Chiral kinetic theory from effective theory and beyond

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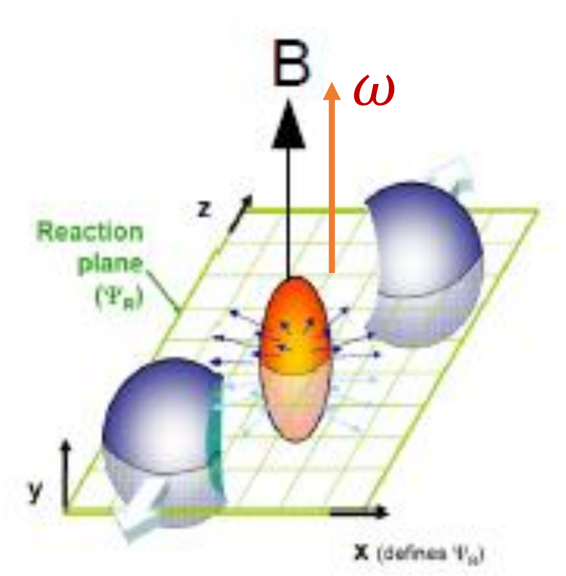
SL, Shukla, JHEP 2019

SL, Yang, 1908.XXXXX

Outline

- Anomalous effect and Chiral kinetic theory (CKT)
- Discrepancy of CKT from field theory and effective field theory (EFT)
- Reparametrization transformation in EFT and implications
- Equivalence of CKTs with different degree of freedom
- Difficulty of CKT at higher order
- Chiral kinetic theory with Landau level basis
- Summary and outlook

Anomalous effects in heavy ion collisions



Chiral Magnetic/Separation Effect(CME/CSE)

$\boldsymbol{j} = C\mu_5 e\boldsymbol{B}$ $\qquad$ $\boldsymbol{j}_5 = C\mu e\boldsymbol{B}$

Kharzeev, Zhitnitsky, NPA 2007
Kharzeev, McLerran, Warringa, NPA 2008
Metlitski, Zhitnitsky, PRD 2005

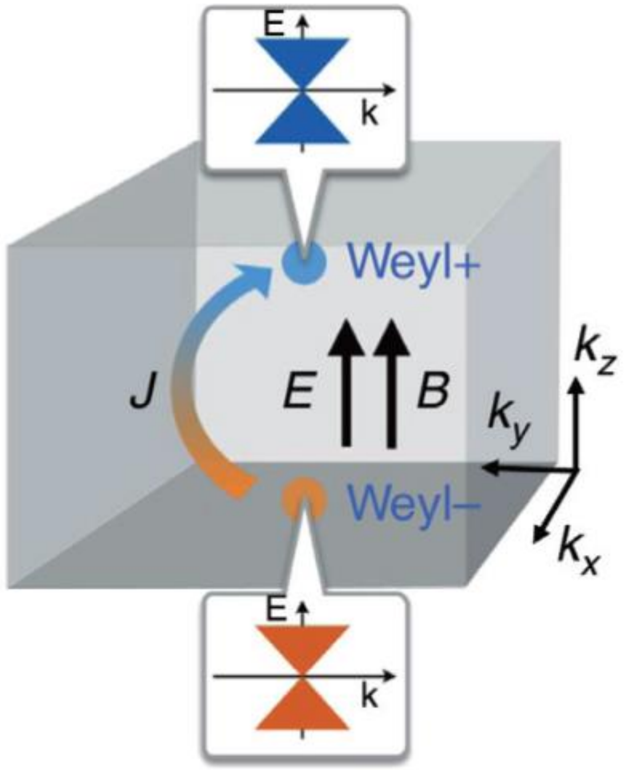
Chiral Vortical Effect(VCVE/ACVE)

$\boldsymbol{j} = C\mu_5\mu\boldsymbol{\omega}$ $\qquad$ $\boldsymbol{j}_5 = C(\mu^2 + \mu_5^2 + \frac{\pi^2 T^2}{3})\boldsymbol{\omega}$

Vilenken, PRD 1980
Erdmenger et al, JHEP 2009
Banerjee et al, JHEP 2011

Anomalous effect in Weyl semimetal

Weyl Semi-metal



Li, Kharzeev et al Nature. Phys. (2016)
Gooth et al Nature. (2017)
Son, Spivak PRB (2013)

Chiral kinetic theory for anomalous effect

Transport equation
$$\dot{n}_{\mathbf{p}} + \frac{1}{1+\mathbf{B}\cdot\mathbf{\Omega}_{\mathbf{p}}}\left[\left(\tilde{\mathbf{E}}+\tilde{\mathbf{v}}\times\mathbf{B}+(\tilde{\mathbf{E}}\cdot\mathbf{B})\mathbf{\Omega}_{\mathbf{p}}\right)\cdot\frac{\partial n_{\mathbf{p}}}{\partial\mathbf{p}}+\left(\tilde{\mathbf{v}}+\tilde{\mathbf{E}}\times\mathbf{\Omega}_{\mathbf{p}}+(\tilde{\mathbf{v}}\cdot\mathbf{\Omega}_{\mathbf{p}})\mathbf{B}\right)\cdot\frac{\partial n_{\mathbf{p}}}{\partial\mathbf{x}}\right]=0,$$

Constitutive equation

$$n = \int\frac{d^3p}{(2\pi)^3}(1+\mathbf{B}\cdot\mathbf{\Omega}_{\mathbf{p}})n_{\mathbf{p}}$$

$$\mathbf{j} = -\int\frac{d^3p}{(2\pi)^3}\left[\epsilon_{\mathbf{p}}\frac{\partial n_{\mathbf{p}}}{\partial\mathbf{p}}+\left(\mathbf{\Omega}_{\mathbf{p}}\cdot\frac{\partial n_{\mathbf{p}}}{\partial\mathbf{p}}\right)\epsilon_{\mathbf{p}}\mathbf{B}+\epsilon_{\mathbf{p}}\mathbf{\Omega}_{\mathbf{p}}\times\frac{\partial n_{\mathbf{p}}}{\partial\mathbf{x}}\right]+\mathbf{E}\times\boldsymbol{\sigma} \qquad \boldsymbol{\sigma} = \int\frac{d^3p}{(2\pi)^3}\mathbf{\Omega}_{\mathbf{p}}n_{\mathbf{p}}$$

Berry curvature $\quad \mathbf{\Omega}_{\mathbf{p}} \equiv \pm\frac{\hat{\mathbf{p}}}{2|\mathbf{p}|^2}$

Son, Yamamoto, PRD (2012) PRL (2012)
Stephanov, Yin, PRL (2012)
Gao, Liang, Pu, Q. Wang, X.-N. Wang, PRL (2012), PRD (2014)
Manuel, Torres-Rincol, PRD (2014)
Hidaka, Pu, Yang, PRD (2016)
Huang, Shi, Jiang, Liao, Zhuang PRD (2018)
Liu, Gao, Mameda, Huang, **1812.10127**

Discrepancy between CKT from different approaches

$$\Delta_\mu = \partial_\mu - \frac{\partial}{\partial p_\nu} F_{\mu\nu}$$

$$[\Delta_0 + \hat{\mathbf{p}}^i \left(1 + \frac{\mathbf{B}\cdot\hat{\mathbf{p}}}{2p^2}\right)\Delta_i - \frac{\epsilon^{ijk}\hat{\mathbf{p}}^j\mathbf{E}^k}{2p^2}\Delta_i]n = 0$$

Poisson bracket, Dirac equation, field theory …

high density effective theory

Son, Yamamoto, PRD 2012

$$[\Delta_0 + \hat{\mathbf{p}}^i \left(1 + \frac{\mathbf{B}\cdot\hat{\mathbf{p}}}{2p^2}\right)\Delta_i - \frac{\epsilon^{ijk}\hat{\mathbf{p}}^j\mathbf{E}^k + \mathbf{B}_\perp^i}{4p^2}\Delta_i]n_v = 0.$$

on-shell effective field theory

Carignano, Manuel, Torres-Rincol, PRD 2018

Discrepancy between CKT from different approaches: revisited

$$[\Delta_0 + \hat{\mathrm{p}}^i \left(1 + \frac{\mathbf{B}\cdot\hat{\mathrm{p}}}{2p^2}\right)\Delta_i - \frac{\epsilon^{ijk}\hat{\mathrm{p}}^j\mathbf{E}^k}{2p^2}\Delta_i]n = 0$$

Poisson bracket, Dirac equation, field theory …

$$[\Delta_0 + \hat{\mathrm{p}}^i \left(1 + \frac{\mathbf{B}\cdot\hat{\mathrm{p}}}{2p^2}\right)\Delta_i - \frac{\epsilon^{ijk}\hat{\mathrm{p}}^j\mathbf{E}^k + \mathbf{B}_\perp^i}{4p^2}\Delta_i]n_v = 0.$$

on-shell effective field theory
Carigonano, Manuel, Torres-Rincol, PRD 2018
high density effective theory
Son, Yamamoto, PRD 2012

High density effective theory (HDET)

$$\mathcal{L}_0 = \bar{\psi}(i\gamma^\mu D_\mu)\psi + \mu\bar{\psi}\gamma^0\psi$$

Finite density, zero temperature,
right-handed fermions

Hong, PLB 1998, NPB 2000
Schafer NPA 2003

$$E_\pm = -\mu \pm |p|$$

low energy dynamics

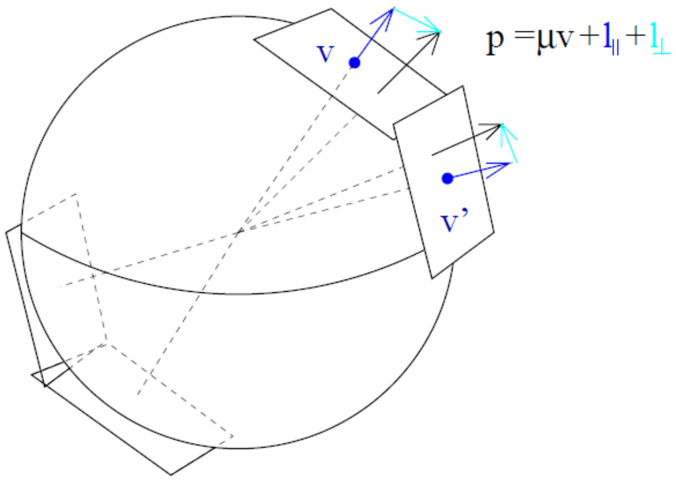
$E_+ \sim 0$ near Fermi surface, slow mode

$E_- \sim -2\mu$ deep in Fermi sea, fast mode

slow mode

$$p^0 = \mu + l^0 \quad \mathbf{p} = \mu\mathbf{v} + \mathbf{l}$$

$$l^0, |\mathbf{l}| \ll \mu$$



$$\psi(x) = \sum_{\mathbf{v}} e^{i\mu\mathbf{v}\cdot\mathbf{x}}\left[\psi_{+v}(x) + \psi_{-v}(x)\right]$$

$$\mathcal{L}_{eff} = \psi^\dagger \sum_n D^{(n)}\psi = \psi^\dagger_{+v}\left[iv\cdot D + \frac{\not{D}_\perp^2}{2\mu} + \frac{\not{D}_\perp(-i\bar{v}\cdot D)\not{D}_\perp}{4\mu^2}\right]\psi_{+v}$$

Wigner function and its EOM

$$G_v(x,y) = \langle \psi_v(x)\psi_v^\dagger(y)\rangle$$

$$\mathcal{D}_x G_v(x,y) = 0, \qquad G_v(x,y)\mathcal{D}_y^\dagger = 0,$$

EOM hold out of equilibrium

$$x = X + \frac{s}{2}, \qquad y = X - \frac{s}{2}$$

gradient expansion $\partial_X \ll \partial_s \sim l \ll \mu$

$$\tilde{G}_v(X,l) = \int d^4s\, e^{il\cdot s} U\left(X, X+\frac{s}{2}\right) G_v\left(X+\frac{s}{2}, X-\frac{s}{2}\right) U\left(X-\frac{s}{2}, X\right)$$

$$I_\pm^{(n)} \equiv \int \frac{d^4s}{(2\pi)^4} e^{il\cdot s}\, \left(\mathcal{D}_x^{(n)} G_v \pm G_v \mathcal{D}_y^{(n)\dagger}\right)$$

Ambiguous dispersion relation

$$I_+^{(0)} = 2v\cdot l\,\tilde{G}_v, \qquad I_-^{(0)} = iv^\mu \Delta_\mu \tilde{G}_v,$$

$$I_+^{(1)} = \frac{1}{\mu}\left[-l_\perp^2 + \mathbf{B}\cdot\mathbf{v}\right]\tilde{G}_v, \qquad I_-^{(0)} = \frac{i}{\mu} l_\perp^\mu \Delta_\mu \tilde{G}_v,$$

$$I_+^{(2)} = \frac{1}{4\mu^2}\left[4l_\parallel l_\perp^2 - 4l_\parallel(\mathbf{B}\cdot\mathbf{v}) + 2\mathbf{B}\cdot\mathbf{l}_\perp + 2(\mathbf{E}\times\mathbf{l})\cdot\mathbf{v}\right]\tilde{G}_v$$

$$I_-^{(2)} = -\frac{i}{4\mu^2}\left[\left(4l_\parallel l^\mu - \bar{v}^\mu(l_\perp^2 - \mathbf{B}\cdot\mathbf{v})\right)\Delta_\mu \tilde{G}_v - \left(\varepsilon^{ijk} v^k \bar{v}_\mu F^{i\mu}\right)\Delta_j\right]\tilde{G}_v$$

disagree with Son, Yamamoto

$$\tilde{G}_v = 2\pi\, P_+\, \delta\Big(l_0 - l_\parallel - \frac{1}{2\mu}[l_\perp^2 - \mathbf{B}\cdot\mathbf{v}] + \frac{1}{2\mu^2}[l_\parallel(l_\perp^2 - \mathbf{B}\cdot\mathbf{v})] + \frac{1}{4\mu^2}[\mathbf{B}\cdot\mathbf{l}_\perp + (\mathbf{E}\times\mathbf{l})\cdot\mathbf{v}]\Big)n_v$$

Not invariant under reparametrization!

$$p^0 = \mu + l^0 \quad \mathbf{p} = \mu\mathbf{v} + \mathbf{l}$$

$p^0 = \epsilon_p(\boldsymbol{p}, \mathbf{v})$ depends on momentum decomposition

Reparametrization in EFTs

High Density ET

$$p^\mu = \mu v^\mu + l^\mu$$

$$v^\mu \longrightarrow v^{\mu\prime} = v^\mu + \delta v^\mu$$

$$l^{\mu\prime} = l^\mu - \mu\,\delta v^\mu$$

$$v^2 = 0$$

Heavy Quark ET

$$p^\mu = m v^\mu + l^\mu$$

$$v^\mu \longrightarrow v^{\mu\prime} = v^\mu + \delta v^\mu$$

$$l^\mu \longrightarrow l^{\mu\prime} = l^\mu - m\,\delta v^\mu$$

$$v^2 = 1$$

Physics independent of parameter v

Reparametrization transformation of Wigner function

$$v^\mu \longrightarrow v^{\mu'} = v^\mu + \delta v^\mu$$

$$\psi_v \longrightarrow \psi'_v = \psi_v + i\mu\delta v\cdot x - \frac{\delta\!\!\!/\psi}{2}\left(1 - \frac{1}{2\mu + i\bar{v}\cdot D} i D\!\!\!\!/_\perp\right)\psi_v$$

dressed particle

anti-particle

$$\mathcal{L} = \psi_v^\dagger iv\cdot D\psi_v + \psi_v^\dagger D\!\!\!\!/_\perp \frac{1}{2\mu + i\bar{v}\cdot D} D\!\!\!\!/_\perp \psi_v$$

Effective Action invariant

$$\mathrm{tr}\delta\tilde{G}_v(X,l) = \frac{1}{4\mu}\delta v_j \Delta_i v^k \epsilon^{ijk}\mathrm{tr}\tilde{G}_v(X,l) + \frac{1}{2\mu}\delta v_j l_i \Delta_{ij}\mathrm{tr}\tilde{G}_v(X,l)$$

$n_v \sim tr\tilde{G}_v$. Distribution function for dressed particle changes with **v**.

Reparametrization transformation of EOM

$$I_{\pm}^{(n)} = \int_s e^{il\cdot s}(\mathcal{D}_x^{(n)} G_v(x,y) \pm G_v(x,y)\mathcal{D}_y^{(n)})$$

$$I_{+}^{(0)} = 2v\cdot l\,\tilde{G}_v, \qquad I_{-}^{(0)} = iv^{\mu}\Delta_{\mu}\tilde{G}_v,$$

$$I_{+}^{(1)} = \frac{1}{\mu}\Big[-l_{\perp}^2 + \mathbf{B}\cdot\mathbf{v}\Big]\,\tilde{G}_v, \qquad I_{-}^{(0)} = \frac{i}{\mu}l_{\perp}^{\mu}\Delta_{\mu}\tilde{G}_v,$$

$$I_{+}^{(2)} = \frac{1}{4\mu^2}\Big[4l_{\|}l_{\perp}^2 - 4l_{\|}(\mathbf{B}\cdot\mathbf{v}) + 2\mathbf{B}\cdot\mathbf{l}_{\perp} + 2(\mathbf{E}\times\mathbf{l})\cdot\mathbf{v}\Big]\tilde{G}_v$$

$$I_{-}^{(2)} = -\frac{i}{4\mu^2}\Big[\Big(4l_{\|}l^{\mu} - \bar{v}^{\mu}(l_{\perp}^2 - \mathbf{B}\cdot\mathbf{v})\Big)\Delta_{\mu}\tilde{G}_v - \Big(\varepsilon^{ijk}v^k\bar{v}_{\mu}F^{i\mu}\Big)\Delta_j\Big]\tilde{G}_v$$

$[D_+]\tilde{G}_v = 0$ $\qquad$ $[D_-]\tilde{G}_v = 0$ $\qquad$ EOM reparametrization invariant

Transformation of $D_{\pm}$ and $\tilde{G}_v$ under reparametrization compensate each other

Expect: dispersion relation and transport equation v-dependent

What is the role of v?

v determines degree of freedom (HDET)

$$\psi_v \longrightarrow \psi'_v = \psi_v + i\mu\delta v\cdot x - \frac{\delta\not{v}}{2}\left(1 - \frac{1}{2\mu + i\bar{v}\cdot D} i\not{D}_\perp\right)\psi_v$$

$$\mathrm{tr}\delta\tilde{G}_v(X,l) = \frac{1}{4\mu}\delta v_j \Delta_i v^k \epsilon^{ijk}\mathrm{tr}\tilde{G}_v(X,l) + \frac{1}{2\mu}\delta v_j l_i \Delta_{ij}\mathrm{tr}\tilde{G}_v(X,l)$$

formally analogous to side-jump, but physically not entirely the same:

side jump: change of distribution function under Lorentz boost
here: change of distribution function under different choice of dof

Equivalence of two CKTs

$$\tilde{G} = \int_s e^{ip\cdot s}\psi(x)\psi^\dagger(y)U(y,x),$$

dof: particle

$$\tilde{G}_v = \int_s e^{il\cdot s}\psi_v(x)\psi_v^\dagger(y)U(y,x).$$

dof: dressed particle

$$\mathrm{tr}\tilde{G} = \mathrm{tr}\tilde{G}_v - \frac{1}{4\mu^2}\mathrm{tr}l_i\Delta_j\tilde{G}_v\epsilon^{ijm}v^m \quad \Rightarrow \quad n = n_v - \frac{1}{4\mu^2}l_i\Delta_j n_v\epsilon^{ijm}v^m.$$

$$[\Delta_0 + \hat{\mathrm{p}}^i\left(1 + \frac{\mathbf{B}\cdot\hat{\mathrm{p}}}{2p^2}\right)\Delta_i - \frac{\epsilon^{ijk}\hat{\mathrm{p}}^j\mathbf{E}^k}{2p^2}\Delta_i]n = 0$$

$$\Updownarrow$$

Although dof different, both use free particle basis

$$[\Delta_0 + \hat{\mathrm{p}}^i\left(1 + \frac{\mathbf{B}\cdot\hat{\mathrm{p}}}{2p^2}\right)\Delta_i - \frac{\epsilon^{ijk}\hat{\mathrm{p}}^j\mathbf{E}^k + \mathbf{B}_\perp^i}{4p^2}\Delta_i]n_v = 0.$$

Difficulty at higher order in $\hbar$ with free particle basis

$O(\hbar^0)$: spinless particle $\partial_t f + \mathbf{v}\cdot\boldsymbol{\nabla}_x f + (\boldsymbol{E} + \mathbf{v}\times\boldsymbol{B})\cdot\boldsymbol{\nabla}_p f = 0$

$O(\hbar)$: particle with magnetic moment + Berry curvature

$$\begin{aligned} &(1+\hbar s Q \boldsymbol{\Omega}_p\cdot\mathbf{B})\,\partial_t f(x,E_p,\mathbf{p}) \\ &+\left[\mathbf{v}+\hbar s Q(\mathbf{E}\times\boldsymbol{\Omega}_p)+\hbar s Q\frac{1}{2|\mathbf{p}|^2}\mathbf{B}\right]\cdot\boldsymbol{\nabla}_x f(x,E_p,\mathbf{p}) \\ &+\left[Q\tilde{\mathbf{E}}+Q\mathbf{v}\times\mathbf{B}+\hbar s Q^2(\mathbf{E}\cdot\mathbf{B})\boldsymbol{\Omega}_p\right]\cdot\boldsymbol{\nabla}_p f(x,E_p,\mathbf{p}) \;=\; 0 \end{aligned}$$

$O(\hbar^2)$: particle no longer on-shell, simple picture lost

$$p_\mu G^\mu_{(0)}\left[f\delta(\tilde{p}^2)\right]+\frac{\hbar s}{2}\mathbf{G}^{(0)}\cdot\left\{\frac{1}{p_0}\mathbf{G}^{(0)}\times\left[\mathbf{p}f\delta(\tilde{p}^2)\right]\right\}+\hbar^2 C(f)=0$$

$C(f)$: off-shell effect

Gao, Liang, Q. Wang, X.N. Wang (2018)

Two alternative formulations

free particle basis: subject to weak electromagnetic field

with increasing B, Landau quantization effect becomes important. CKT more singular

Landau level basis: subject to strong magnetic field + perturbation

with increasing B, lowest Landau level approximation more accurate

Wigner function at lowest Landau level

Wigner function at constant B for right-handed chiral fermion

$$W(X,P)=\int \frac{d^4X'}{(2\pi)^4}\exp(-ip\cdot X')\left\langle \psi\left(X-\frac{1}{2}X'\right)U\left(A,X+\frac{1}{2}X',X-\frac{1}{2}X'\right)\psi^\dagger\left(X-\frac{1}{2}X'\right)\right\rangle$$

LLL approximation

$$W(P)=f(p_0)[\delta(p_0-E_{pz})W_+^{(0)}(p)+\delta(p_0+E_{pz})W_-^{(0)}(p)]$$

$$W_r^{(0)}(P)=\frac{1}{(2\pi)^3}\frac{1}{E_{pz}}(E_{pz}+rp_z)\exp(-\frac{p_T^2}{eB})\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\qquad r=+/-$$

transverse size: $\frac{1}{\sqrt{eB}}$

LLL spin: aligns with B

Sheng, Rischke, Wang, Vasak (2018)

CKT with Landau level basis from perturbation

$$W(X,P) = \int \frac{d^4X'}{(2\pi)^4} \exp(-ip\cdot X') \left\langle \psi\left(X-\frac{1}{2}X'\right) U\left(A, X+\frac{1}{2}X', X-\frac{1}{2}X'\right) \psi^\dagger\left(X-\frac{1}{2}X'\right)\right\rangle$$

$A_\mu \to A_\mu(B) + a_\mu(e,b)$ $F_{\mu\nu} \to F_{\mu\nu}(B) + f_{\mu\nu}(e,b)$

$$\left(\frac{1}{2}\Delta_\mu - ip_\mu\right)\sigma^\mu W = 0$$

$$\left(\frac{1}{2}\Delta_\mu + ip_\mu\right)W\sigma^\mu = 0$$

$$\Delta_\mu = \partial_\mu - \frac{\partial}{\partial p_\nu} f_{\mu\nu}$$

Valid to all order in background B and $O(\partial_X)$ and $O(f_{\mu\nu})$

EOM for components

W Hermitian $W = F \cdot 1 + j_i \sigma^i$

$$J^0 \propto \int \frac{d^4p}{(2\pi)^4} F$$

$$J^i \propto \int \frac{d^4p}{(2\pi)^4} j_i$$

$$\begin{cases} \Delta_0 F + \Delta_i j_i = 0 \\ \Delta_0 j_i + \Delta_i F + 2\epsilon^{ijk} p_j j_k = 0 \\ p_0 F + p_i j_i = 0 \\ -p_0 j_i - p_i F + \frac{1}{2}\epsilon^{ijk}\Delta_j j_k = 0 \end{cases}$$

Transport equations

Constraint equations

CKT with Landau level basis: simple example

$$W = F \cdot 1 + j_i \sigma^i$$

For homogeneous $e \parallel B$, LL remains on shell $\qquad p_0^2 = p_z^2$

$$\Delta_0 F + \Delta_i j_i = 0$$

$$j_z = \frac{p_z}{p_0} F \propto \frac{p_z}{p_0} f(p_0) \delta(p_0 - p_z) \exp(-\frac{p_T^2}{eB}) \qquad j_x = j_y = 0$$

$$\frac{\partial f_\pm}{\partial t} + \dot{z} \frac{\partial f_\pm}{\partial z} + \dot{p_z} \frac{\partial f_\pm}{\partial p_z} = C[f_\pm, f_g]$$

Hattori, Li, Satow, Yee PRD (2018)

p_T dependence unchanged, sufficient for conductivity computation

CKT with Landau level basis: a less trivial example

$W = F \cdot 1 + j_i \sigma^i$

For inhomogeneous $e \parallel B$, LL off shell

$a_z(t, x)$ induces e_z and b_y

spin doesn't respond to e_z but precess in b_y ⟹ $j_x \neq 0, j_y \neq 0$

All four components needed

Summary&outlook

- Reparametrization transformation shows CKT from EFT depends on choice of degree of freedom
- The two CKTs are equivalent: particle vs dressed particle (free particle basis)
- CKT with Landau level basis good at large B
- Examples of CKT at homogeneous and inhomogeneous E field
- Transverse conductivity
- photon self-energy in B field

Thank you!

CKT with a choice of v

$l \parallel v$ Choice of v fixes degree of freedom

Dispersion relation $l_0 = l - \frac{\mathbf{B}\cdot\mathbf{v}}{2\mu} + \frac{\mathbf{B}\cdot\mathbf{v}l}{2\mu^2} \longrightarrow p_0 = p - \frac{\mathbf{B}\cdot\hat{\mathbf{p}}}{2p}$

Transport equation $\left[\Delta_0 + v^i(1 + \frac{\mathbf{B}\cdot\mathbf{v}}{2\mu^2})\Delta_i + \frac{\bar{v}^\nu \epsilon^{ijm} v^m F_{i\nu}\Delta_j}{4\mu^2}\right] n_v(X,l) = 0$

$$\longrightarrow \quad [\Delta_0 + \hat{\mathbf{p}}^i\left(1 + \frac{\mathbf{B}\cdot\hat{\mathbf{p}}}{2p^2}\right)\Delta_i - \frac{\epsilon^{ijk}\hat{\mathbf{p}}^j\mathbf{E}^k + \mathbf{B}^i_\perp}{4p^2}\Delta_i] n_v = 0.$$

Constitutive equation

Within scheme $l \parallel v$

$$j^{\mu(0)} = \frac{1}{(2\pi)^4}\int_l \mathrm{tr}\left[v^\mu \tilde{G}_v(X,l)\right]$$

$$j^{i(1)} = \frac{1}{2\mu}\frac{1}{(2\pi)^4}\int_l \varepsilon^{ijk}\Delta_j v^k \mathrm{tr}\tilde{G}_v(X,l)$$

$$n^{(2)} = \frac{1}{(2\pi)^4}\int_l \frac{1}{2\mu^2}\left[\mathbf{B}\cdot\mathbf{v}\mathrm{tr}\tilde{G}_v(X,l)\right]$$

Berry curvature $\frac{\boldsymbol{B}\cdot\boldsymbol{v}}{2\mu^2} = \frac{\boldsymbol{B}\cdot\widehat{\boldsymbol{p}}}{2p^2} + \cdots$

$$j^{i(2)} = \frac{1}{(2\pi)^4}\int_l \frac{1}{4\mu^2}\left[-\partial_{X^j} l_\nu \varepsilon^{ijm} v^m \bar{v}^\nu - 2\mathbf{B}\cdot\mathbf{v}v^i + F_{\nu j}\bar{v}^\nu v^m \varepsilon^{ijm}\right]\mathrm{tr}\tilde{G}_v(X,l)$$