Two-loop triangle integrals with 4 scales for the HZV vertex

Based on :

Yuxuan Wang, Xiaofeng Xu, Li Lin Yang. arXiv : 1905.11463

Yuxuan Wang, Xiaofeng Xu, Li Lin Yang. arXiv : 1907.xxxxx

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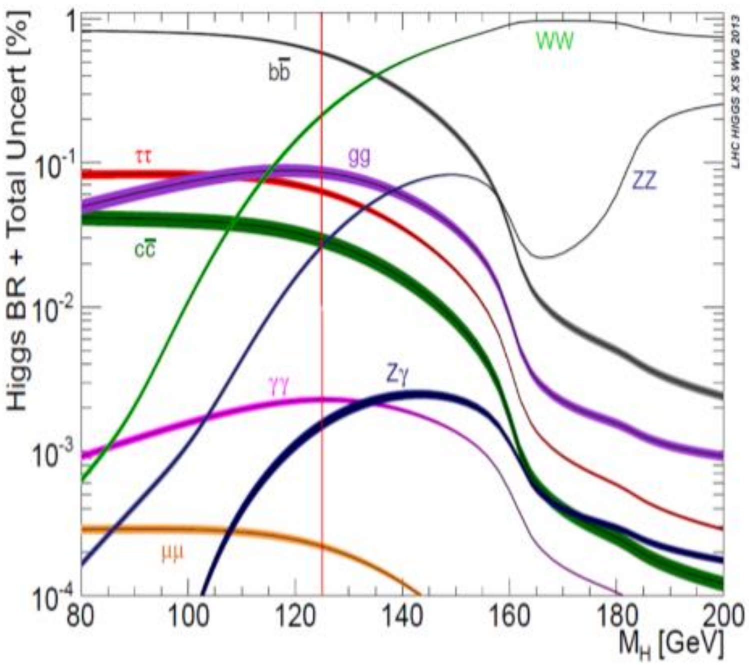
Outline

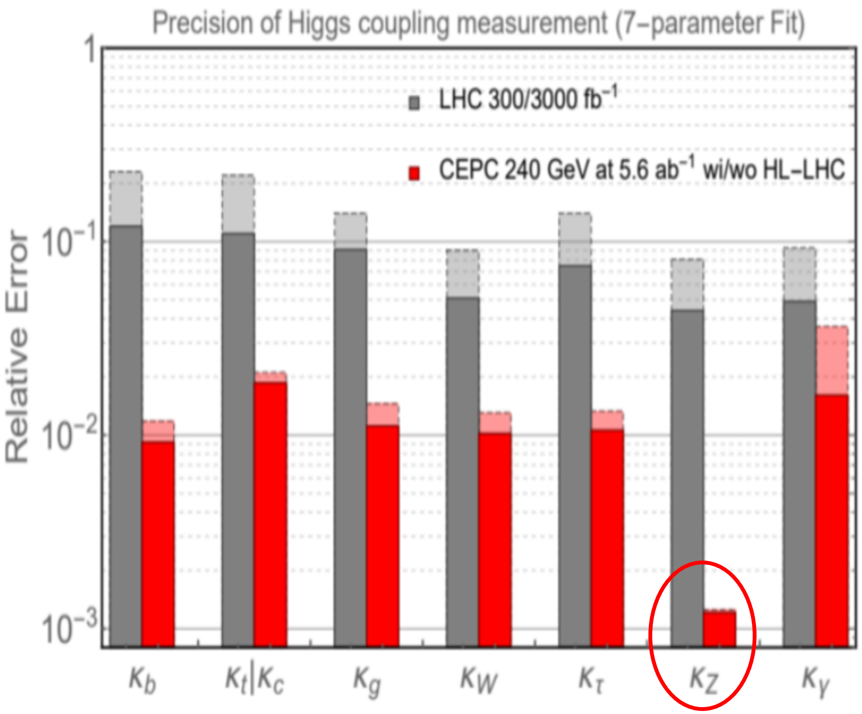
- Motivation
- Solve the Integrals
- One loop example
- Result
- Summary

Motivation

- **Higgs**
  - Found at LHC in 2012, H(125) is a Spin-0 SM-like particle
  - Indirect probe to New Physics

- $\boldsymbol{HZV}$ vertex ($V = Z^*$ or γ^*)
  - decay and production *

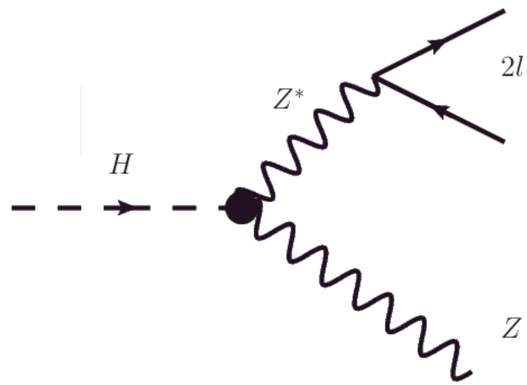




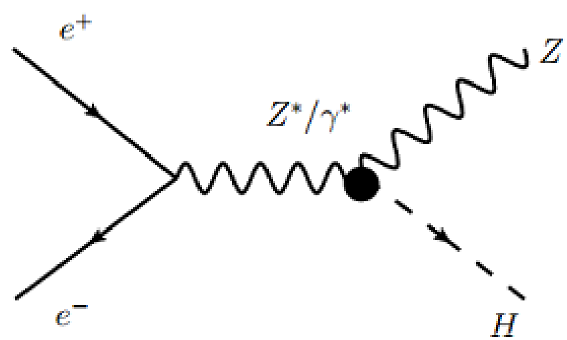
* The CEPC Study Group.... arXiv: 1809.00285

Motivation

- $\boldsymbol{HZV}$ vertex $(V = Z^*$ or $\gamma^*)$
  - Decay : $H \rightarrow Z\,Z^* \rightarrow Z + 2l$

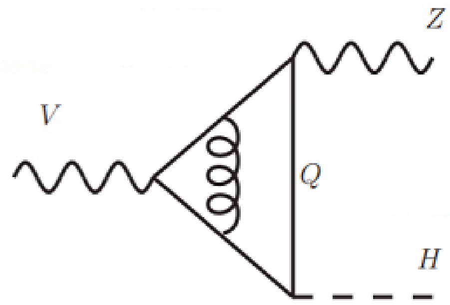


  - Production : $e^+ + e^- \rightarrow V \rightarrow Z + H$



Motivation

- **NNLO $\mathcal{O}(\alpha\alpha_s)$ corrections (EW-QCD)**



- Two-Loop (quark loop and gluon loop)
- Four Scales (V , Z, H and Q)
  - Three independent dimensionless variables

$$x = -\frac{q^2}{4m_Q^2}\,, \quad y = -\frac{p_Z^2}{4m_Q^2}\,, \quad z = -\frac{p_H^2}{4m_Q^2}\,.$$

- Hard to calculate, in both analytic and numerical way.

- We first give the fully analytic results up to weight-3 in terms of multiple polylogarithms, by using the Differential Equations method.

Solve the Integrals

- **Workflow**

  - IBP to a set of master integrals

  - Canonical-form Differential Equations

$$
\begin{aligned}
d\vec{f}(x,y,z,\epsilon) &= \epsilon\, dA(x,y,z)\, \vec{f}(x,y,z;\epsilon) \\
&= \epsilon \sum_i A_i\, d\log(\alpha_i)\, \vec{f}(x,y,z,\epsilon)
\end{aligned}
$$

  - Solution

$$
\vec{f}(x,y,z;\epsilon) = \mathcal{P}\exp\left(\epsilon \int_{\vec{r}_0}^{\vec{r}} dA\right)\vec{f}(x_0,y_0,z_0;\epsilon)
$$

  - Power series at ϵ

$$
\vec{f}(x,y,z;\epsilon) = \sum_{n=0}^{\infty} \vec{f}^{(n)}(x,y,z)\epsilon^n
$$

Integration By Parts (IBP)

- Process include huge amounts of integrals

- IBP give a set of linear relations of integrals in a given family.

$$\int \left(\prod_i d^d k_i\right) \frac{\partial}{\partial k_j^\mu} \left(q_\mu \prod_l \frac{1}{(k_l - p_n)^{a_l}}\right) = 0$$

  - where q_μ can be any internal momentum k_i^μ or external moentum p_n^μ.

- A max set of independent Integrals are called Master Integrals(MIs).

- Auto-computer algebra program : FIRE5 *

- Huge amounts of integrals → much less MIs

* A. V. Smirnov FIRE5: A C++ implementation of Feynman Integral Reduction.ArXiv:1408.2372

Differential Equations

- Given a set of MIs $\vec{f}(\vec{x},\epsilon)$ where $\vec{x}$ are the kinematic variables, we have

$$\frac{\partial}{\partial x_i}\vec{f}(\vec{x},\epsilon) = \bar{A}_i(\vec{x},\epsilon)\,\vec{f}(\vec{x},\epsilon)$$

- By choosing another appropriate MIs $\vec{f_0}(\vec{x},\epsilon)$

$$\frac{\partial}{\partial x_i}\vec{f_0}(\vec{x},\epsilon) = \left[\epsilon\,\tilde{A}_i(\vec{x}) + \tilde{B}_i(\vec{x})\right]\vec{f_0}(\vec{x},\epsilon)$$

- Canonical Form by further making a linear transformation $T(\vec{x},\epsilon)$

$$\frac{\partial}{\partial x_i}\vec{f}(\vec{x},\epsilon) = \epsilon\,A_i(\vec{x})\,\vec{f}(\vec{x},\epsilon)$$

where $\vec{f}(\vec{x},\epsilon) = T(\vec{x},\epsilon)\vec{f_0}(\vec{x},\epsilon)$ are called canonical-basis of MIs

Canonical form of DE

- Canonical form

$$\mathrm{d}\vec{f}(\vec{x},\epsilon) = \epsilon\,\mathrm{d}A(\vec{x})\,\vec{f}(\vec{x},\epsilon)$$

- Chen iterated integrals*

$$\vec{f}(\vec{x},\epsilon) = \mathbb{P}\exp\left(\epsilon\int_\gamma \mathrm{d}A(\vec{x})\right)\vec{f}(\vec{x}_0,\epsilon)$$

- Multiple polylogarithms(GPLs)* *

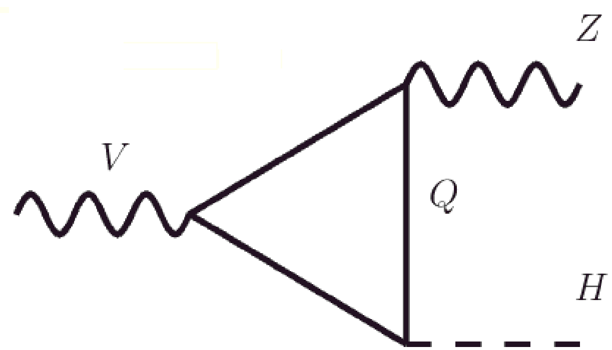
$$G(a_1,\ldots,a_n;x) \equiv \int_0^x \frac{dx'}{x'-a_1} G(a_2,\ldots,a_n;x')\,,$$

with $G(;\beta)\equiv 1$ and $G(\underbrace{0,\ldots,0}_{n\text{ times}};x) \equiv \log^n(x)/n!$, weight is n

$$G(\vec{0}_{n-1},1;x) \equiv -Li_n(x)$$

* Kuo-Tsai Chen. Iterated path integrals. https://projecteuclid.org:443/euclid.bams/1183539443
* * A. B. Goncharov, Math. Res. Lett. 5, 497 (1998) arXiv:1105.2076

Example : One-loop triangle integrals

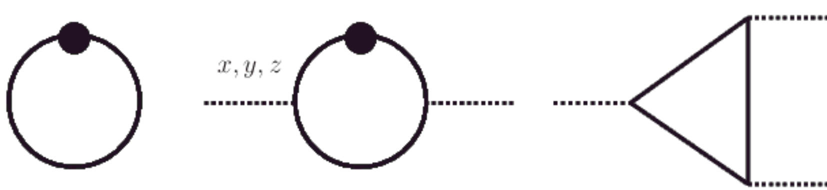


- IBP to only 5 master integrals

- Canonical basis for differential equation

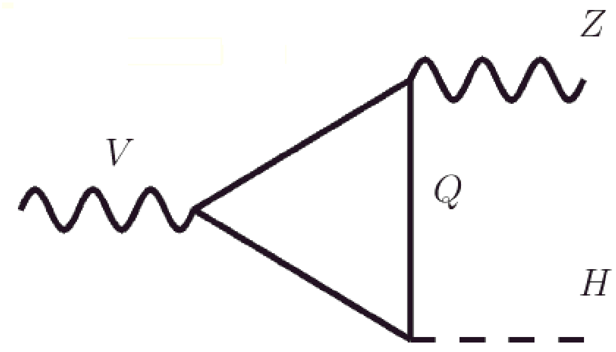
$$x = -\frac{q^2}{4m_Q^2}\,, \quad y = -\frac{p_Z^2}{4m_Q^2}\,, \quad z = -\frac{p_H^2}{4m_Q^2}\,.$$

$$\begin{aligned} d\vec{f}(x,y,z,\epsilon) &= \epsilon\, dA(x,y,z)\, \vec{f}(x,y,z;\epsilon) \\ &= \epsilon \sum_i A_i\, d\log(\alpha_i)\, \vec{f}(x,y,z,\epsilon) \end{aligned}$$



Example : One-loop triangle integrals

$$d\vec{f}(x,y,z,\epsilon) = \epsilon\, dA(x,y,z)\, \vec{f}(x,y,z;\epsilon)$$
$$= \epsilon \sum_i A_i\, d\log(\alpha_i)\, \vec{f}(x,y,z,\epsilon)$$



- Four square roots $R_1(x) \equiv \sqrt{x(x+1)}\,, \quad R_1(y)\,, \quad R_1(z)\,,$

$$R_2 \equiv R_2(x,y,z) \equiv \sqrt{\lambda(x,y,z)}\,.$$

$$\lambda(x,y,z) \equiv x^2 + y^2 + z^2 - 2xy - 2yz - 2zx\,.$$

$$\beta(t) \equiv \frac{R_1(t)-t}{R_1(t)+t}$$

$$dA = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ d\log(\beta(x))/4 & -d\log(1+x) & 0 & 0 & 0 \\ d\log(\beta(y))/4 & 0 & -d\log(1+y) & 0 & 0 \\ d\log(\beta(z))/4 & 0 & 0 & -d\log(1+z) & 0 \\ 0 & -d\log(\frac{x(x-y-z)-R_1(x)R_2}{x(x-y-z)+R_1(x)R_2}) & \cdots & \cdots & \cdots \end{pmatrix}$$

Example : One-loop triangle integrals

- Iterated integrals solution

$$\vec{f}^{(i)}(\vec{x}) = \int_\gamma \mathrm{d}A(\vec{x})\, \vec{f}^{(i-1)}(\vec{x}) + \vec{f}^{(i)}(\vec{x_0})$$

- Boundary Conditions

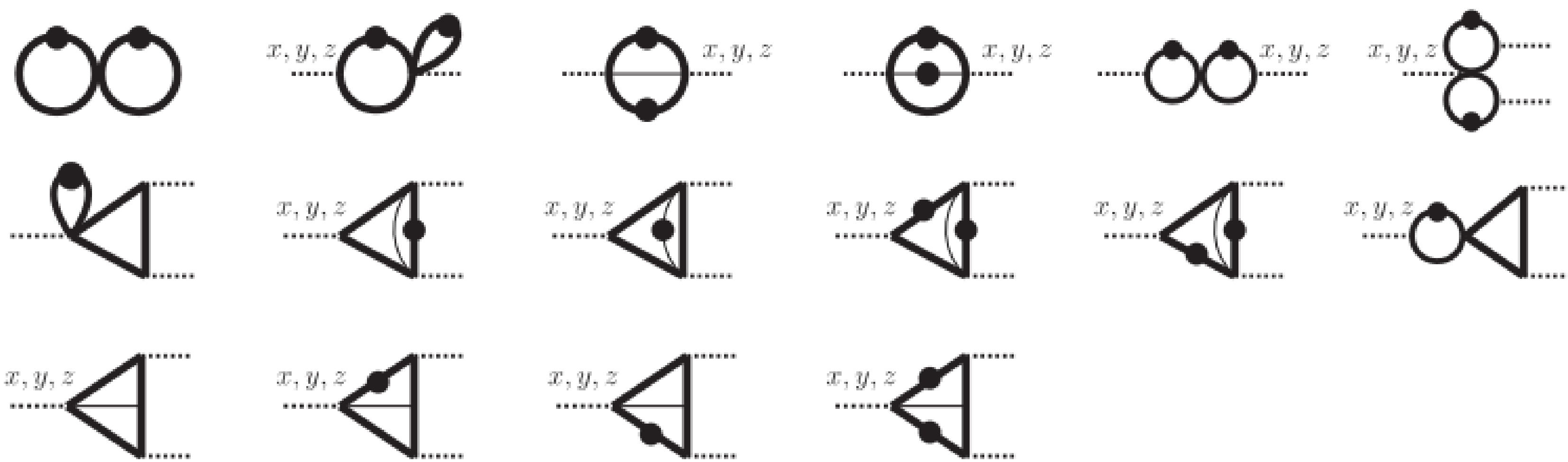
$$\lim_{x,y,z\to 0} f_i(x,y,z,\epsilon) = \delta_{i,1}$$

- Only need to be solved up to weight 2

Solve the Integrals (Two-loop)

- IBP to **41** master integrals

- The canonical basis of MIs



- Need to be solved up to weight 4
  - Using the Symbol *

Claude Duhr, Herbert Gangl, and John R. Rhodes. arXiv: 1110.0458

Symbol representation

- Maps the iterated integrals to their integration kernels

$$\vec{f}^{(i)}(\vec{x}) = \int_\gamma \mathrm{d}A(\vec{x})\, \vec{f}^{(i-1)}(\vec{x}) + \vec{f}^{(i)}(\vec{x_0})$$

$$\mathcal{S}\big(f_n^{(i)}(\vec{x})\big) = \sum_m \mathcal{S}\big(f_m^{(i-1)}(\vec{x})\big) \otimes \mathcal{S}\big(A_{nm}(\vec{x})\big)$$

- Algebraic properties

$$\alpha_1(\vec{x}) \otimes \cdots \otimes \big(\alpha_i(\vec{x})\alpha_{i'}(\vec{x})\big) \otimes \cdots \otimes \alpha_n(\vec{x}) = \alpha_1(\vec{x}) \otimes \cdots \otimes \alpha_i(\vec{x}) \otimes \cdots \otimes \alpha_n(\vec{x}) + \alpha_1(\vec{x}) \otimes \cdots \otimes \alpha_{i'}(\vec{x}) \otimes \cdots \otimes \alpha_n(\vec{x})$$

$$\alpha_1(\vec{x}) \otimes \cdots \otimes \big(c\alpha_i(\vec{x})\big) \otimes \cdots \otimes \alpha_n(\vec{x}) = \alpha_1(\vec{x}) \otimes \cdots \otimes \alpha_i(\vec{x}) \otimes \cdots \otimes \alpha_n(\vec{x})$$

- Shuffle product

$$a \,\text{Ш}\, b = a \otimes b + b \otimes a$$

$$(a_1 \otimes \ldots \otimes a_{n_1}) \,\text{Ш}\, (a_{n_1+1} \otimes \ldots \otimes a_{n_1+n_2}) = \sum_{\sigma \in \Sigma(n_1,n_2)} a_{\sigma^{-1}(1)} \otimes \ldots \otimes a_{\sigma^{-1}(n_1+n_2)}$$

where $\Sigma(n_1, n_2)$ denotes the set of all shuffles of $n_1 + n_2$ elements

Weight 2 solution

$$R_1(x) \equiv \sqrt{x(x+1)}$$
$$R_2 \equiv R_2(x,y,z) \equiv \sqrt{\lambda(x,y,z)}$$

$$\beta(t) \equiv \frac{R_1(t)-t}{R_1(t)+t}\,.$$

- Integrable symbols

$$\beta_i \otimes \beta_i\,, \quad \beta_i \otimes \beta_j + \beta_j \otimes \beta_i\,, \quad \beta_i \otimes r_i\,, \quad \beta_i \otimes (1+r_i)\,,$$
$$\frac{\beta_i\beta_j}{\beta_k} \otimes \left(1 - \frac{\beta_i\beta_j}{\beta_k}\right), \quad (\beta_i\beta_j\beta_k) \otimes (1-\beta_i\beta_j\beta_k)\,, \qquad i \neq j \neq k = x,y,z$$
$$\beta(x) \otimes \frac{x(x-y-z) - R_1(x)R_2}{x(x-y-z)+R_1(x)R_2} + (x \leftrightarrow y) + (x \leftrightarrow z)\,,$$

- Simple example

$$\mathcal{S}\big(G(0,0;u)\big) = \mathcal{S}\left(\frac{1}{2}\log^2(u)\right) = u \otimes u\,,$$
$$\mathcal{S}\big(G(0;u)G(0;v)\big) = \mathcal{S}\big(\log(u)\log(v)\big) = u \otimes v + v \otimes u\,,$$
$$\mathcal{S}\big(G(0,1;u)\big) = -\mathcal{S}\big(\mathrm{Li}_2(u)\big) = (1-u) \otimes u\,.$$

Weight 2 solution

$$R_1(x) \equiv \sqrt{x(x+1)}$$
$$R_2 \equiv R_2(x,y,z) \equiv \sqrt{\lambda(x,y,z)}$$
$$\lambda(x,y,z) \equiv x^2 + y^2 + z^2 - 2xy - 2yz - 2zx\,.$$

$$\beta(t) \equiv \frac{R_1(t) - t}{R_1(t) + t}$$

- **complex symbol**

$$\beta(x) \otimes \frac{x(x-y-z) - R_1(x)R_2}{x(x-y-z) + R_1(x)R_2} + (x \leftrightarrow y) + (x \leftrightarrow z)$$

- **parametrize the path : $t\,\vec{x}$** $\quad R_2(tx,ty,tz) = tR_2(x,y,z) = tR_2$

$$I_1 = \int_{t=0}^{t=1} \log(\beta(tx))\, d\log \frac{tx(x-y-z) - R_1(tx)R_2}{tx(x-y-z) + R_1(tx)R_2}$$

$$= \int_{u=0}^{u=1-\beta(x)} G(1;u)\, d\log \frac{u(x-y-z+R_2) - 2R_2}{u(x-y-z-R_2) + 2R_2}$$

$$= G\left(\frac{2R_2}{R_2 + x - y - z}, 1; 1-\beta(x)\right) - G\left(\frac{2R_2}{R_2 - x + y + z}, 1; 1-\beta(x)\right)$$

Weight 3 solution

$$R_1(x) \equiv \sqrt{x(x+1)}$$

$$R_2 \equiv R_2(x,y,z) \equiv \sqrt{\lambda(x,y,z)}$$

$$\beta(t) \equiv \frac{R_1(t) - t}{R_1(t) + t}$$

- The shuffle algebra (without R_2)

$$\beta(y) \otimes \beta(x) \otimes (1+x) + \beta(x) \otimes \beta(y) \otimes (1+x) + \beta(x) \otimes (1+x) \otimes \beta(y)$$

$$G(1; 1-\beta(y)) * \left(\beta(x) \otimes (1+x)\right)$$

$$= G(1; 1-\beta(y)) * \left(\beta(x) \otimes \frac{\left(1+\beta(x)\right)^2}{4\beta(x)}\right)$$

$$= G(1; u(y)) * \left(2G(2,1; u(x)) - G(1,1; u(x))\right) \qquad u=1-\beta$$

Weight 3 solution

$$R_1(x) \equiv \sqrt{x(x+1)}$$

$$R_2 \equiv R_2(x,y,z) \equiv \sqrt{\lambda(x,y,z)}$$

$$\beta(t) \equiv \frac{R_1(t)-t}{R_1(t)+t}.$$

- Most symbols contain two R_1 with R_2
  - For example $R_1(x)$, $R_1(y)$, $R_2(x,y,z)$

- choose a path parameterized by $t\,\vec{x}$, $R_2(tx,ty,tz) = t * R_2(x,y,z)$
- rationalize $R_1(tx)$ and $R_1(ty)$ simultaneously :

$$t = \frac{v^2(2+v)^2}{4(1+v)\left((1+v)\sqrt{x}+\sqrt{y}\right)\left((1+v)\sqrt{y}+\sqrt{x}\right)}$$

$$\beta(tx) = \frac{(1+v)\sqrt{y}+\sqrt{x}}{(1+v)\left((1+v)\sqrt{x}+\sqrt{y}\right)} \qquad \beta(ty) = \frac{(1+v)\sqrt{x}+\sqrt{y}}{(1+v)\left((1+v)\sqrt{y}+\sqrt{x}\right)}$$

Weight 4 solution

$$R_1(x) \equiv \sqrt{x(x+1)}$$

$$R_2 \equiv R_2(x,y,z) \equiv \sqrt{\lambda(x,y,z)}$$

- Symbols contain all 4 square roots
  - Cannot be rationalized simultaneously and solved analytically

- One-fold numerical integration
  - Parameterize integration path γ with $\boldsymbol{t}\,\vec{\boldsymbol{x}}$, $t \in [0,1]$

$$\vec{f}^{(i)}(\vec{x}) = \int_\gamma \mathrm{d}A(\vec{x})\,\vec{f}^{(i-1)}(\vec{x}) + \vec{f}^{(i)}(\vec{x_0})$$

$$\vec{f}^{(4)}(\vec{x}) = \int_0^1 \partial_t A(t)\,\vec{f}^{(3)}(t)\,dt$$

Result

- **Why analytic result ???**

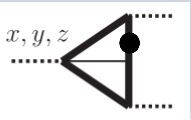
  - Accurate and fast

  - Apply to any phase-point

  - Investigate the behaviors in various asymptotic limit (to be done)

Result

- Accurate and fast

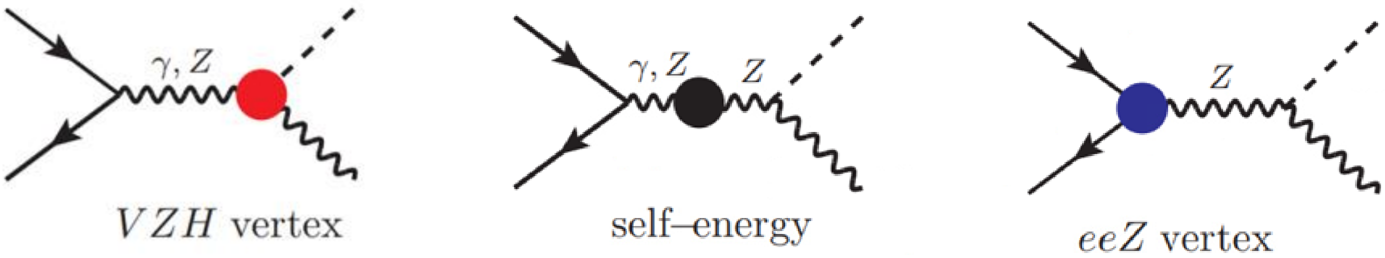
	$\sqrt{s} = 300\ GeV$		$\sqrt{s} = 500\ GeV$	
	ϵ^0	ϵ^1	ϵ^0	ϵ^1
Analytic result	0.329941	0.081257	0.410898 + 0.114525 i	0.103284 + 0.399126 i
pySecDec *	0.329(9)	0.081(2)	0.410(9) + 0.114(5) i	0.103(2) + 0.399(1) i
Time for all 41 MIs	12 s		19 s	
	556 s		517 s	

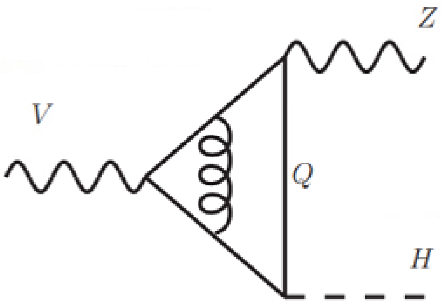
* S. Borowka et al. pySecDec... arXiv:1703.09692

Result

- $e^+ + e^- \rightarrow V \rightarrow Z + H$

$$\sigma^{\alpha\alpha_s} = \sigma_Z^{\alpha\alpha_s} + \sigma_\gamma^{\alpha\alpha_s} + \sigma_{eeZ}^{\alpha\alpha_s}$$





Result

- $e^+ + e^- \to V \to Z + H$
  - Top

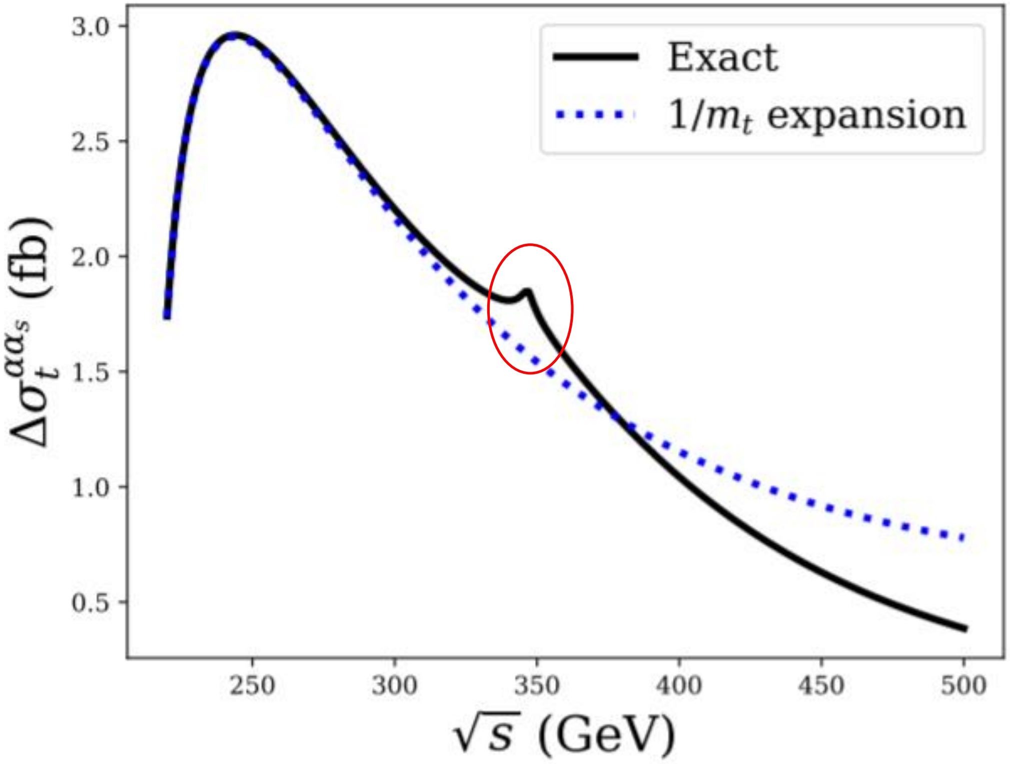


Fig1 : NNLO $\mathcal{O}(\alpha\alpha_s)$ corrections from top quark loops to the $e^+e^- \to ZH$ production cross sections.

Result

- $e^{+} + e^{-} \rightarrow V \rightarrow Z + H$
  - Top + Bottom

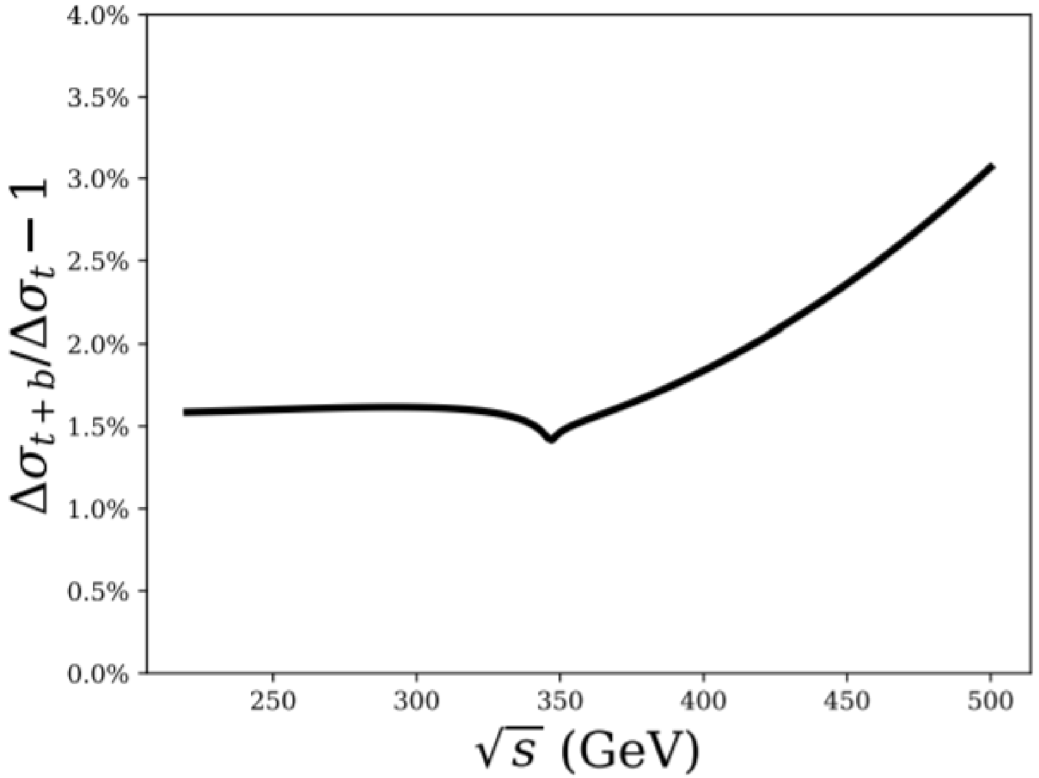


Fig2 : Relative corrections from **bottom** quark loops

Result

- $H \rightarrow Z\, Z^{*} \rightarrow Z + 2l$
  - Top + Bottom

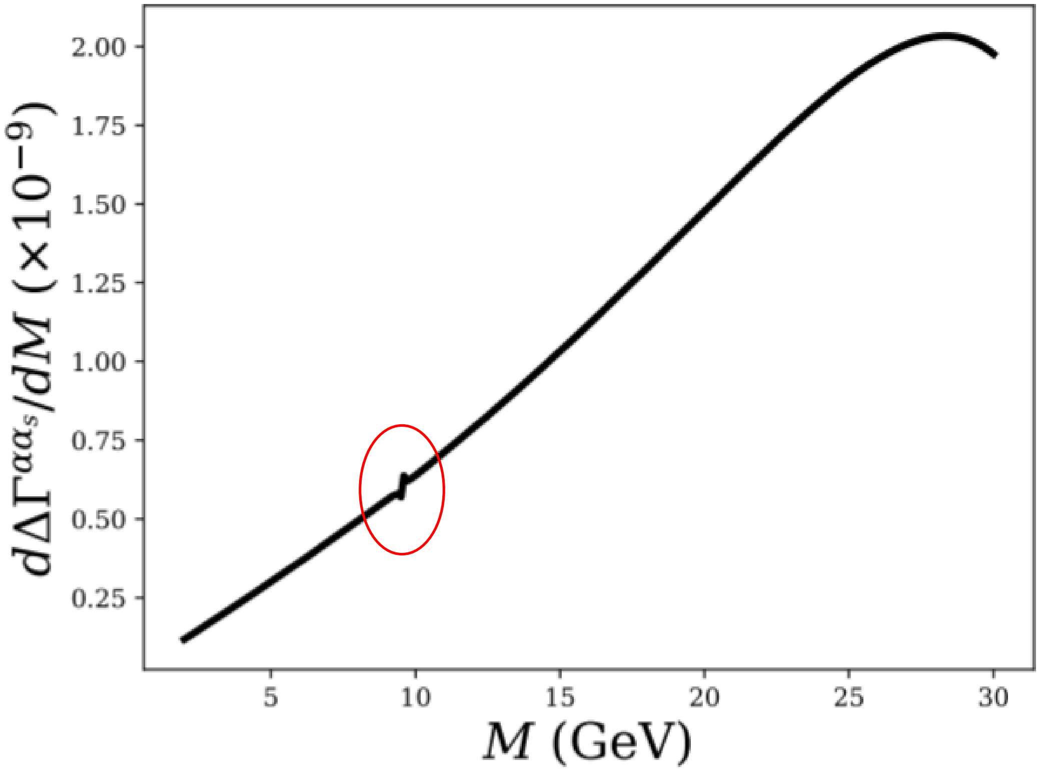


Fig3 : NNLO $\mathcal{O}(\alpha\alpha_s)$ corrections differential decay rate as a function of the lepton pair invariant mass M, including **both top and bottom** quark contributions

Summary

- We give the fully **analytic result** up to weight-3 of Two-loop triangle integrals with 4 scales for the $\boldsymbol{HZV}$ vertex .

- The analytic result we get is **accurate and fast**

- We use our analytic result to calculate the $e^+e^- \to ZH$ production cross section and the $H \to Z\,Z^* \to Z + 2l$ decay width.

Thanks!

Yuxuan Wang