Methods for multiloop calculations

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Based on:

X. Liu, Y.Q. Ma, C. Y. Wang, Phys.Lett. B779 (2018) 353

X. Liu, Y.Q. Ma, Phys.Rev.D99 (2019) no.7, 071501

CEPC Physics Workshop
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Outline

I. Introduction

II. Current methods for multiloop calculations

II. A new method

IV. Summary and outlook

CEPC/SPPC physics

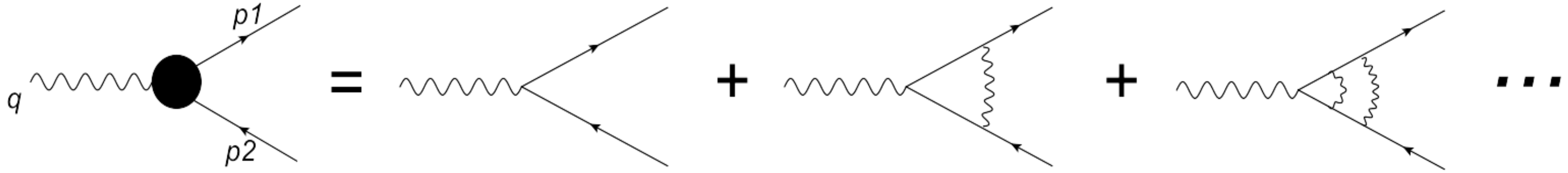
➢ Experiment versus Theory

- **Test for the standard model**
- **Search for signals of new physics**

➢ Various processes at future colliders

CEPC-SPPC study group, 2015

- $e^+ + e^- \to Z + H$ **(CEPC)**
- $e^+ + e^- \to \mu^+ + \mu^- + H$ **(CEPC)**
- $g + g \to H + H$ **(SPPC)**
- $g + g \to t + \bar{t} + H$ **(SPPC)**
- …



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Scattering amplitudes

➢ Construct the amplitude

$$\mathcal{M} = \sum_{i=1}^{\mathcal{O}(10^4)} f_i \times \mathcal{M}_i$$

- Feynman rules, unitarity-based method, etc.

➢ Reduce the amplitude to master integrals

$$\mathcal{M}_i = \sum_{j=1}^{\mathcal{O}(10^2)} c_{ij} \times I_j$$

- Unitarity-based method, integration-by-parts reduction, etc.

➢ Calculate master integrals

- Sector decomposition, Mellin-Barnes representation, differential equations method, etc.

Integration-by-parts reduction

➢ Integration-by-parts (IBP)

Chetyrkin, Tkachov, NPB (1981)
Laporta, 0102033

$$0 = \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \frac{\partial}{\partial \ell_j^\mu} \left(\frac{v_k^\mu}{\mathcal{D}_1^{\nu_1} \cdots \mathcal{D}_m^{\nu_m}} \right)$$

- Laporta's algorithm: huge coupled linear system $A(\vec{s}, D)\ \vec{I}(\vec{s}, D) = 0$
- Gaussian elimination ⇒ master integrals
- Implementations:
  - AIR — Anastasiou and Lazopoulos, 0404258
  - FIRE — Smirnov, 0807.3243
  - Reduze — Studerus, 0912.2546
  - LiteRed — Lee, 1212.2685
  - Kira — Maierhoefer, Usovitsch and Uwer, 1705.05610
- New development: "syzygy equations"
  Gluza, Kajda and Kosower, 1009.0472
  Ita, 1510.05626
  Larsen and Zhang, 1511.01071

Generalized Unitarity Cuts

➢ Generalized unitarity

Bern, Dixon, Dunbar, Kosower, 9403226
Britto, Cachazo, Feng, 0412103

$$\mathcal{M}^{(1)}(2\to 2) = \int \frac{\mathrm{d}^D\ell}{\mathrm{i}\pi^{D/2}}\left(\frac{\Delta_4(\ell)}{\mathcal{D}_1\mathcal{D}_2\mathcal{D}_3\mathcal{D}_4} + \sum_{i_1 i_2 i_3}\frac{\Delta_{3,i_1 i_2 i_3}(\ell)}{\mathcal{D}_{i_1}\mathcal{D}_{i_2}\mathcal{D}_{i_3}} + \sum_{i_1 i_2}\frac{\Delta_{2,i_1 i_2}(\ell)}{\mathcal{D}_{i_1}\mathcal{D}_{i_2}} + \text{tadpoles}\right)$$

$$D_1 = D_2 = D_3 = D_4 = 0 \quad \Rightarrow \quad \Delta_4$$

$$D_{i_1} = D_{i_2} = D_{i_3} = 0 \quad \Rightarrow \quad \Delta_{3,i_1 i_2 i_3}$$

$$\cdots$$

- **Hugely successful at one loop**

Ossola, Papadopoulos, Pittau, 0609007
Ossola, Papadopoulos, Pittau, 0711.3596

- **Extension beyond one loop**

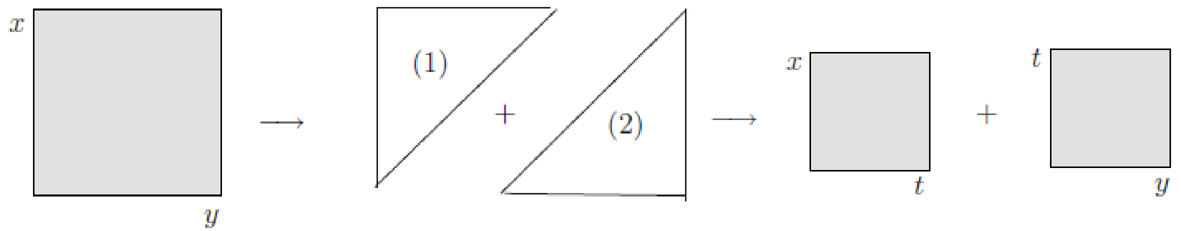
Mastrolia and Ossola, 1107.6041
Badger, Frellesvig and Zhang, 1202.2019
Zhang, 1205.5707
Mastrolia et al., 1205.7087
Ita, 1510.05626
…

Sector decomposition

➢ Sector decomposition

Binoth and Heinrich, 0004013
Heinrich, 0803.4177

$$I = \int_0^1 dx \int_0^1 dy\, x^{-1-a\epsilon}\, y^{-b\epsilon} \Big(x + (1-x)\,y\Big)^{-1}$$



$$I = \int_0^1 dx\, x^{-1-(a+b)\epsilon} \int_0^1 dt\, t^{-b\epsilon} \Big(1 + (1-x)\,t\Big)^{-1}$$
$$+ \int_0^1 dy\, y^{-1-(a+b)\epsilon} \int_0^1 dt\, t^{-1-a\epsilon} \Big(1 + (1-y)\,t\Big)^{-1}$$

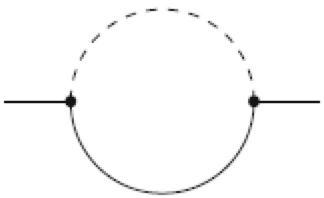
- The idea can be applied to Feynman parametric representation.
- Implementations: FIESTA Smirnov and Tentyukov, 0807.4129 SecDec Carter and Heinrich, 1011.5493

  pySecDec Borowka et al., 1703.09692

Mellin-Barnes Representation

➢ **Mellin-Barnes representation** Usyukina, 1975; Smirnov, 9905323

$$\frac{1}{(X+Y)^\lambda} = \frac{1}{\Gamma(\lambda)}\frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \Gamma(\lambda+z)\Gamma(-z)\frac{Y^z}{X^{\lambda+z}}$$



$$= \int \frac{\mathrm{d}^D\ell}{\mathrm{i}\pi^{D/2}} \frac{1}{(\ell^2-m^2)(\ell+p)^2}$$

$$\longrightarrow \int \mathrm{d}z \frac{\Gamma(\epsilon+z)\Gamma(-z)\Gamma(1-\epsilon-z)}{\Gamma(2-2\epsilon-z)}\left(-\frac{m^2}{p^2}\right)^z$$

- The integration can be carried out numerically after resolving the singularities in ϵ.
- Mathematica packages: MB.m Czakon, 2005 AMBRE.m Gluza, Kajda and Riemann, 0704.2423
  MBresolve.m A.Smirnov and V.Smirnov, 0901.0386
  MBnumerics Usovitsch, Dubovyk and Riemann, 1810.04580

Differential equations

- **Differential equations** Kotikov, PLB 267 (1991) 123-127

$$\frac{\partial}{\partial x}\vec{I}(x;\epsilon) = A(x;\epsilon)\vec{I}(x;\epsilon)$$

$$\vec{I}(x;\epsilon) \overset{x\to x_0}{\sim} \cdots$$

  - x: $p^2, m^2, \ldots$

- **Canonical form** Henn, 1304.1806 Henn, 1412.2296

$$\vec{I}(x;\epsilon) = T(x;\epsilon)\vec{J}(x;\epsilon) \;\Rightarrow\; \frac{\partial}{\partial x}\vec{J}(x;\epsilon) = \epsilon A(x)\vec{J}(x;\epsilon)$$

  - Solutions can be systematically expressed by iterated integrals.

State of the art

- Two-loop massive $2 \rightarrow 2$ and massless $2 \rightarrow 3$ processes are already the frontier
  - $g+g \rightarrow t+\bar{t},\ g+g \rightarrow H+H,\ g+g \rightarrow g+g+g, \ldots$
- Very time-consuming
  - Two-loop $g+g \rightarrow H+H\ (g)$: complete IBP reduction cannot be achieved within tolerable time

    Borowka et. al., 1604.06447
    Jones, Kerner, Luisoni, 1802.00349
  - Two-loop decay $Q+\bar{Q} \rightarrow g+g$, masters cost $O(10^5)$ CPU core-hour

    Feng, Jia, Sang, 1707.05758
  - …
- Even more complicated processes in future? Nearly hopeless!

New ideas are badly needed

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Synopsis

- **A new representation of Feynman integrals**

$$\mathcal{M}(D,\vec{s},\eta) \equiv \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \prod_{\alpha=1}^{N} \frac{1}{(q_\alpha^2 - m_\alpha^2 + \mathrm{i}\eta)^{\nu_\alpha}}$$

$$\mathcal{M}(D,\vec{s},\eta) = \eta^{LD/2-\nu} \sum_{\mu_0=0}^{\infty} \eta^{-\mu_0} \mathcal{M}_{\mu_0}^{\mathrm{vac}}(D,\vec{s}) \qquad |\eta| \to \infty$$

- **Reduction to MIs through the new rep.**

$$\sum_{i=1}^{n} Q_i(D,\vec{s},\eta)\, \mathcal{M}_i(D,\vec{s},\eta) = 0$$

- **Analytic continuation for MIs**

$$\frac{\partial}{\partial\eta} \vec{I}(D;\eta) = A(D;\eta)\vec{I}(D;\eta)$$ **with known** $\vec{I}(D;\infty)$

Feynman integrals with η

➢ Dimensionally regularized Feynman integral

$$\mathcal{M}(D,\vec{s},\eta) \equiv \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \prod_{\alpha=1}^{N} \frac{1}{(q_\alpha^2 - m_\alpha^2 + \mathrm{i}\eta)^{\nu_\alpha}}$$

- **Analytic function of η**
- **Physical result is defined by the limit**

$$\mathcal{M}(D,\vec{s},0) \equiv \lim_{\eta\to 0^+} \mathcal{M}(D,\vec{s},\eta)$$

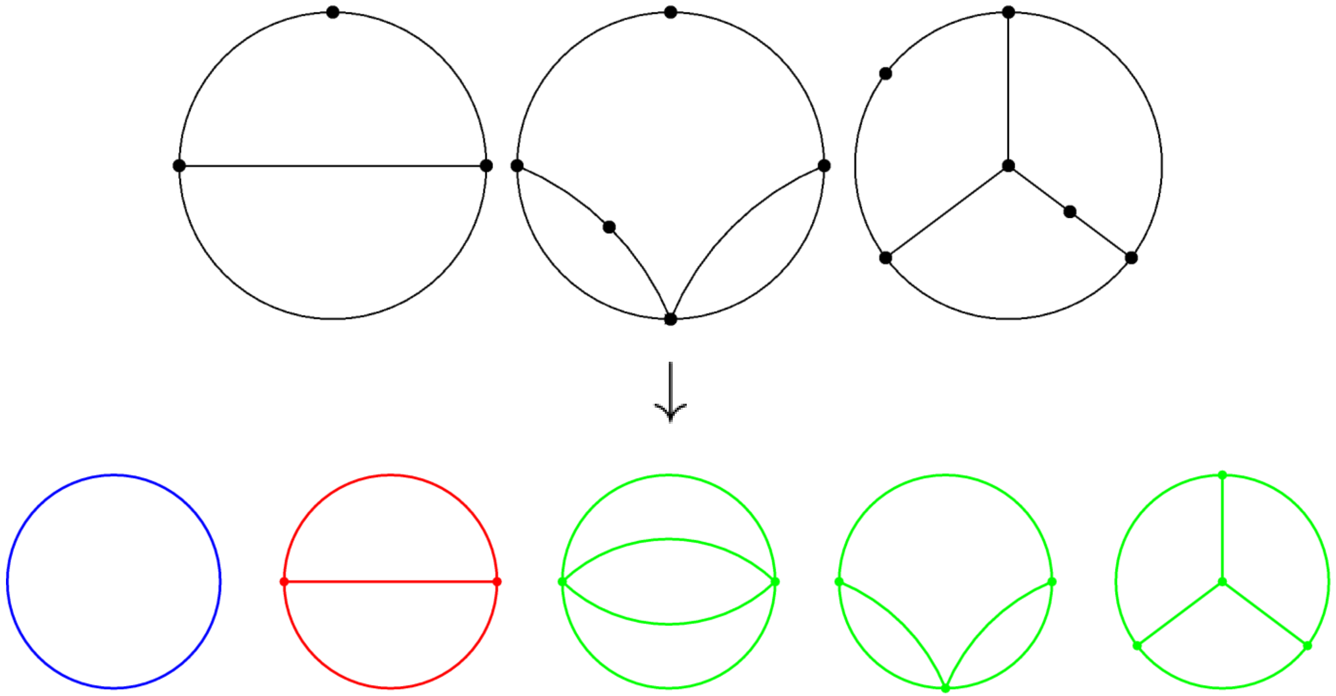
➢ Expansion around $\eta = \infty$

$$\frac{1}{[(\ell+p)^2 - m^2 + \mathrm{i}\eta]^\nu} = \frac{1}{(\ell^2+\mathrm{i}\eta)^\nu} \sum_{n=0}^{\infty} \frac{(\nu)_n}{n!} \left(\frac{-2\ell\cdot p - p^2 + m^2}{\ell^2 + \mathrm{i}\eta}\right)^n$$

- **Only one region: $l^\mu \sim |\eta|^{1/2}$**
- **No external momenta in denominator, vacuum integrals**

Vacuum integrals

➢ "Building blocks"



- Analytical results are known up to 3-loop
- Numerical results are known up to 5-loop

Davydychev,Tausk, NPB(1993)
Broadhurst, 9803091
Kniehl, Pikelner, Veretin, 1705.05136

Schroder, Vuorinen, 0503209
Luthe, PhD thesis (2015)
Luthe, Maier, Marquard, Ychroder, 1701.07068

A new representation

➢ Asymptotic expansion

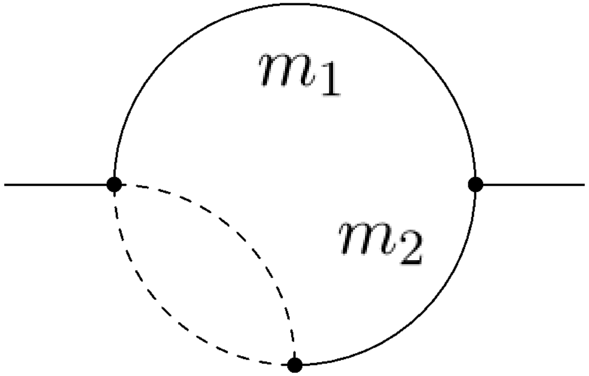
$$\mathcal{M}(D,\vec{s},\eta)=\eta^{LD/2-\nu}\sum_{\mu_0=0}^{\infty}\eta^{-\mu_0}\mathcal{M}^{\text{vac}}_{\mu_0}(D,\vec{s})$$

$$\mathcal{M}^{\text{vac}}_{\mu_0}(D,\vec{s})=\sum_{k=1}^{B_L}I^{\text{vac}}_{L,k}(D)\sum_{\vec{\mu}\in\Omega^r_{\mu_0}}C_k^{\mu_0\cdots\mu_r}(D)s_1^{\mu_1}\cdots s_r^{\mu_r}$$

- $I^{vac}_{L,k}(D)$: known vacuum master integrals
- $C_k^{\mu_0\dots\mu_r}(D)$: known rational functions of D
- Physical Feynman integral can be obtained by analytic continuation of this calculable asymptotic series: a new representation

Example

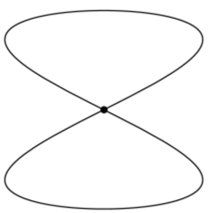
- **The seagull integral**

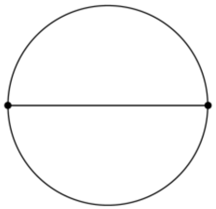


$$\mathcal{D}_1 = (\ell_1 - p)^2 - m_1^2\,, \quad \mathcal{D}_2 = \ell_1^2 - m_2^2\,,$$

$$\mathcal{D}_3 = \ell_2^2\,, \quad \mathcal{D}_4 = (\ell_1 + \ell_2)^2$$

$$\mathcal{M}(D, \{p^2, m^2\}, \eta) \equiv \eta^{D-4} \tilde{\mathcal{M}}(D, \{p^2, m^2\}, \eta)$$

$\tilde{\mathcal{M}} =$  $\times \left[\frac{(D-16)(D-2)^2}{108\,\mathrm{i}D}\frac{p^2}{\eta} + \frac{(D-2)^2}{12\,\mathrm{i}}\frac{m_1^2+m_2^2}{\eta} + \mathcal{O}(\eta^{-2})\right]$

$+$  $\times \left[\frac{3-D}{3\,\mathrm{i}} + \frac{(D-3)(5D^2-24D-32)}{162D}\frac{p^2}{\eta} - \frac{(D-8)(D-3)}{18}\frac{m_1^2+m_2^2}{\eta} + \mathcal{O}(\eta^{-2})\right]$

Reduction part 1: linear relations

➢ Reduction

- Set up linear relations among loop integrals;
- Use the relations to express Feynman integrals in terms of masters.

➢ Linear relations among $G \equiv \{M_1, M_2, \ldots, M_n\}$

$$\sum_{i=1}^{n} Q_i(D, \vec{s}, \eta)\, \mathcal{M}_i(D, \vec{s}, \eta) = 0$$

- $Q_i(D, \vec{s}, \eta)$: homogeneous polynomials of $\vec{s}, \eta$ of degree d_i

➢ Constraints from mass dimension

$$2d_1 + \mathrm{Dim}(\mathcal{M}_1) = \cdots = 2d_n + \mathrm{Dim}(\mathcal{M}_n)$$

- Only 1 degree of freedom in $\{d_i\}$, chosen as $d_{\max} \equiv \mathrm{Max}\,\{d_i\}$

Reduction part 1: linear relations

- **Decomposition of** $Q_i(D,\vec{s},\eta)$

$$Q_i(D,\vec{s},\eta) = \sum_{(\lambda_0,\vec{\lambda})\in\Omega_{d_i}^{r+1}} \tilde{Q}_i^{\lambda_0\ldots\lambda_r}(D)\,\eta^{\lambda_0}s_1^{\lambda_1}\cdots s_r^{\lambda_r}$$

- **Linear constraints for** $\tilde{Q}$

$$\sum_{i=1}^{n} Q_i(D,\vec{s},\eta)\,\mathcal{M}_i(D,\vec{s},\eta) = 0$$

$$\sum_{k,\rho,\vec{\rho}} f_k^{\rho_0\cdots\rho_r}[\tilde{Q}] \times I_{L,k}^{\text{vac}}(D)\eta^{\rho_0}\cdots s_r^{\rho_r} = 0$$

$$\Downarrow$$

$$f_k^{\rho_0\cdots\rho_r}[\tilde{Q}] = 0$$

- With enough constraints ⇒ $\tilde{Q}_i^{\lambda_0\ldots\lambda_r}(D)$
- With finite field technique, only integers in a finite field are involved, equations can be efficiently solved

Manteuffel and Schabinger, 1406.4513
Peraro, 1608.01902

Reduction part 2: algorithm

- **Reduction:** $\text{desired integrals } G_1 \longrightarrow \text{target integrals } G_2$

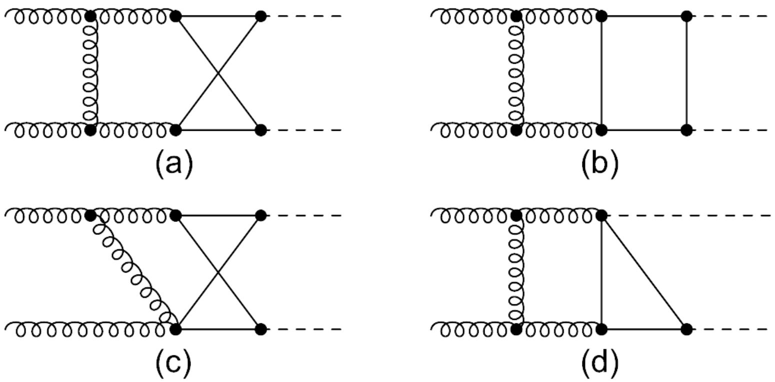
- **Algorithm to reduce G_1** *Search for simplest relations*

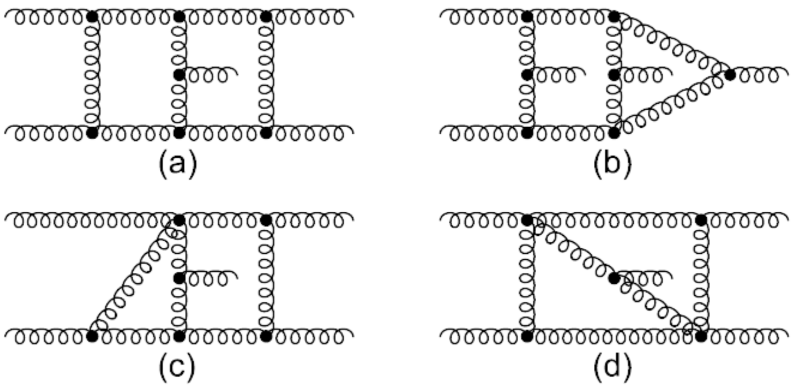
1. Set $d_{\max} = 0$
2. Find out all relations in $G = G_1 \cup G_2$ with fixed $d_{\max}$
3. If obtained relations are enough to reduce G_1, stop; else $d_{\max}++$ and go to step 2

With an appropriate choice of G_1 and G_2, one can realize a step-by-step reduction!

Examples

➢ 2-loop $g+g\to H+H$ and $g+g\to g+g+g$





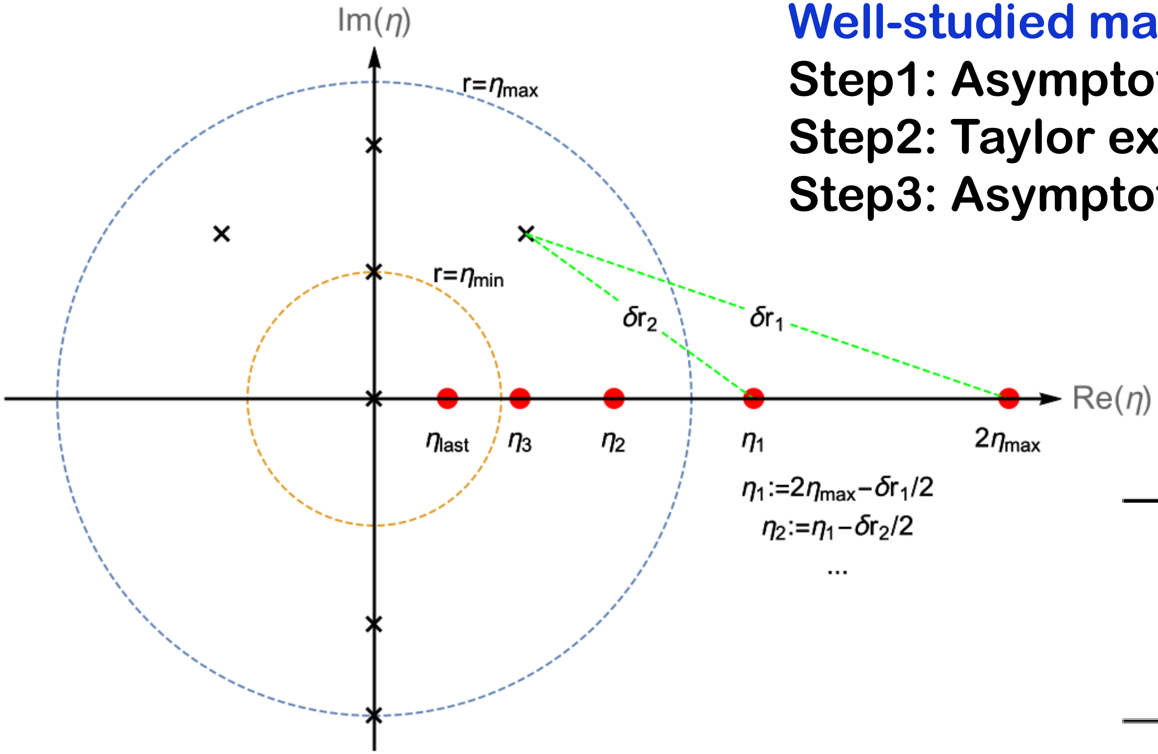
$g+g\to H+H$					$g+g\to g+g+g$				
Sector	Type	$d^{\mathrm{T}}_{\max}$	$\mathbf{m}^+$	Time/s	Sector	Type	$d^{\mathrm{T}}_{\max}$	$\mathbf{m}^+$	Time/s
(a)	7-NP	1	$\mathbf{3}^+$	3444	(a)	8-NP	1	$\mathbf{3}^+$	18281
(b)	7-P	1	$\mathbf{3}^+$	1771	(b)	8-NP	3	$\mathbf{3}^+$	54353
(c)	6-NP	5	$\mathbf{3}^+$	59354	(c)	7-NP	4	$\mathbf{3}^+$	102629
(d)	6-P	3	$\mathbf{2}^+$	462	(d)	6-NP	2	$\mathbf{3}^+$	832

- **Once the block triangular analytic relations are obtained, numerical calculation can be very efficient.** $O(N)$!

Analytic continuation

➢ **Solve differential equations of master integrals**

$$\frac{\partial}{\partial\eta}\vec{I}(D;\eta) = A(D;\eta)\vec{I}(D;\eta) \quad \textbf{with known } \vec{I}(D;\infty)$$

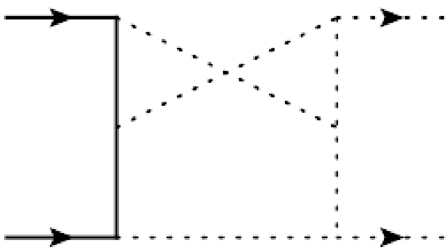


Well-studied mathematic problem:

Step1: Asymptotic expansion at $\eta = \infty$

Step2: Taylor expansion at analytical points

Step3: Asymptotic expansion at $\eta = 0$



Faster than FIESTA by 10^5 times

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Summary and outlook

- **Multiloop calculation is by any means necessary for CEPC/SPPC physics. However, traditional methods may not meet our requirements any more.**
- **We propose a new method for multiloop calculations and prove its power by several cutting edge examples.**
- $O(\alpha^2)$ **correction for** $e^+ + e^- \to Z + H$**,**
  $O(\alpha_s^2)$ **correction for** $g + g \to t + \bar{t} + H$ …

Thank you!

Analytic structure at infinity

- **Feynman parametric rep.**

$$I(\eta) = (-1)^{\nu} \frac{\Gamma(\nu - LD/2)}{\prod_i \Gamma(\nu_i)} \int \prod_{\alpha} (x_{\alpha}^{\nu_\alpha - 1} \mathrm{d}x_\alpha)\, \delta\left(1 - \sum_j x_j\right) \frac{\mathcal{U}^{-D/2}}{(\mathcal{F}/\mathcal{U} - \mathrm{i}\eta)^{\nu - LD/2}}$$

  - $\mathcal{U}$: graph polynomial of 1-tree
  - $\mathcal{F}$: graph polynomial of 2-tree

- **Observation: $|\mathcal{F}/\mathcal{U}|$ is bounded in the Feynman parameter space!**

$$|\mathcal{F}_i| < |t_i||\mathcal{U}_i| < |t_i||\mathcal{U}| \text{ and } |\mathcal{F}| < \sum_i |t_i||\mathcal{U}|$$

- **Thus:** $J(D;\eta) \equiv \eta^{\nu - LD/2} I(D;\eta)$ **is analytic at** $\eta = \infty$

Example

- $G = \{M_1, M_2, M_3\}$

$$\mathcal{M}_1 \equiv \int \frac{\mathrm{d}^D\ell}{\mathrm{i}\pi^{D/2}} \frac{1}{\mathcal{D}_1 + \mathrm{i}\eta}, \quad \mathcal{M}_2 \equiv \int \frac{\mathrm{d}^D\ell}{\mathrm{i}\pi^{D/2}} \frac{1}{(\mathcal{D}_1 + \mathrm{i}\eta)(\mathcal{D}_2 + \mathrm{i}\eta)}, \quad \mathcal{M}_3 \equiv \int \frac{\mathrm{d}^D\ell}{\mathrm{i}\pi^{D/2}} \frac{1}{(\mathcal{D}_1 + \mathrm{i}\eta)(\mathcal{D}_2 + \mathrm{i}\eta)^2},$$

$$\mathcal{D}_1 = \ell^2,\ \mathcal{D}_2 = (\ell + p)^2$$

- **Asymptotic Series**

$$\mathcal{I}_{1,1}^{\mathrm{bub}}(D) \equiv \int \frac{\mathrm{d}^D\tilde{\ell}}{\mathrm{i}\pi^{D/2}} \frac{1}{\tilde{\ell}^2 + \mathrm{i}}.$$

$$\begin{aligned}
\mathcal{M}_1 =& \eta^{D/2-1}\mathcal{I}_{1,1}^{\mathrm{bub}}(D), \\
\mathcal{M}_2 =& \eta^{D/2-2}\mathcal{I}_{1,1}^{\mathrm{bub}}(D)\left[-\frac{D-2}{2\,\mathrm{i}} + \frac{(D-4)(D-2)}{24}\frac{p^2}{\eta} + \frac{(D-6)(D-4)(D-2)}{480\,\mathrm{i}}\frac{p^4}{\eta^2}\right. \\
&\left. -\frac{(D-8)(D-6)(D-4)(D-2)}{13440}\frac{p^6}{\eta^3} - \frac{(D-10)(D-8)(D-6)(D-4)(D-2)}{483840\,\mathrm{i}}\frac{p^8}{\eta^4} + \cdots\right], \\
\mathcal{M}_3 =& \eta^{D/2-3}\mathcal{I}_{1,1}^{\mathrm{bub}}(D)\left[-\frac{(D-4)(D-2)}{8} - \frac{(D-6)(D-4)(D-2)}{96\,\mathrm{i}}\frac{p^2}{\eta} + \frac{(D-8)(D-6)(D-4)(D-2)}{1920}\frac{p^4}{\eta^2}\right. \\
&\left. +\frac{(D-10)(D-8)(D-6)(D-4)(D-2)}{53760\,\mathrm{i}}\frac{p^6}{\eta^3} - \frac{(D-12)(D-10)(D-8)(D-6)(D-4)(D-2)}{1935360}\frac{p^8}{\eta^4} + \cdots\right],
\end{aligned}$$

Example

- $Q_1 M_1 + Q_2 M_2 + Q_3 M_3 = 0$ **with** $d_{max} = 2$

$$Q_1(D,p^2,\eta) = Q_1^{00}\,, \quad Q_2(D,p^2,\eta) = Q_2^{10}\,\eta + Q_2^{01}\,p^2\,, \quad Q_3(D,p^2,\eta) = Q_3^{20}\,\eta^2 + Q_3^{11}\,\eta\,p^2 + Q_3^{02}\,p^4$$

- **Linear equations**

$$\mathcal{I}_{1,1}^{\text{bub}} \quad : \quad Q_1^{00} - \frac{D-2}{2\mathrm{i}} Q_2^{10} - \frac{(D-4)(D-2)}{8} Q_3^{20} = 0\,,$$

$$\mathcal{I}_{1,1}^{\text{bub}}\frac{p^2}{\eta} \quad : \quad \frac{D-4}{24} Q_2^{10} - \frac{1}{2\mathrm{i}} Q_2^{01} - \frac{(D-6)(D-4)}{96\,\mathrm{i}} Q_3^{20} - \frac{D-4}{8} Q_3^{11} = 0\,,$$

$$\mathcal{I}_{1,1}^{\text{bub}}\frac{p^4}{\eta^2} \quad : \quad \frac{D-6}{480\,\mathrm{i}} Q_2^{10} + \frac{1}{24} Q_2^{01} + \frac{(D-8)(D-6)}{1920} Q_3^{20} - \frac{D-6}{96\,\mathrm{i}} Q_3^{11} - \frac{1}{8} Q_3^{02} = 0\,,$$

$$\mathcal{I}_{1,1}^{\text{bub}}\frac{p^6}{\eta^3} \quad : \quad -\frac{D-8}{13440} Q_2^{10} + \frac{1}{480\,\mathrm{i}} Q_2^{01} + \frac{(D-10)(D-8)}{53760\,\mathrm{i}} Q_3^{20} + \frac{D-8}{1920} Q_3^{11} - \frac{1}{96\,\mathrm{i}} Q_3^{02} = 0\,,$$

$$\mathcal{I}_{1,1}^{\text{bub}}\frac{p^8}{\eta^4} \quad : \quad -\frac{D-10}{483840\,\mathrm{i}} Q_2^{10} - \frac{1}{13440} Q_2^{01} - \frac{(D-12)(D-10)}{1935360} Q_3^{20} + \frac{D-10}{53760\,\mathrm{i}} Q_3^{11} + \frac{1}{1920} Q_3^{02} = 0\,,$$

$$\{Q_1^{00}, Q_2^{10}, Q_2^{01}, Q_3^{20}, Q_3^{11}, Q_3^{02}\} \sim \{D-2, 2\,\mathrm{i}(D-3), 0, -8, 2\,\mathrm{i}, 0\}\,.$$

Denominator reduction scheme

- **Powers of denominators:** $\vec{\nu} = (\nu_1, \dots, \nu_N), \nu_i \geq 0$
  - $\mathbf{1}^+(5,1,0,3) = \{(6,1,0,3), (5,2,0,3), (5,1,0,4)\}$
  - $\mathbf{1}^-(5,1,0,3) = \{(4,1,0,3), (5,0,0,3), (5,1,0,2)\}$
  - $\mathbf{0}^\pm \equiv \textbf{Identity}, \ \mathbf{m}^\pm \equiv (\mathbf{m}-\mathbf{1})^\pm \mathbf{1}^\pm$
- **One-loop:** $G_1 = \mathbf{1}^+\vec{\nu}, G_2 = \mathbf{1}^-\mathbf{1}^+\vec{\nu}$ Duplancic and Nizic, hep-ph/0303184
- **Multiloop:**

$$G_1 = \mathbf{m}^+\vec{\nu}, G_2 = \{\mathbf{1}^-\mathbf{m}^+, \mathbf{1}^-(\mathbf{m}-\mathbf{1})^+, \dots, \mathbf{1}^-\mathbf{1}^+\}\vec{\nu}$$

  - The size of G_1 is not too large, about dozens of integrals
  - Relations among G_1 and G_2 are not too complicated, see examples

A step-by-step reduction is realized!