. We incorporate algorithmic improvements developed in our previous work. The calculation was performed on the $48^3 \times 96$ ensemble generated with a physical pion mass and a 5.5 fm spatial extent by the RBC and UKQCD Collaborations using the chiral, domain wall fermion formulation. We find $a_\mu^{\mathrm{HLbL}} = 5.35(1.35) \times 10^{-10}$, where the error is statistical only. The finite-volume and finite lattice-spacing errors could be quite large and are the subject of ongoing research. The omitted disconnected graphs, while expected to give a correction of order 10%, also need to be computed.

DOI: 10.1103/PhysRevLett.118.022005

Introduction.—The lattice calculation of the hadronic light-by-light contribution to the muon anomalous magnetic moment is part of the ongoing effort to better understand the approximately 3 standard deviation difference between the extremely accurate BNL E821 experimental result and the current theoretical calculation with similar accuracy

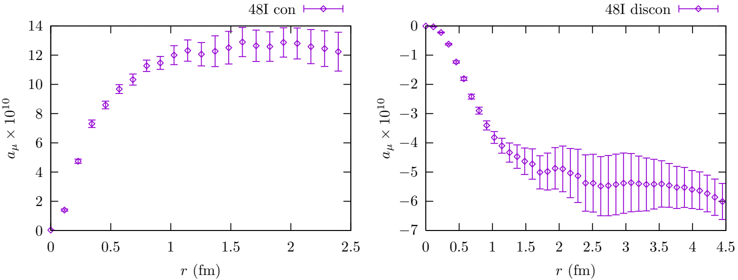
constant. However, new, high-energy phenomena that appear at an energy scale Λ can introduce additional structure, leading to new contributions to a_l that are typically suppressed by the ratio $(m_l/\Lambda)^2$, where $l = e$, μ, or τ and m_l is the mass of the corresponding lepton. The muon anomaly may be the best place to search for such



- Left: **connected diagrams**. Right: leading **disconnected diagrams**.
- $48^3 \times 96$ lattice, with $a^{-1} = 1.73\,\mathrm{GeV}$, $m_\pi = 139\,\mathrm{MeV}$, $m_\mu = 106\,\mathrm{MeV}$.
- We use Lanczos, AMA, and zMobius techniques to speed up the computations.
- 65 configurations are used. They each are separated by 20 MD time units.

$$a_\mu^{\mathrm{cHLbL}} = (11.60 \pm 0.96) \times 10^{-10} \quad (30)$$
$$a_\mu^{\mathrm{dHLbL}} = (-6.25 \pm 0.80) \times 10^{-10} \quad (31)$$
$$a_\mu^{\mathrm{cHLbL}} = (5.35 \pm 1.35) \times 10^{-10} \quad (32)$$

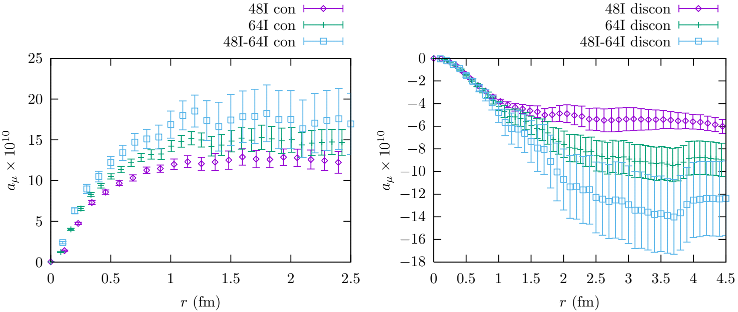
- We are using **Domain wall fermion (DWF)** in all our lattice calculations for HLbL. DWF respects Chiral symmetry, which helps systematically eliminating the $\mathcal{O}(a)$ discretization error. The remaing discretization error are in general quite small. The fifth dimension is needed in order to fullfil the Chiral symmetry. This results numerical cost proportion to the length in the fifth dimension, L_s, and large L_s is needed to reach the Chiral limit.
- **Mobius DWF** allows us to use a smaller value for L_s and having almost the same Chiral property. For 48I, we use $L_s = 24$ (also in evolution) to mimic the original $L_s = 48$ DWF.
- The **zMobius** formulation allows us to obtain a very good approximate of the original (M)DWF propagator with a significantly reduced L_s. For 48I, we further reduce the L_s from 24 to 10. **PoS LATTICE2015, 019 (2016)**
- **Multigrid Lanczos** algorithm help us efficiently generate the low modes of the DWF operator which accelerate the inversion roughly by a factor of 20 for light quarks. **arXiv:1710.06884**
- **All-mode-averaging (AMA)** allows us to perform the inversion with much less iterations most of the time, and only compute the small correction term for a small portion of the entire calculation. This can bring an addition factor of 5 speed up. **Phys. Rev. D 91, no. 11, 114511 (2015)**
- We use highly optimized DWF Dirac operator inverter from the **BFM** and **Grid** to perform the inversion. `https://github.com/paboyle`



- Left: **connected diagrams**. Right: leading **disconnected diagrams**.
- $48^3 \times 96$ lattice, with $a^{-1} = 1.73\,\mathrm{GeV}$, $m_\pi = 135\,\mathrm{MeV}$ (corrected from $139\,\mathrm{MeV}$).
- 65 configs for connected, 99 configs for disconnected diagrams.

$$a_\mu^{\mathrm{cHLbL}} = (12.35 \pm 1.18) \times 10^{-10} \quad (33)$$
$$a_\mu^{\mathrm{dHLbL}} = (-6.15 \pm 0.61) \times 10^{-10} \quad (34)$$
$$a_\mu^{\mathrm{cHLbL}} = (6.21 \pm 1.41) \times 10^{-10} \quad (35)$$

This is slightly partial quenched calculation performed on the $139\,\mathrm{MeV}$ pion mass ensemble.



- Left: **connected diagrams**. Right: leading **disconnected diagrams**.
- $48^3 \times 96$ lattice, with $a^{-1} = 1.73\,\mathrm{GeV}$, $m_\pi = 135\,\mathrm{MeV}$ (corrected from $139\,\mathrm{MeV}$).
- $64^3 \times 128$ lattice, with $a^{-1} = 2.36\,\mathrm{GeV}$, $m_\pi = 135\,\mathrm{MeV}$.

$$a_\mu^{\mathrm{cHLbL}} = (16.94 \pm 3.77) \times 10^{-10} \tag{36}$$
$$a_\mu^{\mathrm{dHLbL}} = (-12.29 \pm 3.34) \times 10^{-10} \tag{37}$$
$$a_\mu^{\mathrm{HLbL}} = (4.66 \pm 4.38) \times 10^{-10} \tag{38}$$

HLbL: hybrid continuum RBC-UKQCD prelim

- Very large statistical error (48I: $(6.21 \pm 1.41) \times 10^{-10}$, 48I-64I: $(4.66 \pm 4.38) \times 10^{-10}$).
- Can we do better than this? => Split the a_μ into two parts:

$$a_\mu = a_\mu^{\text{short}} + a_\mu^{\text{long}} \tag{39}$$

- $a_\mu^{\text{short}} = a_\mu(r \leqslant 1\,\text{fm})$: most of the contribution, small statistical error.
- $a_\mu^{\text{long}} = a_\mu(r > 1\,\text{fm})$: small contribution, large statistical error.

Treat a_μ^{short} and a_μ^{long} separately:

- a_μ^{short}: just like before, continuum extrapolation assuming a^2 scaling.
- a_μ^{long}: simply use the results from 48I, estimate the $\mathcal{O}(a^2)$ error.

$$a_\mu^{\text{cHLbL}} = (16.94 \pm 3.77) \times 10^{-10} \Rightarrow (17.35 \pm 1.97_{\text{stat}} \pm 0.20_{\text{sys},a^2}) \times 10^{-10} \tag{40}$$

$$a_\mu^{\text{dHLbL}} = (-12.29 \pm 3.34) \times 10^{-10} \Rightarrow (-7.21 \pm 1.03_{\text{stat}} \pm 1.02_{\text{sys},a^2}) \times 10^{-10} \tag{41}$$

$$a_\mu^{\text{HLbL}} = (4.66 \pm 4.38) \times 10^{-10} \Rightarrow (5.06 \pm 3.67_{\text{stat}} \pm 0.20_{\text{sys},a^2}) \times 10^{-10} \tag{42}$$

- The above systematic error is the estimated $\mathcal{O}(a^2)$ error. In addition, there can be additional $\mathcal{O}(a^4)$ error.

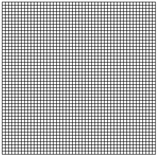
HLbL: RBC-UKQCD lattices

- Photons: Feynman gauge, QED_L [Hayakawa and Uno, 2008] (omit all modes with $\vec{q} = 0$)
- Gluons: Iwasaki (I) (+DSDR) gauge action (RG improved, plaquette+rectangle)
- muons: $L_s = \infty$ free domain-wall fermions (DWF)
- quarks: Möbius-DWF
- Lanczos, AMA, and zMöbius techniques used to speed up the calculation

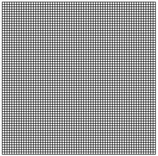
2+1f Möbius-DWF, I and I-DSDR physical point QCD ensembles (RBC/UKQCD) [Blum et al., 2014]

	48I	64I	24D	32D	48D
a^{-1} (GeV)	1.73	2.36	1.0	1.0	1.0
a (fm)	0.114	0.084	0.2	0.2	0.2
L (fm)	5.47	5.38	4.8	6.4	9.6
L_s	48	64	24	24	24
m_π (MeV)	139	135	140	140	140
m_μ (MeV)	106	106	106	106	106
meas (con, disco)	65, 99	43, 44	158, 157	71, 70	64, 0

HLbL: RBC-UKQCD lattices

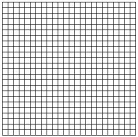


48I: $48^3 \times 96$, $5.5\,\mathrm{fm}$ box

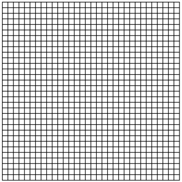


64I: $64^3 \times 128$, $5.5\,\mathrm{fm}$ box

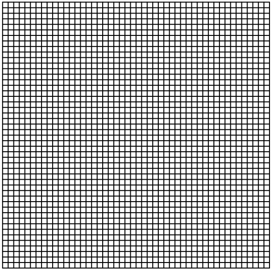
Phys. Rev. D 93, 074505 (2016)



24D: $24^3 \times 64$, $4.8\,\mathrm{fm}$ box



32D: $32^3 \times 64$, $6.4\,\mathrm{fm}$ box

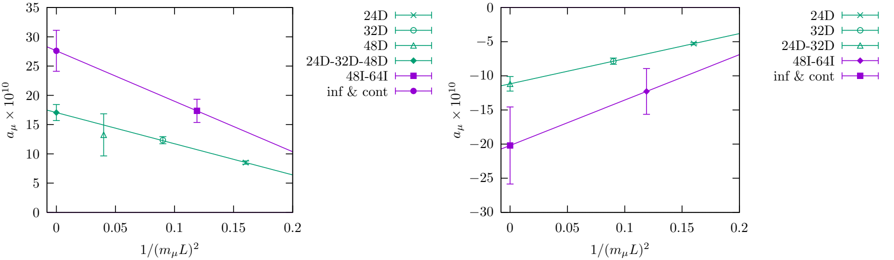


48D: $48^3 \times 64$, $9.6\,\mathrm{fm}$ box

32Dfine: $32^3 \times 64$, $4.8\,\mathrm{fm}$ box

- MDWF+Iwasaki: continuum limit ($5.4\,\mathrm{fm}$)
- MDWF+DSDR: $a^{-1} = 1.015\,\mathrm{GeV}$: $24^3 \times 64$ ($4.8\,\mathrm{fm}$), $32^3 \times 64$ ($6.4\,\mathrm{fm}$), $48^3 \times 64$ ($9.6\,\mathrm{fm}$).
- MDWF+DSDR: $a^{-1} = 1.371\,\mathrm{GeV}$: $32^3 \times 64$ ($4.6\,\mathrm{fm}$).

$$F_2(a, L) = F_2\left(1 - \frac{c_1}{(m_\mu L)^2}\right)(1 - c_2\, a^2) \qquad (43)$$



Connected diagrams　　Disconnected diagrams

$$a_\mu^{\mathrm{cHLbL}} = (27.61 \pm 3.51_{\mathrm{stat}} \pm 0.32_{\mathrm{sys},a^2}) \times 10^{-10} \qquad (44)$$

$$a_\mu^{\mathrm{dHLbL}} = (-20.20 \pm 5.65_{\mathrm{stat}}) \times 10^{-10} \qquad (45)$$

$$a_\mu^{\mathrm{HLbL}} = (7.41 \pm 6.32_{\mathrm{stat}} \pm 0.32_{\mathrm{sys},a^2}) \times 10^{-10} \qquad (46)$$

Outline

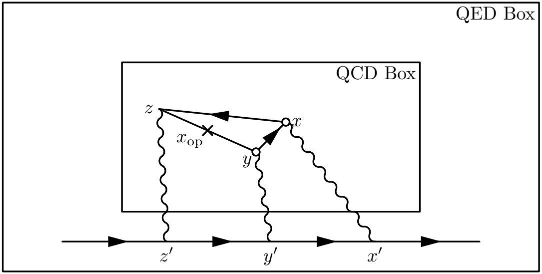
1. Introduction
2. HLbL: QED_L approach
3. **HLbL: QED_∞ approach**
   - QED kernel and subtraction
   - QCD calculation
4. HVP

HLbL: QCD box inside QED box

$$\mathcal{F}^C_\nu(x,y,z,x_{\rm op}) \;=\; (-ie)^6\,\mathcal{G}_{\rho,\sigma,\kappa}(x,y,z)\mathcal{H}_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\rm op})$$

The QED part, $\mathcal{G}_{\rho,\sigma,\kappa}(x,y,z)$ can be evaluated in infinite volume QED box.

The QCD part, $\mathcal{H}_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\rm op})$ can be evaluated in a finite volume QCD box.



$$i^3\,\mathcal{G}_{\rho,\sigma,\kappa}(x,y,z) \;=\; \mathfrak{G}_{\rho,\sigma,\kappa}(x,y,z)+\mathfrak{G}_{\sigma,\kappa,\rho}(y,z,x)+\text{other 4 permutations.} \qquad (47)$$

$$\mathfrak{G}_{\rho,\sigma,\kappa}(x,y,z) \;=\; e^{m_\mu(t_{\rm snk}-t_{\rm src})}\sum_{x',y',z'} G_{\rho,\rho'}(x,x')G_{\sigma,\sigma'}(y,y')G_{\kappa,\kappa'}(z,z') \qquad (48)$$
$$\times\sum_{\vec{x}_{\rm snk},\vec{x}_{\rm src}} S_\mu(x_{\rm snk},x')i\gamma_{\rho'}S_\mu(x',y')i\gamma_{\sigma'}S_\mu(y',z')i\gamma_{\kappa'}S_\mu(z',x_{\rm src})$$

Position-space approach to hadronic light-by-light scattering in the muon $g-2$ on the lattice

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The anomalous magnetic moment of the muon currently exhibits a discrepancy of about three standard deviations between the experimental value and recent Standard Model predictions. The theoretical uncertainty is dominated by the hadronic vacuum polarization and the hadronic light-by-light (HLbL) scattering contributions, where the latter has so far only been fully evaluated using different models. To pave the way for a lattice calculation of HLbL, we present an expression for the HLbL contribution to $g-2$ that involves a multidimensional integral over a position-space QED kernel function in the continuum and a lattice QCD four-point correlator. We describe our semi-analytic calculation of the kernel and test the approach by evaluating the π^0-pole contribution in the continuum.

PoS(LATTICE2016)164

The RBC & UKQCD collaborations

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Outline

1. Introduction
2. HLbL: QED_L approach
3. HLbL: QED_∞ approach
   - **QED kernel and subtraction**
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How to evaluate $\mathfrak{G}_{\rho,\sigma,\kappa}(x,y,z)$? arXiv:1705.01067.

First, we need to regularize the infrared divergence in $\mathfrak{G}_{\rho,\sigma,\kappa}(x,y,z)$.

$$\mathfrak{G}_{\rho,\sigma,\kappa}(x,y,z) \;=\; \frac{1+\gamma_0}{2}[(a_{\rho,\sigma,\kappa}(x,y,z))_k\Sigma_k + i\,b_{\rho,\sigma,\kappa}(x,y,z)]\frac{1+\gamma_0}{2} \tag{49}$$

where $a_{\rho,\sigma,\kappa}(x,y,z)$ and $b_{\rho,\sigma,\kappa}(x,y,z)$ are real functions.

$$\mathfrak{G}^{(1)}_{\rho,\sigma,\kappa}(x,y,z) \;=\; \frac{1}{2}\mathfrak{G}_{\rho,\sigma,\kappa}(x,y,z) + \frac{1}{2}[\mathfrak{G}_{\kappa,\sigma,\rho}(z,y,x)]^\dagger \tag{50}$$

It turned out that $\mathfrak{G}^{(1)}_{\rho,\sigma,\kappa}(x,y,z)$ is infrared finite.

$$\begin{aligned}\mathfrak{G}^{(1)}_{\rho,\sigma,\kappa}(x,y,z) \;=\;& \frac{\gamma_0+1}{2}\,i\gamma_\sigma\big(\partial\!\!\!/_\zeta+\gamma_0+1\big)i\,\gamma_\kappa\big(\partial\!\!\!/_\xi+\gamma_0+1\big)i\,\gamma_\rho\,\frac{\gamma_0+1}{2} \\ &\times\int \frac{d^4\eta}{4\,\pi^2}\,\frac{f(\eta-y+\zeta)\,f(x-\eta+\xi)-f(y-\eta+\zeta)\,f(\eta-x+\xi)}{2(\eta-z)^2}\bigg|_{\xi=\zeta=0}\end{aligned} \tag{51}$$

$$f(x)=f(|x|,x_t/|x|) \;=\; \frac{1}{8\pi^2}\int_0^1 d\,y\,e^{-yx_t}K_0(y\,|x|) \tag{52}$$

The 4 dimensional integral is calculated numerically with the CUBA library cubature rules.

Eventually, we need to compute

$$\sum_{x,y,z} \mathfrak{G}_{\rho,\sigma,\kappa}(x,y,z)\,\mathcal{H}^{C}_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\mathrm{op}}) \tag{53}$$

$\mathcal{H}^{C}_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\mathrm{op}})$ satisfies current conservation condition, which implies:

$$\sum_{x} \mathcal{H}^{C}_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\mathrm{op}}) \;=\; 0 \tag{54}$$

$$\sum_{z} \mathcal{H}^{C}_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\mathrm{op}}) \;=\; 0 \tag{55}$$

So, we have some freedom in changing $\mathfrak{G}_{\rho,\sigma,\kappa}(x,y,z)$. One choice we find particularly helpful is:

$$\mathfrak{G}^{(2)}_{\rho,\sigma,\kappa}(x,y,z) \;=\; \mathfrak{G}^{(1)}_{\rho,\sigma,\kappa}(x,y,z) - \mathfrak{G}^{(1)}_{\rho,\sigma,\kappa}(y,y,z) - \mathfrak{G}^{(1)}_{\rho,\sigma,\kappa}(x,y,y) + \mathfrak{G}^{(1)}_{\rho,\sigma,\kappa}(y,y,y)$$

Consequence of current conservation

Consider a vector field $J_\rho(x)$. It satisfies two conditions:

- $\partial_\rho J_\rho(x) = 0$.
- $J_\rho(x) = 0$ if $|x|$ is large.

We can conclude (the result is a little bit unexpected, but actually correct):

$$\int d^4x \, J_\rho(x) \; = \; \int d^4x \, \partial_\sigma(x_\rho J_\sigma(x)) = 0 \tag{56}$$

In three dimension, this result have a consequence which is well-known.

Consider a finite size system with stationary current. We then have

- $\vec{\nabla} \cdot \vec{j}(\vec{x}) = 0$, because of current conservation.
- $\vec{j}(\vec{x}) = 0$ if $|\vec{x}|$ large, because the system if of finite size.

Within a constant external magnetic field $\vec{B}$, the total magnetic force should be

$$\int \, [\vec{j}(\vec{x}) \times \vec{B}] \, d^3x \; = \; \left[\int \vec{j}(\vec{x}) \, d^3x \right] \times \vec{B} = 0 \tag{57}$$

PHYSICAL REVIEW D **96,** 034515 (2017)

Using infinite-volume, continuum QED and lattice QCD for the hadronic light-by-light contribution to the muon anomalous magnetic moment

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(Received 21 May 2017; published 22 August 2017)

In our previous work, Blum *et al.* [Phys. Rev. Lett. **118**, 022005 (2017)], the connected and leading disconnected hadronic light-by-light contributions to the muon anomalous magnetic moment $(g-2)$ have been computed using lattice QCD ensembles corresponding to physical pion mass generated by the RBC/UKQCD Collaboration. However, the calculation is expected to suffer from a significant finite-volume error that scales like $1/L^2$ where L is the spatial size of the lattice. In this paper, we demonstrate that this problem is cured by treating the muon and photons in infinite-volume, continuum QED, resulting in a weighting function that is precomputed and saved with affordable cost and sufficient accuracy. We present numerical results for the case when the quark loop is replaced by a muon loop, finding the expected exponential approach to the infinite volume limit and consistency with the known analytic result. We have implemented an improved weighting function which reduces both discretization and finite-volume effects arising from the hadronic part of the amplitude.

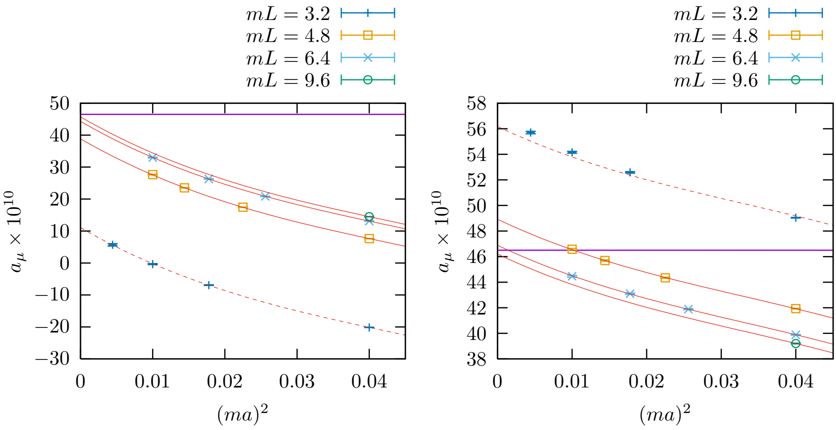
DOI: 10.1103/PhysRevD.96.034515

I. INTRODUCTION

Precision measurements of lepton magnetic dipole moments provide a powerful tool for testing the standard model (SM) of particle physics at high precision. The

which outweighs a loss in experimental precision. With the τ being experimentally inaccessible, a_μ is the most promising channel to reveal physics beyond the standard model.

- Compare the two $\mathfrak{G}_{\rho,\sigma,\kappa}(x, y, z)$ in **pure QED computation**.

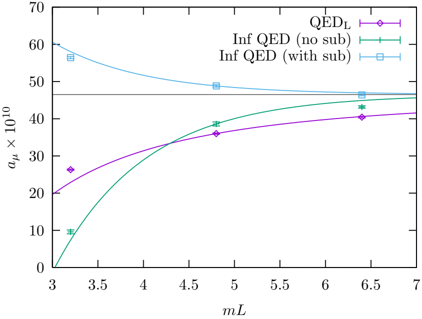


Original QED kernel

Subtracted QED kernel

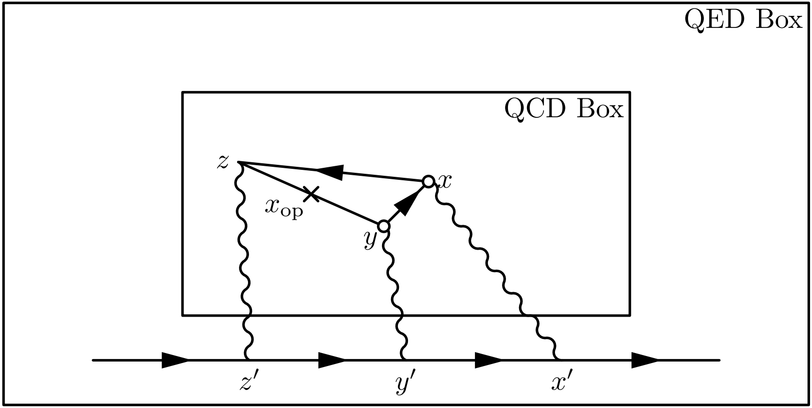
- Notice the vertical scales in the two plots are different.

- Compare the finite volume effects in different approaches in **pure QED computation**,



- QED$_\mathrm{L}$: $\mathcal{O}(1/L^2)$ finite volume effect, because the photons are emitted from a conserved loop. Phys.Rev. D93 (2016) 1, 014503.
- Inf QED (no sub): $\mathcal{O}(e^{-mL})$ finite volume effect. Everything except the four-point-correlation function is evaluated in infinite volume. arXiv:1705.01067.
- Inf QED (with sub): smaller $\mathcal{O}(e^{-mL})$ finite volume effect. arXiv:1705.01067.

Outline

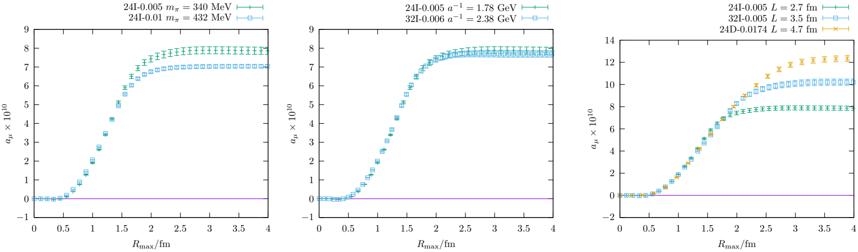


We will plot the data based on $R_{\max} = \max\{|x-y|, |y-z|, |z-x|\}$.

For π^0 exchange contribution:

- $R_{\max}$ is the limit on the distance the π^0 travels.

HLbL: $m_\pi = 340$ MeV RBC-UKQCD prelim



Pion mass dependence | Discretization effects | Finite volume effects

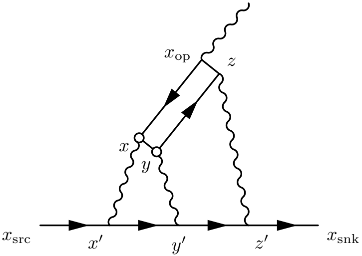
Figure 3. RBC-UKQCD preliminary results. $R_{\max} = \max\{|x-y|, |y-z|, |z-x|\}$.

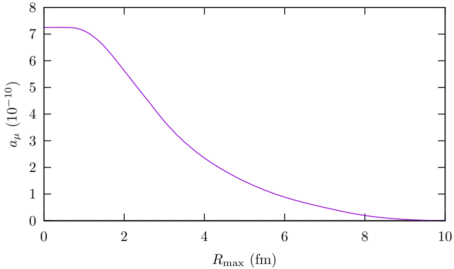
$$a_\mu^{\text{cHLbL}}(m_\pi = 340\,\text{MeV}, L = 2.66\,\text{fm}, a = 0) = 8.67(47) \times 10^{-10} \tag{58}$$

$$a_\mu^{\text{cHLbL}}(m_\pi = 340\,\text{MeV}, L = 3.54\,\text{fm}, a = 0) = 11.02(71) \times 10^{-10} \tag{59}$$

There is sizable difference between these two volumes. Further study is required to extrapolate to infinite volume limit.

HLbL: Long distance π^0-pole RBC-UKQCD prelim





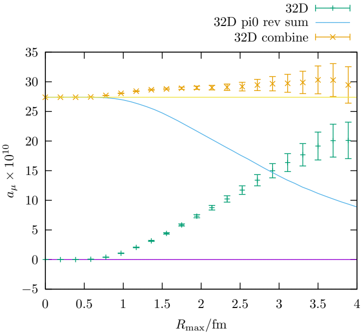
$$R_{\rm max} = \max\{|x-y|, |y-z|, |z-x|\} \quad (60)$$

Currently, we use the LMD (Lowest Meson Dominance) pion TFF model [Talk by Luchang Jin and Taku Izubuchi at Mainz $g-2$ workshop in June 2018] with the following parameters:

$$m_V = 770\,\mathrm{MeV} \qquad F_\pi = 93\,\mathrm{MeV} \quad (61)$$

- Connected diagram: multiply by $34/9$.
- Leading disconnected diagram: multiply by $-25/9$.
- [JHEP 1609 (2016) 113, PoS LATTICE2016 (2016) 181]

For the QED part, we use the weighting function developed in [arXiv:1705.01067] by us.



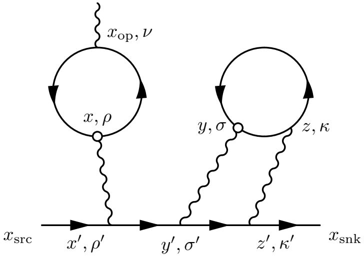
Combine

Lattice with

Pion pole

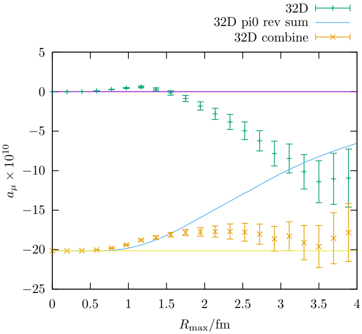
LMD model.

- $\max(|x-y|, |x-z|, |y-z|) = R_{\max}$
- Short distance: lattice calculation with 32D (6.4 fm, 1.015 GeV) (partial sum upto $R_{\max}$).
- Long distance: LMD model multiplied by $34/9$ (partial sum from $R_{\max}$ upto infinity).
- At $R_{\max} = 2.5\,\mathrm{fm}$, the combined result is $a_\mu^{\mathrm{cHLbL}} = 29.19(0.73)_{\mathrm{stat}} \times 10^{-10}$.
- Previous extrapolated results with $\mathrm{QED_L}$ is $a_\mu^{\mathrm{cHLbL}} = 27.61(3.51)_{\mathrm{stat}}(0.32)_{\mathrm{sys},a^2} \times 10^{-10}$.



- For QED_L, we can compute the QED function for all x given the y location fixed and z summed over. Allow us to compute all combination of x, y with little cost.
- For QED_∞, although we can compute all the function $\mathcal{G}_{\rho,\sigma,\kappa}(x, y, z)$ simply by interpolate, we cannot easily compute this function (even after fixing y) for all x and z, simply because of its cost is proportion to Volume^2.
- However, we with QED_∞ and interpolation, we can freely choose which coordinates we compute. For example, we may compute all z for $|z - y| \leqslant 5$, and sample z for $|z - y| > 5$.

Leading discon diagram $m_\pi = 140$ MeV RBC-UKQCD prelim



Combine

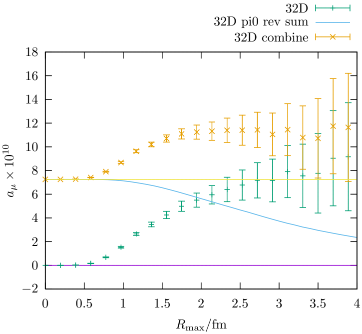
Lattice with

Pion pole

LMD model.

- $\max(|x-y|, |x-z|, |y-z|) = R_{\max}$
- Short distance: lattice calculation with 32D (6.4 fm, 1.015 GeV) (partial sum upto $R_{\max}$).
- Long distance: LMD model multiplied by $-25/9$. (partial sum from $R_{\max}$ upto infinity).
- At $R_{\max} = 2.5\,\mathrm{fm}$, the combined result is $a_\mu^{\mathrm{discon}} = -17.79(1.13)_{\mathrm{stat}} \times 10^{-10}$.

  Previous extrapolated results with $\mathrm{QED_L}$ is $a_\mu^{\mathrm{discon}} = -20.20(5.65)_{\mathrm{stat}} \times 10^{-10}$.



Combine

Lattice with

Pion pole

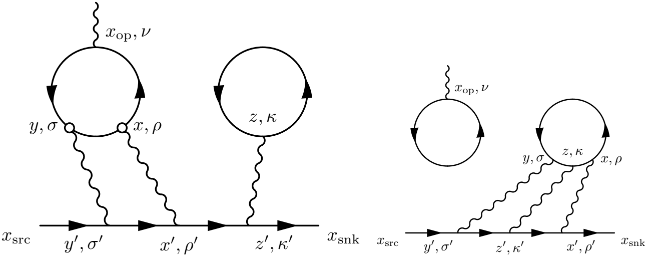
LMD model.

- $\max(|x-y|, |x-z|, |y-z|) = R_{\max}$
- Short distance: lattice calculation with 32D (6.4 fm, 1.015 GeV) (partial sum upto $R_{\max}$).
- Long distance: LMD model. (partial sum from $R_{\max}$ upto infinity).
- At $R_{\max} = 2.5\,\text{fm}$ the combined results is $a_\mu^{\text{total}} = 11.40(1.27)_{\text{stat}} \times 10^{-10}$

  the part from lattice is $6.78(1.27)_{\text{stat}} \times 10^{-10}$.

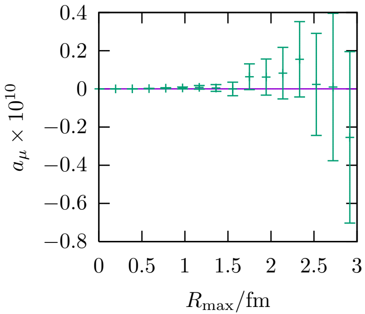
  Previous extrapolated results with QED_L is $a_\mu^{\text{total}} = 7.41(6.32)_{\text{stat}}(0.32)_{\text{sys},a^2} \times 10^{-10}$.

HLbL: subleading discon



- These are the subleading disconnected diagrams in the SU(3) limit.
- The right diagram has a factor of $1/3$ suppression from the multiplicity of the diagram compare with the left diagram, i.e. the external photon is more likely to be on the loop with three photons.
- For the left diagram, the moment method works just like the connected case. We can sample x, y and sum over z. The M^2 trick can be used for the x, y sampling. Low-modes-averaging for the loop with z.
- For the right diagram, The moment method still works, however, we have to use a point on the other loop as the reference point, which may be more noisy. But as mentioned above, the right diagram is more suppressed.

Subleading discon $m_\pi = 140$ MeV RBC-UKQCD prelim



- $\max(|x-y|, |x-z|, |y-z|) = R_{\mathrm{max}}$
- Lattice calculation with 24D (4.8 fm, 1.015 GeV) (partial sum upto R_{max}).

Outline

The RBC & UKQCD collaborations

BNL and BNL/RBRC

Yasumichi Aoki (KEK)
Taku Izubuchi
Yong-Chull Jang
Chulwoo Jung
Meifeng Lin
Aaron Meyer
Hiroshi Ohki
Shigemi Ohta (KEK)
Amarjit Soni

UC Boulder

Oliver Witzel

CERN

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Columbia University

Ryan Abbot
Norman Christ
Duo Guo
Christopher Kelly
Bob Mawhinney
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Luigi Del Debbio
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Vera Gülpers
Tadeusz Janowski
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Antonin Portelli
Tobias Tsang
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KEK

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University of Liverpool

Nicolas Garron

MIT

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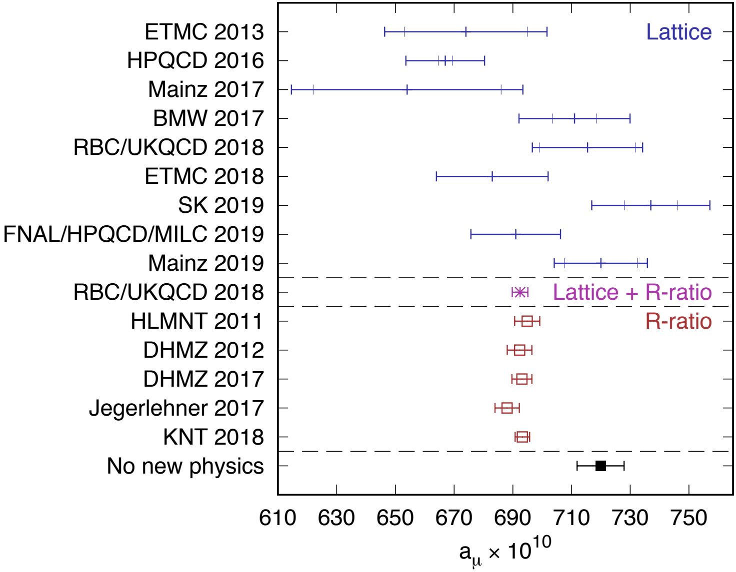
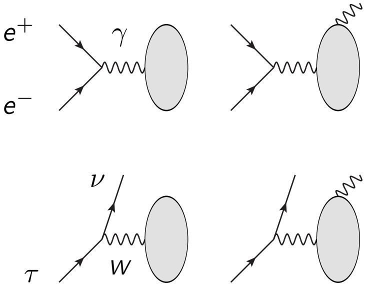
Chris Sachrajda

Stony Brook University

Jun-Sik Yoo
Sergey Syritsyn (RBRC)


ETMC 2013
HPQCD 2016
Mainz 2017
BMW 2017
RBC/UKQCD 2018
ETMC 2018
SK 2019
FNAL/HPQCD/MILC 2019
Mainz 2019
RBC/UKQCD 2018
HLMNT 2011
DHMZ 2012
DHMZ 2017
Jegerlehner 2017
KNT 2018
No new physics
Lattice
Lattice + R-ratio
R-ratio
610 630 650 670 690 710 730 750
$a_\mu \times 10^{10}$

HVP: dispersive method - overview



$e^+e^- \to \mathrm{hadrons}(\gamma)$

$$J_\mu = V_\mu^{I=1,I_3=0} + V_\mu^{I=0,I_3=0}$$

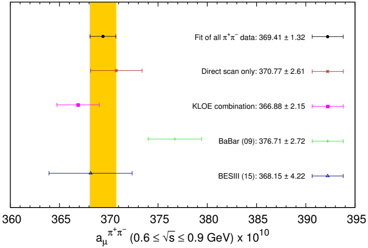
$\tau \to \nu\mathrm{hadrons}(\gamma)$

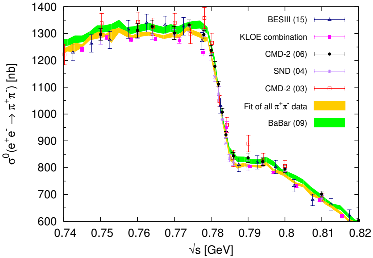
$$J_\mu = V_\mu^{I=1,I_3=\pm1} - A_\mu^{I=1,I_3=\pm1}$$

Knowledge of isospin-breaking corrections and separation of vector and axial-vector components needed to use τ decay data. Can do this from LQCD+QED (Bruno, Izubuchi, CL, Meyer, 1811.00508)!

Can have both energy-scan and ISR setup.

Tension in 2π experimental input. BaBar and KLOE central values differ by $\delta a_\mu = 9.8(3.5) \times 10^{-10}$, compare to quoted total uncertainties of dispersive results of order $\delta a_\mu = 3 \times 10^{-10}$.





Conflicting input limits the precision and reliability of the dispersive results. Can we replace some of this data with LQCD+QED?

Looking for more data and insight: energy-scans update from CMD-3 in Novosibirsk and ISR updates from KLOE2, BaBar, Belle, BESIII and BelleII.

Lattice Calculation of the Lowest-Order Hadronic Contribution to the Muon Anomalous Magnetic Moment

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(Received 18 December 2002; published 30 July 2003)

We present a quenched lattice calculation of the lowest order $[\mathcal{O}(\alpha^2)]$ hadronic contribution to the anomalous magnetic moment of the muon which arises from the hadronic vacuum polarization. A general method is presented for computing entirely in Euclidean space, obviating the need for the usual dispersive treatment which relies on experimental data for e^+e^- annihilation to hadrons. While the result is not yet of comparable precision to those state-of-the-art calculations, systematic improvement of the quenched lattice computation to this level is straightforward and well within the reach of present computers. Including the effects of dynamical quarks is conceptually trivial; the computer resources required are not.

DOI: 10.1103/PhysRevLett.91.052001 PACS numbers: 12.38.Gc, 13.40.Em, 14.60.Ef, 14.65.Bt

The magnetic moment of the muon is defined by the $q^2 \to 0$ (static) limit of the vertex function which describes the interaction of the electrically charged muon with the photon,

$$\Gamma_\rho(p_2, p_1) = \gamma_\rho F_1(q^2) - \frac{i}{4m_\mu}(\gamma_\rho \not{q} - \not{q}\gamma_\rho)F_2(q^2), \quad (1)$$

where m_μ is the muon mass, $q = p_2 - p_1$ is the photon momentum, and p_1, p_2 are the incoming and outgoing momentum of the muon. Lorentz invariance and current conservation have been used in obtaining Eq. (1). Form factors $F_1(q^2)$ and $F_2(q^2)$ contain all information about the muon's interaction with the electromagnetic field. In particular, $F_1(0) = 1$ is the electric charge of the muon in

though a discrepancy with a calculation that uses τ decay data may indicate a theory error as large as 5% [2] and reduces the disagreement with experiment to roughly 1.6 standard deviations. A purely theoretical, first principles, calculation has been lacking and is desirable, and also has several advantages over the conventional approach. For instance, the separation of QED effects from hadronic corrections is automatic, as is the treatment of isospin corrections if different quark masses are used in the simulation. Thus, it is possible that lattice calculations may eventually help to settle the above-mentioned discrepancy between $e^+ e^-$ annihilation and τ decay.

The method described here is simple and direct. We begin with Ref. [5] which describes the computation of

Two-Flavor QCD Correction to Lepton Magnetic Moments at Leading Order in the Electromagnetic Coupling

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(Received 28 March 2011; published 17 August 2011)

We present a reliable nonperturbative calculation of the QCD correction, at leading order in the electromagnetic coupling, to the anomalous magnetic moment of the electron, muon, and tau leptons using two-flavor lattice QCD. We use multiple lattice spacings, multiple volumes, and a broad range of quark masses to control the continuum, infinite-volume, and chiral limits. We examine the impact of the commonly ignored disconnected diagrams and introduce a modification to the previously used method that results in a well-controlled lattice calculation. We obtain $1.513(43) \times 10^{-12}$, $5.72(16) \times 10^{-8}$, and $2.650(54) \times 10^{-6}$ for the leading-order two-flavor QCD correction to the anomalous magnetic moment of the electron, muon, and tau, respectively, each accurate to better than 3%.

DOI: 10.1103/PhysRevLett.107.081802 PACS numbers: 13.40.Em, 12.38.Gc, 14.60.Ef

Introduction.—The experimental [1] and theoretical [2] determinations of the anomalous magnetic moment of the muon a_μ have both reached an accuracy that is better than six parts per million. This high precision reveals a discrepancy of over 3 standard deviations (3σ), which raises the possibility of physics beyond the standard model. However, the dominant error in the theory computation is due to hadronic effects that are currently not calculated but are instead either separately measured or simply modeled. This obscures the significance of the 3σ effect and makes it

perturbative expansion in the electromagnetic coupling α. Contributions from QCD first occur at the order α^2 and can be written as [4]

$$a_l^{\mathrm{hvp}} = \alpha^2 \int_0^\infty dQ^2 \frac{1}{Q^2} w(Q^2/m_l^2)\Pi_R(Q^2), \qquad (1)$$

where m_l is the mass of the lepton, Q is the Euclidean momentum, and $w(Q^2/m_l^2)$ is a known function. The combination $\Pi_R(Q^2) = \Pi(Q^2) - \Pi(0)$ is the renormalized hadronic vacuum polarization function $\Pi(Q^2)$, which is

The 2011 KWLA panel is proud to award

The 2011 Ken Wilson Lattice Award

To: Xu Feng, Marcus Petschlies, Karl Jansen, and Dru B. Renner

In recognition of their paper titled

Two-flavor QCD Correction to Lepton Magnetic Moments at Leading-Order in the Electromagnetic Coupling

The 2011 KWLA Panel Members

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Andre Walker-Loud
Joe Wasem

Design by Mayia A. Vranas

Outline

1. Introduction
2. HLbL: QED_L approach
3. HLbL: QED_∞ approach
4. HVP
   - **Master formula**
   - Long distance part

HVP: master formula

Introduce the vector correlatiors in momentum space:

$$(\delta_{\mu,\nu} q^2 - q_\mu q_\nu)\Pi(q^2) = \sum_x e^{iq\cdot x}\langle J_\mu(x) J_\nu(0)\rangle$$

$$J_\mu(x) = i\sum_f Q_f \Psi_f(x)\gamma_\mu \Psi_f(x)$$

Obtain the LO HVP contribution to muon $g-2$:

$$a_\mu^{\text{HVP LO}} = 4\alpha^2 \int_0^\infty dq^2 f(q^2)[\Pi(q^2) - \Pi(q^2 = 0)]$$

where $f(q^2)$ is from perturbative calculation:

$$f(q^2) = \frac{m_\mu^2 q^2 Z^3(q^2)(1 - q^2 Z(q^2))}{1 + m_\mu^2 q^2 Z^2(q^2)}$$

$$Z(q^2) = \frac{\sqrt{q^4 + 4m_\mu^2 q^2} - q^2}{2m_\mu^2 q^2}$$

HVP: master formula

$$a_\mu^{\text{HVP LO}} = 4\alpha^2 \int_0^\infty dq^2 f(q^2)[\Pi(q^2) - \Pi(q^2 = 0)]$$

$$(\delta_{\mu,\nu} q^2 - q_\mu q_\nu)\Pi(q^2) = \sum_x e^{iq\cdot x} \langle J_\mu(x) J_\nu(0) \rangle$$

- The $a_\mu^{\text{HVP LO}}$ is a linear combination of $\Pi(q^2)$, which is a linear combination of $\langle J_\mu(x) J_\nu(0) \rangle$
- It should be possible to express $a_\mu^{\text{HVP LO}}$ as a linear combination of $\langle J_\mu(x) J_\nu(0) \rangle$ directly?
- Target: obtain $w(t)$ such that

$$a_\mu = \sum_{t=0}^{+\infty} w(t) C(t), \qquad C(t) = \frac{1}{3} \sum_{\vec{x}} \sum_{j=0,1,2} \langle J_j(\vec{x}, t) J_j(0) \rangle.$$

Obtain $\Pi(q^2)$ from $C(t)$:

$$(\delta_{\mu,\nu}q^2 - q_\mu q_\nu)\Pi(q^2) = \sum_x e^{iq\cdot x}\langle J_\mu(x)J_\nu(0)\rangle$$

$$\Downarrow$$

$$3q^2\Pi(q^2) = \sum_t e^{iqt}\sum_{\vec{x}}\sum_{j=0,1,2}\langle J_j(\vec{x},t)J_j(0)\rangle$$

$$\Downarrow$$

$$\begin{aligned}\Pi(q^2) &= \sum_t \frac{e^{iqt}}{q^2}\left[\frac{1}{3}\sum_{\vec{x}}\sum_{j=0,1,2}\langle J_j(\vec{x},t)J_j(0)\rangle\right] \\ &= \sum_t \frac{e^{iqt}}{q^2}C(t)\end{aligned}$$

At small t, $C(t) \sim \log(1/t)/t^3$. Therefore, the above summation over t will be badly divergent, i.e. $\Pi(q^2) \sim \log(1/a)/a^2$.

Consequence of current conservation

Consider a vector field $J_\rho(x)$. It satisfies two conditions:

- $\partial_\rho J_\rho(x) = 0$.
- $J_\rho(x) = 0$ if $|x|$ is large.

We can conclude (the result is a little bit unexpected, but actually correct):

$$\int d^4x \, J_\rho(x) \; = \; \int d^4x \, \partial_\sigma(x_\rho J_\sigma(x)) = 0 \tag{12}$$

In three dimension, this result have a consequence which is well-known.

Consider a finite size system with stationary current. We then have

- $\vec{\nabla} \cdot \vec{j}(\vec{x}) = 0$, because of current conservation.
- $\vec{j}(\vec{x}) = 0$ if $|\vec{x}|$ large, because the system if of finite size.

Within a constant external magnetic field $\vec{B}$, the total magnetic force should be

$$\int \left[\vec{j}(\vec{x}) \times \vec{B} \right] d^3x \; = \; \left[\int \vec{j}(\vec{x}) \, d^3x \right] \times \vec{B} = 0 \tag{13}$$

- In field theory, if you recall, the power divergence should not appear for vacuum polarization calculations. The reason is *current conservation*.

$$\partial^x_\mu \langle J_\mu(x) J_\nu(0)\rangle = 0 \quad \text{and} \quad \lim_{|x|\to\infty} \langle J_\mu(x) J_\nu(0)\rangle = 0$$

$$\Downarrow$$

$$\sum_x \langle J_\mu(x) J_\nu(0)\rangle = 0 \quad \Rightarrow \quad \sum_t C(t) = 0$$

$$\Downarrow$$

$$\Pi(q^2) = \sum_t \frac{e^{iqt} - 1}{q^2} C(t) \quad \Rightarrow \quad \Pi(q^2) = \sum_t \frac{\cos(qt) - 1}{q^2} C(t)$$

$$\Downarrow$$

$$\Pi(q^2) - \Pi(q^2 = 0) = \sum_t \left(\frac{\cos(qt) - 1}{q^2} + \frac{1}{2} t^2 \right) C(t)$$

- You can verify that $\Pi(q^2) \sim \log(1/a)$ and $\Pi(q^2) - \Pi(q^2 = 0)$ is finite in the $a \to 0$ limit.

HVP: master formula

Combine

$$a_\mu^{\text{HVP LO}} = 4\alpha^2 \int_0^\infty dq^2 f(q^2)[\Pi(q^2) - \Pi(q^2 = 0)]$$

and

$$\Pi(q^2) - \Pi(q^2 = 0) = \sum_t \left(\frac{\cos(qt) - 1}{q^2} + \frac{1}{2} t^2 \right) C(t)$$

One can obtain:

$$a_\mu = \sum_{t=0}^{+\infty} w(t) C(t)$$

Eur. Phys. J. A (2011) **47**: 148
DOI 10.1140/epja/i2011-11148-6

THE EUROPEAN PHYSICAL JOURNAL A

Regular Article – Theoretical Physics

Vector correlators in lattice QCD: Methods and applications

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Published online: 28 November 2011 –
Communicated by S. Hands

Abstract. We discuss the calculation of the leading hadronic vacuum polarization in lattice QCD. Exploiting the excellent quality of the compiled experimental data for the $e^+e^- \to$ hadrons cross-section, we predict the outcome of large-volume lattice calculations at the physical pion mass, and design computational strategies for the lattice to have an impact on important phenomenological quantities such as the leading hadronic contribution to $(g-2)_\mu$ and the running of the electromagnetic coupling constant. First, the $R(s)$ ratio can be calculated directly on the lattice in the threshold region, and we provide the formulae to do so with twisted boundary conditions. Second, the current correlator projected onto zero spatial momentum, in a Euclidean time interval where it can be calculated accurately, provides a potentially critical test of the experimental $R(s)$ ratio in the region that is most relevant for $(g-2)_\mu$. This observation can also be turned around: the vector correlator at intermediate distances can be used to determine the lattice spacing in fm, and we make a concrete proposal in this direction. Finally, we quantify the finite-size effects on the current correlator coming from low-energy two-pion states and provide a general parametrization

Editors' Suggestion

Calculation of the Hadronic Vacuum Polarization Contribution to the Muon Anomalous Magnetic Moment

T. Blum,[1] P. A. Boyle,[2] V. Gülpers,[3] T. Izubuchi,[4,5] L. Jin,[1,5] C. Jung,[4] A. Jüttner,[3] C. Lehner,[4,*] A. Portelli,[2] and J. T. Tsang[2]

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(Received 25 January 2018; published 12 July 2018)

We present a first-principles lattice QCD + QED calculation at physical pion mass of the leading-order hadronic vacuum polarization contribution to the muon anomalous magnetic moment. The total contribution of up, down, strange, and charm quarks including QED and strong isospin breaking effects is $a_\mu^{\mathrm{HVP\ LO}} = 715.4(18.7) \times 10^{-10}$. By supplementing lattice data for very short and long distances with R-ratio data, we significantly improve the precision to $a_\mu^{\mathrm{HVP\ LO}} = 692.5(2.7) \times 10^{-10}$. This is the currently most precise determination of $a_\mu^{\mathrm{HVP\ LO}}$.

DOI: 10.1103/PhysRevLett.121.022003

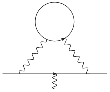
Introduction.—The anomalous magnetic moment of the muon a_μ is defined as the deviation of the Landé factor g_μ from Dirac's relativistic quantum mechanics result, $a_\mu = [(g_\mu - 2)/2]$. It is one of the most precisely determined quantities in particle physics and is currently known both experimentally (BNL E821) [1] and from a standard model

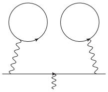
uncertainty by a factor of 4. First results of the E989 experiment may be available before the end of 2018 [9] such that a reduction in uncertainty of the $a_\mu^{\mathrm{HVP\ LO}}$ contribution is of timely interest.

In the following, we perform a complete first-principles calculation of $a_\mu^{\mathrm{HVP\ LO}}$ in lattice QCD + QED at physical

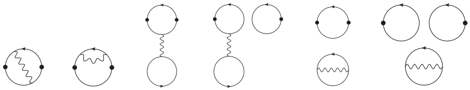
HVP: all diagrams

Isospin limit

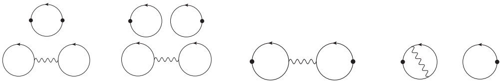




QED corrections

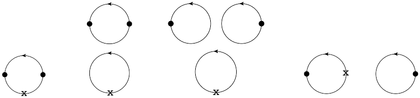


(a) V (b) S (c) T (d) T_d (e) D1 (f) $D1_d$



(g) D2 (h) $D2_d$ (i) F (j) D3

Strong isospin breaking



(a) M (b) R (c) R_d (d) O

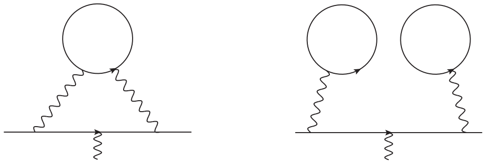
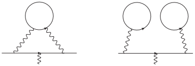
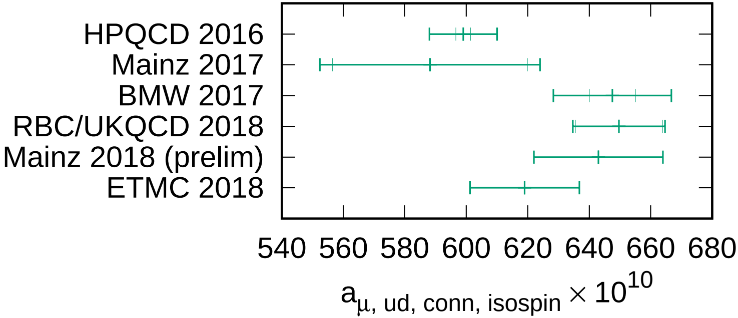
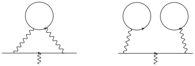
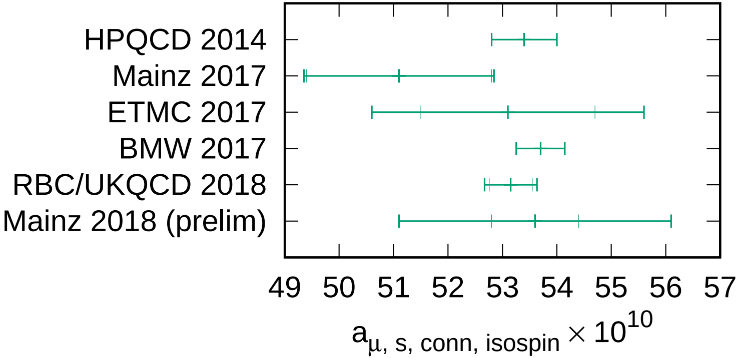


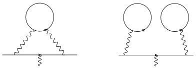
FIG. 1. Quark-connected (left) and quark-disconnected (right) diagram for the calculation of $a_\mu^{\mathrm{HVP\ LO}}$. We do not draw gluons but consider each diagram to represent all orders in QCD.

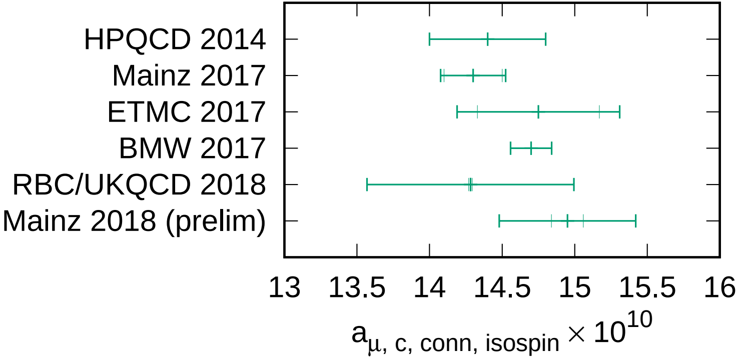


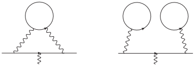

HPQCD 2016
Mainz 2017
BMW 2017
RBC/UKQCD 2018
Mainz 2018 (prelim)
ETMC 2018
540 560 580 600 620 640 660 680
a_{μ, ud, conn, isospin} × 10^10

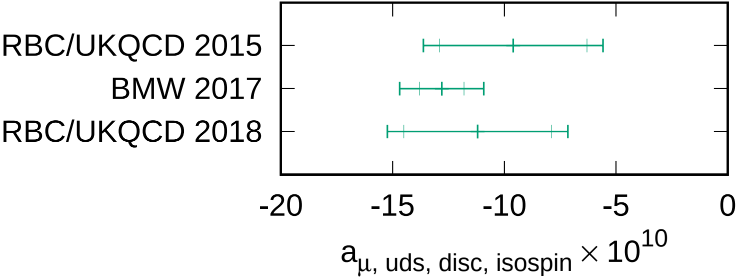


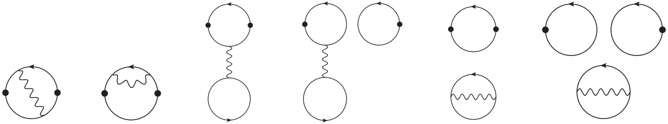

HPQCD 2014
Mainz 2017
ETMC 2017
BMW 2017
RBC/UKQCD 2018
Mainz 2018 (prelim)
49
50
51
52
53
54
55
56
57
$a_{\mu,\ s,\ conn,\ isospin} \times 10^{10}$



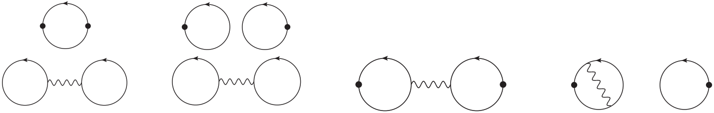

HPQCD 2014
Mainz 2017
ETMC 2017
BMW 2017
RBC/UKQCD 2018
Mainz 2018 (prelim)
13
13.5
14
14.5
15
15.5
16
$a_{\mu,\text{ c, conn, isospin}} \times 10^{10}$




RBC/UKQCD 2015
BMW 2017
RBC/UKQCD 2018
-20
-15
-10
-5
0
$a_{\mu,\text{ uds, disc, isospin}} \times 10^{10}$

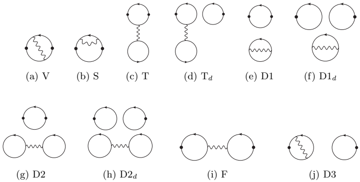


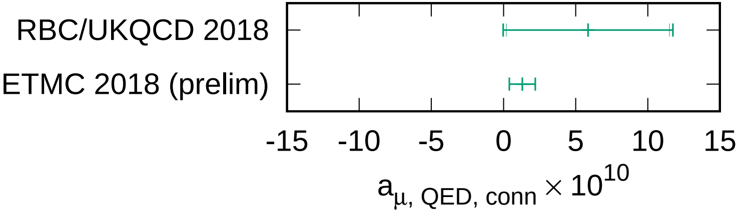
(a) V (b) S (c) T (d) T_d (e) D1 (f) $D1_d$

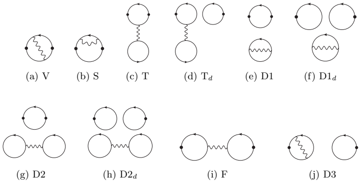


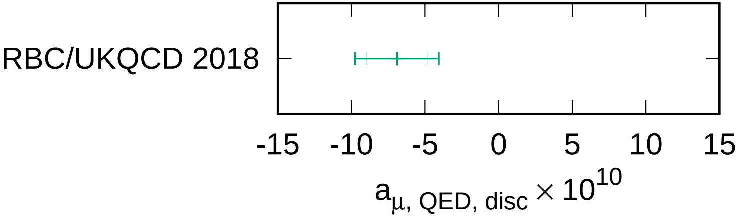
(g) D2 (h) $D2_d$ (i) F (j) D3

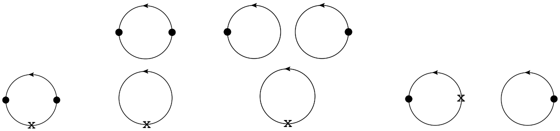
For diagram F we enforce exchange of gluons between the quark loops as otherwise a cut through a single photon line would be possible. This single-photon contribution is counted as part of the HVP NLO and not included for the HVP LO.


(a) V
(b) S
(c) T
(d) T_d
(e) D1
(f) $D1_d$
(g) D2
(h) $D2_d$
(i) F
(j) D3


RBC/UKQCD 2018
ETMC 2018 (prelim)
-15
-10
-5
0
5
10
15
$a_{\mu,\text{ QED, conn}} \times 10^{10}$

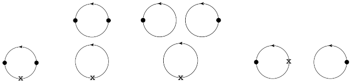

(a) V
(b) S
(c) T
(d) T_d
(e) D1
(f) $D1_d$
(g) D2
(h) $D2_d$
(i) F
(j) D3


RBC/UKQCD 2018
-15
-10
-5
0
5
10
15
$a_{\mu,\text{ QED, disc}} \times 10^{10}$

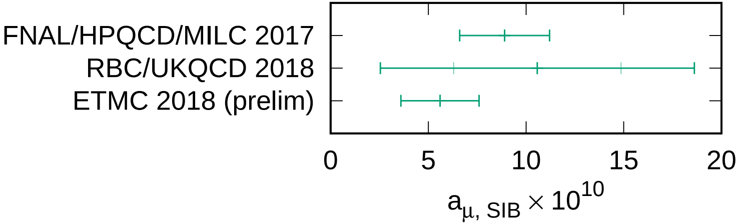


(a) M (b) R (c) R_d (d) O

For the HVP R is negligible since $\Delta m_u \approx -\Delta m_d$ and O is SU(3) and $1/N_c$ suppressed.


x
x
x
x

(a) M (b) R (c) R_d (d) O


FNAL/HPQCD/MILC 2017
RBC/UKQCD 2018
ETMC 2018 (prelim)
0
5
10
15
20
$a_{\mu,\ \mathrm{SIB}} \times 10^{10}$

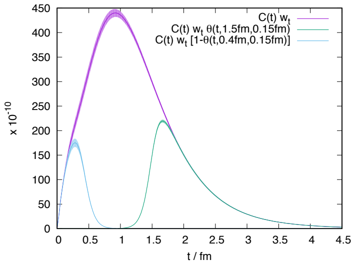
We therefore also consider a window method. Following Meyer-Bernecker 2011 and smearing over t to define the continuum limit we write

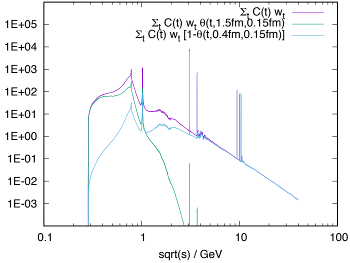
$$a_\mu = a_\mu^{\mathrm{SD}} + a_\mu^{\mathrm{W}} + a_\mu^{\mathrm{LD}}$$

with

$$\begin{aligned}
a_\mu^{\mathrm{SD}} &= \sum_t C(t) w_t [1 - \Theta(t, t_0, \Delta)]\,, \\
a_\mu^{\mathrm{W}} &= \sum_t C(t) w_t [\Theta(t, t_0, \Delta) - \Theta(t, t_1, \Delta)]\,, \\
a_\mu^{\mathrm{LD}} &= \sum_t C(t) w_t \Theta(t, t_1, \Delta)\,, \\
\Theta(t, t', \Delta) &= [1 + \tanh[(t - t')/\Delta]]/2\,.
\end{aligned}$$

In this version of the calculation, we use $C(t) = \frac{1}{12\pi^2}\int_0^\infty d(\sqrt{s}) R(s) s e^{-\sqrt{s}t}$ with $R(s) = \frac{3s}{4\pi\alpha^2}\sigma(s, e^+e^- \to \mathrm{had})$ to compute a_μ^{SD} and a_μ^{LD}.





Most of $\pi\pi$ peak is captured by window from $t_0 = 0.4$ fm to $t_1 = 1.5$ fm, so replacing this region with lattice data reduces the dependence on BaBar versus KLOE data sets.

Outline

- Main idea is that: one does not have to calculate the long distance part of the correlation function directly.

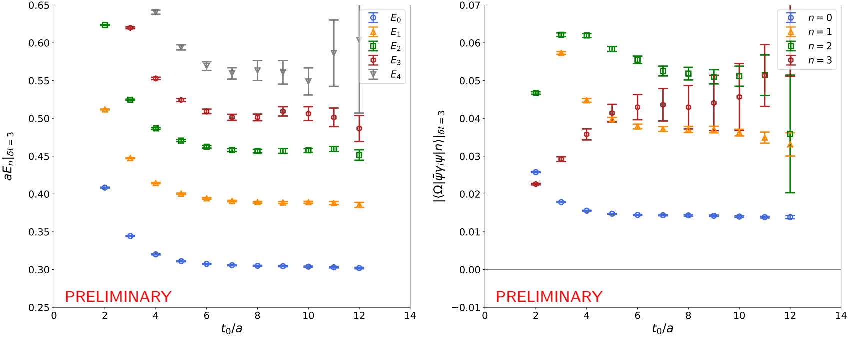
$$\begin{aligned} C(t) &= \frac{1}{3}\sum_{\vec{x}}\sum_{j=0,1,2}\langle J_j(\vec{x},t)J_j(0)\rangle \\ &= \sum_n \frac{V}{3}\sum_{j=0,1,2}\langle 0|J_j(0)|n\rangle\langle n|J_j(0)|0\rangle e^{-E_n t} \end{aligned}$$

- The summation over n is limited to zero momentum states and states are normalized to "1".

- At large t, only lowest few states contribute. We only need the matrix elements $\langle n|J_j(0)|0\rangle$ and the corresponding energy E_n.

- Need to study the spectrum of the $\pi\pi$ system!

HVP: GEVP Results - $J_\mu + 2\pi$ operators only



6-operator basis on 48I ensemble: local+smeared vector, $4\times(2\pi)$

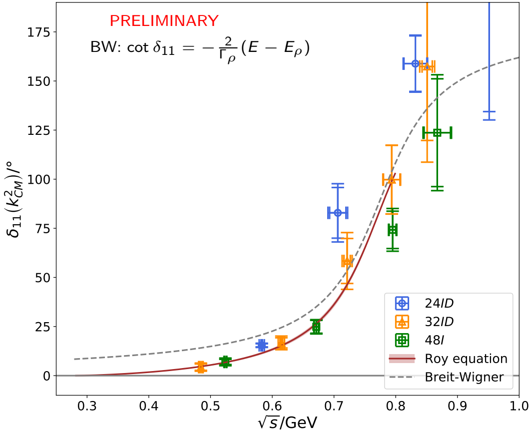
Data points from solving GEVP at fixed δt

$$C(t_0)\,V = C(t_0+\delta t)\,V\,\Lambda(\delta t)\,, \quad \Lambda_{nn}(\delta t) \sim e^{+E_n\delta t}$$

Excited state contaminations decay as $t_0, \delta t \to \infty$
moving right on plot $\implies$ asymptote to lowest states' spectrum & overlaps

Left: Spectrum; Right: Overlap with local vector current

Phase Shift



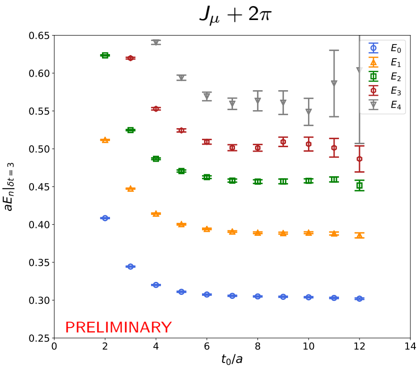
From spectrum, can compute pion scattering phase shifts in $I = 1$ channel
Statistics + systematic uncertainties included

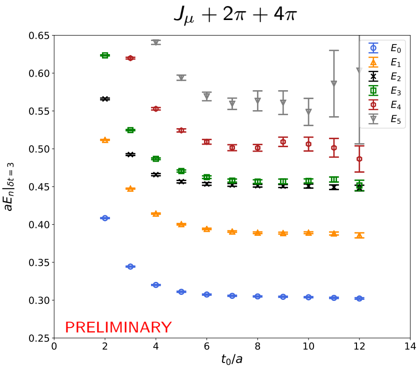
Used to explicitly calculate FV corrections at physical M_π (C.Lehner, Lattice 2018)

Compare to simple Breit-Wigner parametrization and pheno (courtesy of M.Bruno)
Good agreement with pheno for 32ID, 48I
24ID data not at plateau, but improved with fit to data

Scattering phase shift results to appear as part of series of papers by RBC+UKQCD

HVP: GEVP Results - $J_\mu + 2\pi + 4\pi$ operators

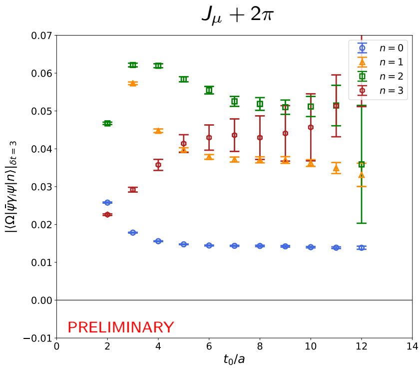


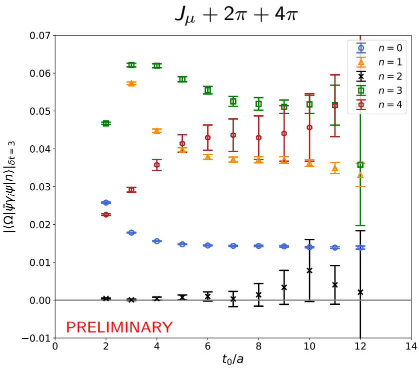


Breakdown of formalism for phase shifts +FVC could occur at 4π threshold
Compute $2\pi \to 4\pi$ and $4\pi \to 4\pi$ correlation functions and check explicitly
- $4\pi \to 4\pi$ has ~ 1000 independent Wick contractions

Spectrum unaffected by inclusion of 4π operator, but state is resolvable

HVP: GEVP Results - $J_\mu + 2\pi + 4\pi$ operators





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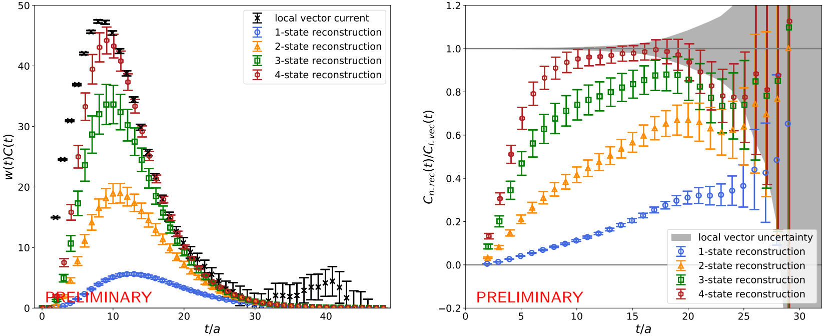
Compute $2\pi \to 4\pi$ and $4\pi \to 4\pi$ correlation functions and check explicitly

- $4\pi \to 4\pi$ has ~ 1000 independent Wick contractions

Spectrum unaffected by inclusion of 4π operator, but state is resolvable

Overlap of 4π state with local vector current unresolvable



GEVP results to reconstruct long-distance behavior of local vector correlation function needed to compute connected HVP

Explicit reconstruction good estimate of correlation function at long-distance, missing excited states at short-distance

More states $\Longrightarrow$ better reconstruction, can replace $C(t)$ at shorter distances

- Target precision for HVP is of $O(1 \times 10^{-10})$ in a few years; for now consolidate error at $O(3 \times 10^{-10})$
- Dispersive result from $e^+e^- \to$ hadrons right now is at 3×10^{-10} but limited by experimental tensions
- Two-pion channel from DHMZ17, KNT18 (e^+e^-) and DHMYZ13 (τ) are scattered by 12.5×10^{-10}

  Experimental updates and first-principles calculation of isospin-breaking corrections desirable. Combination of dispersive and lattice results can in short term lessen dependence on contested experimental data.
- New methods to reduce statistical and systematic errors and a lot of additional data.
- By end of this year, first-principles lattice result could have error of $O(5 \times 10^{-10})$

Summary

1. Introduction
2. HLbL: QED_L approach
3. HLbL: QED_∞ approach
4. HVP

Thank You!