Helicity amplitudes for bremsstrahlung and other

Yahor Dydyshka[a,b]

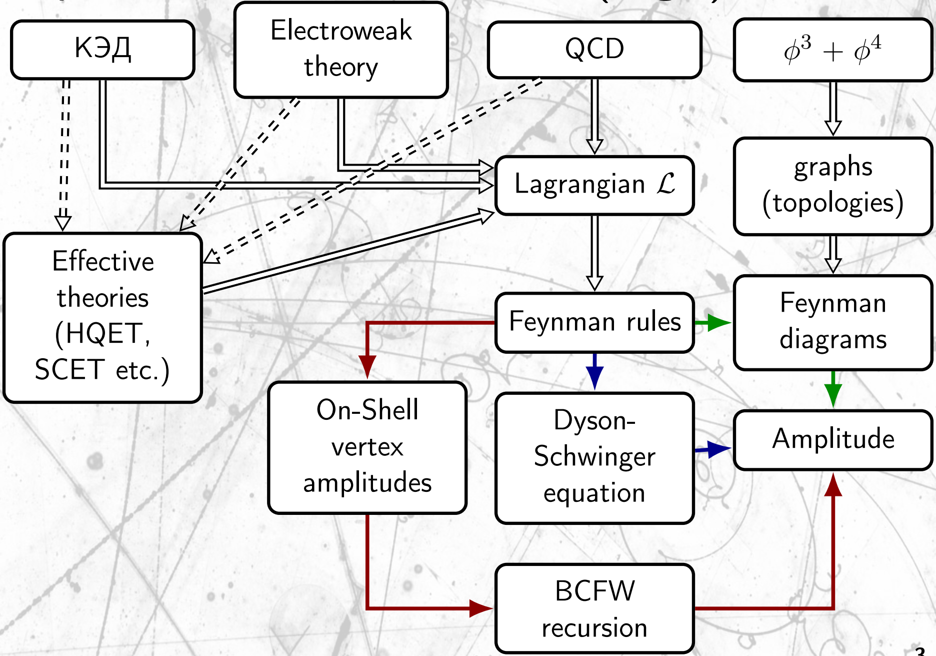
[a] *DLNP, JINR, Dubna*
[b] *INP BSU, Minsk*

IHEP, 4 April 2019

Outline

- General steps of theoretical calculations
- Two types of HA methods
- Limitations and complications
- The formalism
- Light-cone kinematics and $e^+e^- \to ZH\gamma$
- Numerical behaveour of singular terms;
- General-helicity amplitude for $e^+e^- \to f\bar{f}\gamma$ and $e^+e^- \to e^+e^-\gamma$;
- Further development;
- Conclusion;

Steps of theoretical calculations (begin)



Two types of HA methods

Lorentz-non-covariant HA

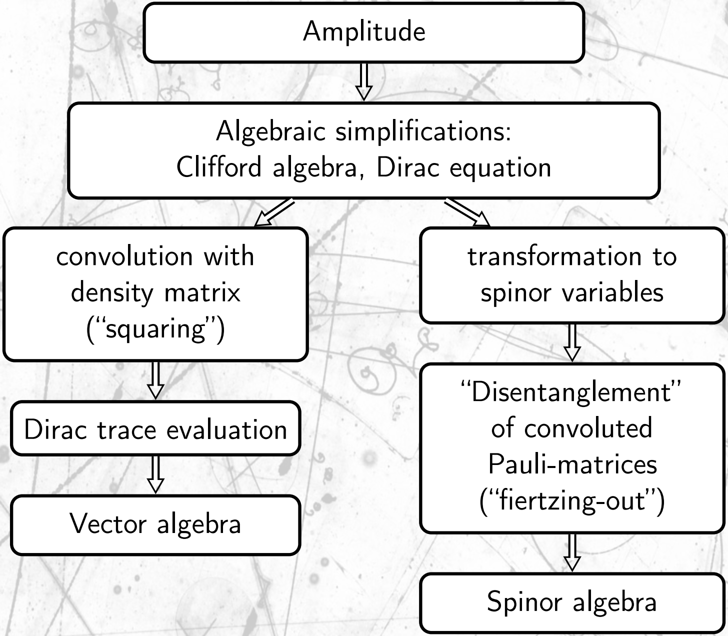
- Dirac-matrix multiplication in some reference frame;
- Klein–Nishina formula obtained by this approach;
- Nowadays used by HELAS and SHERPA.AMEGIC++ at numerical level;
- Applied to solve Dyson-Schwinger Equations (DSE) numerically by ALPGEN and WHIZARD/O'Mega.
- Extended to 1-loop by Madgraph and RECOLA

Two types of HA methods

Lorentz-covariant HA

- Reference frame is not fixed;
- Pioneered by N. Fedorov at Minsk in 1956;
- Becomes popular after works of `CALCUL` group in 1980s;
- Extended by Bern-Dixon-Kosower to d-dimensions and thus to loop integrands;
- Nowadays used by `GoSam/Golem` and `FormCalc` packages for automated 1-loop calculations;
- Applied for 2-loop calculations in $\mathcal{N} = 4$ SYM.

Steps of theoretical calculations (continue)



Limitations and complications

Vector algebra

- Momentum conservation is linear;
- On-shell conditions are quadratic in momenta;
- On-shell conditions are linear in invariants;
- 4-dimensionness is polynomial in invariants;

Spinor algebra

- On-shell conditions are solved explicitly;
- 4-dimensionness of space-time is equivalent to 2-dimensionness of spinor space;
- 2-dimensionness of spinor space is quadratic in spinor products;
- Momentum conservation is quadratic in spinors;
- Momentum conservation can be solved explicitly with help of momentum-twistors;

Limitations and complications

Collinear singularities

- HAs are especially elegant for massless particles;
- In collinear regions there may be "singular terms" of the from $\frac{m^2}{(2pk)^2} \sim \delta(2pk)$ giving nonzero contribution to x-section but absent in massless amplitude;
- Manual addition of such terms is not technologized for polarized x-sections;
- Conclusion: Massless HAs $\neq$ polarized matrix elements.

Helicity of massive particle is not Lorents-invariant

- spin-quantization axis should be specified;
- The helicity states are defined relative to reference frame;
- Auxilary vectors needed to decompose massive momenta onto sum of massless vectors;

Spinors and massless vectors

Vector as a spin-tensor

In spinoral space any vector represented by 2×2-tensor

$$p_{A\dot{A}} = \begin{pmatrix} p_0 + p_z & p_x - ip_y \\ p_x + ip_y & p_0 - p_z \end{pmatrix}, \quad p^{\dot{A}A} = \begin{pmatrix} p_0 - p_z & -p_x + ip_y \\ -p_x - ip_y & p_0 + p_z \end{pmatrix}. \quad (1)$$

It has the only Lorentz invatiant $\frac{1}{2!}\epsilon^{AB}\epsilon^{\dot{A}\dot{B}}p_{A\dot{A}}p_{B\dot{B}} = \det(p_{A\dot{A}}) = m_i^2$.
So for massless vector we have rank-1 matrix, i. e.

$$p_{A\dot{A}} = \lambda_A \widetilde{\lambda}_{\dot{A}} \quad (2)$$

Explicit solutions for λ and $\bar{\lambda}$

$$p_{A\dot{A}} = \begin{pmatrix} \sqrt{p_0 + p_z} \\ \dfrac{p_x + ip_y}{\sqrt{p_0 + p_z}} \end{pmatrix} \otimes \begin{pmatrix} \sqrt{p_0 + p_z} & \dfrac{p_x - ip_y}{\sqrt{p_0 + p_z}} \end{pmatrix} \quad (3)$$

Dirac spinors

Chiral representation of γ-matrices

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^{\pm} = \frac{1 \pm \gamma_5}{2} = \begin{pmatrix} 1(0) & 0 \\ 0 & 0(1) \end{pmatrix}, \tag{4}$$

$$u(p_i, \pm) = \begin{pmatrix} \acute{u}_A(i\pm) \\ \grave{u}^{\dot{A}}(i\pm) \end{pmatrix}, \qquad \bar{u}(p_i, \pm) = \left(\bar{\acute{u}}^A(i\pm), \quad \bar{\grave{u}}_{\dot{A}}(i\pm)\right) \tag{5}$$

Dirac equation $\hat{p}_i u(p_i) = m_i u(p_i)$

$$\begin{cases} p_{A\dot{B}} \grave{u}^{\dot{B}} = m \acute{u}_A \\ p^{\dot{A}B} \acute{u}_B = m \grave{u}^{\dot{A}} \end{cases} \tag{6}$$

Massive momenta

Decomposition to massless vectors

Whe can decompose any massive vector with $p_i^2 = m_i^2$ onto 2 massless vectors

$$p_i = p_{i+} + p_{i-}, \qquad p_{i+}^2 = p_{i-}^2 = 0, \qquad 2p_{i+}p_{i-} = m_i^2, \tag{7}$$

Arbitrariness of such a decomposition exactly captures spin degree of freedom. Indeed, for a given vector of spin n_i with $n_i \cdot p_i = 0$, $n_i^2 = -1$ we just define

$$p_{i\pm} = \frac{p_i \pm m_i n_i}{2}. \tag{8}$$

We can also reversely define n_i from $p_{i\pm}$

$$n_i = \frac{p_{i+} - p_{i-}}{m_i}. \tag{9}$$

Helicity states

Special choice of quantization axis

If spin quantization axis defined by Pauli-Lubansky vector q_i, with $q_i^2 = 0$, we can proceed in following way

$$p_{i-} = \frac{m_i^2}{2p_i \cdot q_i} q_i \qquad p_{i+} = p_i^\flat = p_i - \frac{m_i^2}{2p_i \cdot q_i} q_i = \frac{p_i q_i p_i}{2p_i \cdot q_i} \tag{10}$$

To obtain helicity state in frame, where $p_i = \{E_i, \vec{p}_i\}$, $E_i > 0$, we can just set $q_i = \{|\vec{p}_i|, -\vec{p}_i\}$. For $p_{i\pm}$ we obtain

$$p_{i\pm} = \frac{E_i \pm |\vec{p}_i|}{2|\vec{p}_i|} \{|\vec{p}_i|, \pm\vec{p}_i\}. \tag{11}$$

For decomposition (10) and in particular for (11) the limit $m \to 0$ gives $p_{i+} = O(1)$ and $p_{i-} = O(m^2)$.

Dirac spinors

Solution for the Dirac equation

$$\acute{u}_A(i\pm) = \lambda_A(i\pm), \quad \grave{u}^{\dot{A}}(i\pm) = \frac{p^{\dot{A}B}}{m_i}\acute{u}_B(i\pm) = \mp\frac{\langle i+|i-\rangle}{m_i}\widetilde{\lambda}^{\dot{A}}(i\mp), \quad (12)$$

$$\grave{\bar{u}}_{\dot{A}}(i\pm) = \widetilde{\lambda}_{\dot{A}}(i\pm), \quad \acute{\bar{u}}^{A}(i\pm) = \frac{p^{A\dot{B}}}{m_i}\grave{\bar{u}}_{\dot{B}}(i\pm) = \mp\frac{[i-|i+]}{m_i}\lambda^{A}(i\mp), \quad (13)$$

Spinor label notation

$$
\begin{aligned}
\lambda_A(i) &= |i\rangle\,, & \lambda^A(i) &= \epsilon^{AB}\lambda_B(i) = \langle i| \\
\tilde{\lambda}^{\dot{A}}(i) &= |i]\,, & \tilde{\lambda}_{\dot{A}}(i) &= \epsilon_{\dot{A}\dot{B}}\tilde{\lambda}^{\dot{B}}(i) = [i| \\
\langle i|j\rangle &= \lambda^A(i)\lambda_A(j), & [j|i] &= \tilde{\lambda}_{\dot{A}}(j)\tilde{\lambda}^{\dot{A}}(i) \\
\langle i|P|j] &= \lambda^A(i)P_{A\dot{A}}\tilde{\lambda}^{\dot{A}}(j)
\end{aligned}
\tag{14}
$$

Spinor diada

Outer products of spinors are related to complex light-like 4-vectors:

$$|1\rangle\,[1| = \frac{1+\gamma_5}{2}\hat{k}_1 \quad |1]\,\langle 1| = \frac{1-\gamma_5}{2}\hat{k}_1$$

$$|1\rangle\,[1| + |1]\,\langle 1| = \hat{k}_1$$

$$\langle 1|\gamma^\mu|1] = [1|\gamma^\mu|1\rangle = 2k_1^\mu$$

Spinor products

Inner products of spinors are complex Lorentz invariants

$$\begin{aligned} \langle a|b\rangle &= \langle k_a|k_b\rangle \quad \langle a|b] = 0 \\ [a|b] &= [k_a|k_b] \quad [a|b\rangle = 0 \\ \langle a|b\rangle &= -\langle b|a\rangle \quad \langle a|a\rangle = 0 \\ [b|a] &= -[a|b] \quad [a|a] = 0 \end{aligned} \tag{15}$$

$$[b|a] = \overline{\langle a|b\rangle} \tag{16}$$

$$\langle a|b\rangle [b|a] = \left|\langle a|b\rangle\right|^2 = 2k_a \cdot k_b = (k_a + k_b)^2 = z_{ab} \tag{17}$$

Spinor decomposition

Schouten identity

Spinor is 2-dimenstional complex vector and can be decomposed onto basis

$$|a_2\rangle \langle b_1|b_2\rangle = |b_1\rangle \langle a_2|b_2\rangle - |b_2\rangle \langle a_2|b_1\rangle = \begin{vmatrix} |b_1\rangle & |b_2\rangle \\ \langle a_2|b_1\rangle & \langle a_2|b_2\rangle \end{vmatrix}$$

The Schouten identity

$$\langle a_1|a_2\rangle \langle b_1|b_2\rangle = \langle a_1|b_1\rangle \langle a_2|b_2\rangle - \langle a_1|b_2\rangle \langle a_2|b_1\rangle = \begin{vmatrix} \langle a_1|b_1\rangle & \langle a_1|b_2\rangle \\ \langle a_2|b_1\rangle & \langle a_2|b_2\rangle \end{vmatrix}$$

Anti-Schouten algorithm

We have implemented an computer-algebra algorithm to automatically apply the rule to reduce expressions

$$\langle a_1|b_1\rangle \langle a_2|b_2\rangle - \langle a_1|b_2\rangle \langle a_2|b_1\rangle \to \langle a_1|a_2\rangle \langle b_1|b_2\rangle$$

Polarization vectors

For massless vector boson with momentum k_1 in axial gauge (fixed by light-like vector k_2) we can construct polarization vectors explicitly in terms of spinor diada

$$\epsilon_\mu(k_1,+,k_2) = \frac{\langle 2|\gamma_\mu|1]}{\sqrt{2}\,\langle 2\,1\rangle}$$

$$\epsilon_\mu(k_1,-,k_2) = \frac{[2|\gamma_\mu|1\rangle}{\sqrt{2}\,[2\,1]}$$

$$\hat{\epsilon}(k_1,+,k_2) = \sqrt{2}\frac{|2\rangle[1| + |1]\langle 2|}{\langle 2\,1\rangle}$$

$$\hat{\epsilon}(k_1,-,k_2) = \sqrt{2}\frac{|2]\langle 1| + |1\rangle[2|}{[2\,1]}$$

Anti-Schouten algorithm allow us to explicitly cancel the gauge-fixing vector! This makes expressions maximally compact.

Massless kinematics

Light-cone projection

In this section we consider all momenta incoming, so that $\sum p_i = 0$.
We project all massive momenta p_i with $p_i^2 = m_i^2$ to the light-cone of photon p_5 and introduce associated "momenta" k_i:

$$k_i = p_i - \frac{m_i^2}{2p_i \cdot p_5} p_5, \quad k_i^2 = 0, \quad i = 1..4, \tag{18}$$

$$\begin{aligned} k_5 &= -\sum_{i=1}^{4} k_i = K p_5, \quad K = 1 + \sum_{i=1}^{4} \frac{m_i^2}{2p_i \cdot p_5} = 1 + \sum_{i=1}^{4} \frac{m_i^2}{2k_i \cdot p_5} \\ p_5 &= -\sum_{i=1}^{4} p_i = K' k_5, \quad K' = 1 - \sum_{i=1}^{4} \frac{m_i^2}{2p_i \cdot k_5} = 1 - \sum_{i=1}^{4} \frac{m_i^2}{2k_i \cdot k_5} \end{aligned} \tag{19}$$

Vector k_5 appear to be light-like, so we left with "momentum conservation" of associated vectors.

HA for $e^+e^- \to ZH\gamma$ (Bremsstrahlung)

$$\mathcal{M}_{--++} = 2em_1 M_Z N(s')\left(\frac{\delta_e}{s_{15}} + \frac{\sigma_e}{s_{25}}\right)[1|2]\,\frac{\langle 3|5\rangle}{[3|5]},$$

$$\mathcal{M}_{++-+} = 2em_1 M_Z N(s')\left(\frac{\sigma_e}{s_{15}} + \frac{\delta_e}{s_{25}}\right)[1|2]\,\frac{[3|5]\,\langle 1|5\rangle\,\langle 2|5\rangle}{\langle 3|5\rangle\,[1|5]\,[2|5]},$$

$$\mathcal{M}_{-+-+} = -2eM_Z N(s')\frac{\sigma_e}{s_{15}}\,\frac{[1|2]\,[1|3]\,\langle 1|5\rangle\,\langle 2|5\rangle}{\langle 3|5\rangle\,[2|5]},$$

$$\mathcal{M}_{+--+} = -2eM_Z N(s')\frac{\delta_e}{s_{25}}\,\frac{[1|2]\,[2|3]\,\langle 2|5\rangle\,\langle 1|5\rangle}{\langle 3|5\rangle\,[1|5]},$$

$$\mathcal{M}_{--0+} = \sqrt{2}em_1 N(s')\left(\frac{\delta_e}{s_{15}}\frac{[2|3]}{[2|5]} + \frac{\sigma_e}{s_{25}}\frac{[1|3]}{[1|5]}\right)[1|2]\,\langle 3|5\rangle\,,$$

$$\mathcal{M}_{-+++} = -2eM_Z N(s')\sigma_e\left(\frac{[1|2]\,\langle 2|3\rangle\,\langle 2|5\rangle}{s_{25}\,[3|5]} + \frac{[1|5]\,\langle 3|5\rangle}{[2|5]\,[3|5]}\right),$$

$$\mathcal{M}_{+-++} = -2eM_Z N(s')\delta_e\left(\frac{[1|2]\,\langle 1|3\rangle\,\langle 1|5\rangle}{s_{15}\,[3|5]} - \frac{[2|5]\,\langle 3|5\rangle}{[1|5]\,[3|5]}\right),$$

HA for $e^+e^- \to ZH\gamma$ (Bremsstrahlung)

$$\mathcal{M}_{++0+} = \sqrt{2}em_1 N(s')\Bigg([1|2] \left(\frac{\sigma_e}{s_{15}} \langle 1|5\rangle \langle 2|3\rangle + \frac{\delta_e}{s_{25}} \langle 2|5\rangle \langle 1|3\rangle \right) + \langle 3|5\rangle \Big(\sigma_e - \delta_e\Big)\Bigg) \frac{[3|5]}{[1|5]\,[2|5]},$$

$$\mathcal{M}_{-+0+} = -\sqrt{2}eN(s')\Bigg(\sigma_e\Big(\frac{[1|3]\,\langle 2|3\rangle \langle 1|5\rangle}{s_{15}} + \frac{[1|3]\,\langle 3|5\rangle}{[1|2]} + \frac{M_Z^2 \langle 2|5\rangle}{s_{45}}\Big) + \delta_e \frac{m_1^2 s_{45}\langle 2|5\rangle}{s_{15}s_{25}}\Bigg)\frac{[1|2]}{[2|5]},$$

$$\mathcal{M}_{+-0+} = -\sqrt{2}eN(s')\Bigg(\delta_e\Big(\frac{[2|3]\,\langle 1|3\rangle \langle 2|5\rangle}{s_{25}} - \frac{[2|3]\,\langle 3|5\rangle}{[1|2]} + \frac{M_Z^2 \langle 1|5\rangle}{s_{45}}\Big) + \sigma_e \frac{m_1^2 s_{45}\langle 1|5\rangle}{s_{15}s_{25}}\Bigg)\frac{[1|2]}{[1|5]},$$

where $s_{i5} = 2k_i \cdot p_5 = K' \langle i|5\rangle \, [5|i]\,.$

Spin quantization axis

Freedom in the light-cone projection choice corresponds to arbitrariness of spin quantization direction. We exploit it to make expressions compact. To obtain amplitudes for specified direction of polarization spin-rotation matrices should be applied.

Transformation to helicity basis

$$\mathcal{H}_{a_i} = C_{a_i}^{\;\;b_i} \mathcal{M}_{b_i}$$

$$C_{a_i}^{\;\;b_i} = \begin{bmatrix} \frac{[i^\flat|5]}{[i|5]} & \frac{m_i \langle i^*|5\rangle}{\langle i^*|i^\flat\rangle \langle i|5\rangle} \\ \frac{m_i [i^*|5]}{[i^*|i^\flat][i|5]} & \frac{\langle i^\flat|5\rangle}{\langle i|5\rangle} \end{bmatrix} = \begin{bmatrix} \frac{\langle i^*|i\rangle}{\langle i^*|i^\flat\rangle} & \frac{m_i \langle i^*|5\rangle}{\langle i^*|i^\flat\rangle \langle i|5\rangle} \\ \frac{m_i [i^*|5]}{[i^*|i^\flat][i|5]} & \frac{[i^*|i]}{[i^*|i^\flat]} \end{bmatrix}$$

$$p_i = \{E_i, p_i^x, p_i^y, p_i^z\}, \qquad p_i^2 = m_i^2$$

$$k_{i^*} = \{|\vec{p}_i|, -p_i^x, -p_i^y, -p_i^z\}, \quad k_{i^*}^2 = 0$$

$$k_{i^\flat} = p_i - \frac{m_i^2}{2 p_i \cdot k_{i^*}} k_{i^*}, \qquad k_{i^\flat}^2 = 0$$

$e^+e^- \to ZH$: WHIZARD vs SANC (Born), fb

$\sqrt{s}$=250 GeV

P_{e^-}, P_{e^+}	0,0	-1,-1	-1,1	1,-1	1,1
WHIZARD	225.59	6.368E-8	552.34	350.01	6.368E-8
CalcHEP	225.59	4.411E-8	552.34	350.02	4.411E-8
SANCee	225.59	0	552.34	350.01	0

$\sqrt{s}$=500 GeV

P_{e^-}, P_{e^+}	0,0	-1,-1	-1,1	1,-1	1,1
WHIZARD	53.738	3.762E-7	131.57	83.377	3.762E-7
CalcHEP	53.738	5.994E-8	131.57	83.377	5.994E-8
SANCee	53.737	0	131.57	83.377	0

$\sqrt{s}$=1000 GeV

P_{e^-}, P_{e^+}	0,0	-1,-1	-1,1	1,-1	1,1
WHIZARD	12.054	4.801E-7	29.515	18.703	4.801E-7
CalcHEP	12.054	2.639E-8	29.515	18.703	2.639E-8
SANCee	12.054	0	29.515	18.703	0

$e^+e^- \to ZH$: WHIZARD and CalcHEP vs SANC (hard), fb

$\sqrt{s}$=250 GeV

P_{e^-}, P_{e^+}	0,0	-1,-1	-1,1	1,-1	1,1
WHIZARD	82.00(1)	0.009143(1)	200.7(2)	127.2(1)	0.01470(1)
CalcHEP	82.00(1)	0.02596(1)	200.8(1)	127.2(1)	0.02596(1)
SANCee	82.00(1)	0.02596(1)	200.7(1)	127.2(1)	0.02597(1)

$\sqrt{s}$=500 GeV

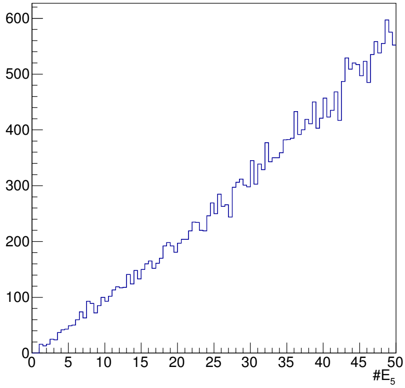
P_{e^-}, P_{e^+}	0,0	-1,-1	-1,1	1,-1	1,1
WHIZARD	38.96(1)	0.1256(1)	95.10(8)	60.27(1)	0.1169(1)
CalcHEP	38.96(1)	0.2201(1)	95.12(1)	60.27(1)	0.2198(1)
SANCee	38.96(1)	0.2200(1)	95.10(1)	60.25(1)	0.2199(1)

$\sqrt{s}$=1000 GeV

P_{e^-}, P_{e^+}	0,0	-1,-1	-1,1	1,-1	1,1
WHIZARD	11.67(1)	0.07051(1)	28.41(1)	18.00(1)	0.07018(1)
CalcHEP	11.67(1)	0.1326(1)	28.41(1)	18.00(1)	0.1326(1)
SANCee	11.67(1)	0.1327(1)	28.40(1)	18.00(1)	0.1326(1)

$e^+e^- \to HZ$: σ distributions on $E\gamma$

$\frac{d\sigma}{dE\gamma}$, with $(P_{e^+}, P_{e^-}) = (-1, -1)$, $\sqrt{s} = 500$GeV



HA for $e^+e^- \to f\bar{f}\gamma$ (Bremsstrahlung)

$$\mathcal{H}^{\text{hard}} = \mathcal{H}^{\text{isr}} + \mathcal{H}^{\text{fsr}}$$

$$\mathcal{H}^{\text{fsr}}_{\chi_1\chi_2\chi_3\chi_4\chi_5}(p_1,p_2,p_3,p_4) = +\mathcal{H}^{\text{isr}}_{-\chi_4-\chi_3-\chi_2-\chi_1\chi_5}(-p_4,-p_3,-p_2,-p_1)$$

HA for $e^+e^- \to e^+e^-\gamma$ (Bremsstrahlung)

$$\mathcal{H}^{\text{hard}} = \mathcal{H}^{\text{isr}} + \mathcal{H}^{\text{fsr}} + \mathcal{H}^{\text{esr}} + \mathcal{H}^{\text{psr}}$$

$$\mathcal{H}^{\text{esr}}_{\chi_1\chi_2\chi_3\chi_4\chi_5}(p_1,p_2,p_3,p_4) = -\mathcal{H}^{\text{isr}}_{+\chi_1-\chi_3-\chi_2+\chi_4\chi_5}(+p_1,-p_3,-p_2,+p_4)$$
$$\mathcal{H}^{\text{psr}}_{\chi_1\chi_2\chi_3\chi_4\chi_5}(p_1,p_2,p_3,p_4) = -\mathcal{H}^{\text{isr}}_{-\chi_4+\chi_2+\chi_3-\chi_1\chi_5}(-p_4,+p_2,+p_3,-p_1)$$

CP-symmetry

$$\mathcal{H}^{\text{hard}}_{\chi_1\chi_2\chi_3\chi_4\chi_5} = -\chi_1\chi_2\chi_3\chi_4\overline{\mathcal{H}}^{\text{hard}}_{-\chi_1-\chi_2-\chi_3-\chi_4-\chi_5}$$

All-helicity amplitude

ISR sub-amplitude

$$\mathcal{H}^{\mathsf{isr}}_{abcd+} = \sqrt{2} Q_2 \frac{1}{s_{34} - M_Z^2} \Bigg\{$$

$$+ g_+^2 \left(\frac{[2_b|3_c] \langle 1_a|4_d\rangle \langle 5|1|2|5\rangle}{(1\cdot 5)(2\cdot 5)} + \frac{[2_b|3_c] \langle 5|1_a\rangle \langle 5|4_d\rangle}{(1\cdot 5)} \right)$$

$$+ g_+ g_- \left(\frac{\left[2_b|\check{2}|4_d\right\rangle \left[3_c|\check{1}|1_a\right\rangle \langle 5|1|2|5\rangle}{(1\cdot 5)(2\cdot 5)} + \frac{\left[2_b|\check{2}|5\right\rangle \left[3_c|\check{1}|1_a\right\rangle \langle 5|4_d\rangle}{(2\cdot 5)} \right)$$

$$+ g_+ g_- \left(\frac{\left[2_b|\check{4}|4_d\right\rangle \left[3_c|\check{3}|1_a\right\rangle \langle 5|1|2|5\rangle}{(1\cdot 5)(2\cdot 5)} + \frac{\left[3_c|\check{3}|5\right\rangle \left[2_b|\check{4}|4_d\right\rangle \langle 5|1_a\rangle}{(1\cdot 5)} \right)$$

$$+ g_-^2 \left(\frac{\left[2_b|\check{2}|\check{3}|3_c\right] \left\langle 1_a|\check{1}|\check{4}|4_d\right\rangle \langle 5|1|2|5\rangle}{(1\cdot 5)(2\cdot 5)} + \frac{\left\langle 1_a|\check{1}|\check{4}|4_d\right\rangle \left\langle 5|\check{2}|2_b\right] \left\langle 5|\check{3}|3_c\right]}{(2\cdot 5)} \right) \Bigg\}$$

with shortcut $\check{p}_i = p_i/m_i$

Future development

- For the processes with multiple-photons in final state momentum-conservation becomes nontrivial and cannot be solved manually;
- Momentum-twistors should help with momentum-conservation and may highlight some hidden conformal symmetry structures;
- Bose-symmetry of photons expected to source another king of cancellations;
- Experience from $\mathcal{N} = 4$ SYM should be applied in NNLO SM calculation;

Conclusion

- Analytical Helicity Amplitudes are much powerful then numerical ones;
- Compact expressions are achievable by traditional computer-algebra algorithms;
- Masses of all particles should be held to avoid collinear artifacts;
- Some interesting possibilities are opened in phase-space parametrization task; May be useful for MC;

Phase space variables

Phase space volume

$$dR_3 = d^4p_3\delta(p_3^2 - m_3^2)d^4p_4\delta(p_4^2 - m_4^2)d^4p_5\delta(p_5^2)\delta^4(\sum_{i=1}^{5} p_i)$$

$$dR_3 = d^4k_3\delta(k_3^2)d^4k_4\delta(k_4^2)d^4k_5\delta(k_5^2)\delta^4(\sum_{i=1}^{5} k_i)K'$$

Some notations

$$p_{i\ldots j} = p_i + \cdots + p_j \quad s_{i\ldots j} = p_{i\ldots j}^2$$
$$k_{i\ldots j} = k_i + \cdots + k_j \quad z_{i\ldots j} = k_{i\ldots j}^2$$